Complex Function Theory and Differential Equations

Kompleks funksjonsteori og differensialligninger

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Chapter 1

The Complex Plane

Complex numbers appeared already in the 16th century, after Cardano's progress on the resolution of the generic cubic equation:

$$x^3 + ax^2 + bx + c = 0, \quad x \in \mathbb{R}. \tag{1.0.1}$$

With the substitution $x' = x - \frac{a}{3}$, this equation can be reduced to

$$x^3 + 3px + 2q = 0, \tag{1.0.2}$$

for new parameters $p, q \in \mathbb{R}$. Cardano showed that if $q^2 + p^3 \geq 0$, the equation (1.0.2) has exactly one real solution, given by the formula

$$x = \sqrt[3]{-q + \sqrt{q^2 + p^3}} + \sqrt[3]{-q - \sqrt{q^2 + p^3}}. \tag{1.0.3}$$

This formula is, a priori, not well-defined in the case where $q^2 + p^3 < 0$. Some years later, Bombelli made the following observation: the cubic equation (1.0.2) with $p = -5$ and $q = -2$, namely,

$$x^3 - 15x - 4 = 0,$$

has three different *real* solutions, despite the fact that $q^2 + p^3 < 0$. It turns out that whenever $q^2 + p^3 < 0$, the equation (1.0.2) has three distinct real solutions, whose values can be obtained from Cardano's formula (1.0.3), if one makes the correct interpretation of the *imaginary number* $\sqrt{q^2 + p^3}$. Note that, even if we are able to manage $\sqrt{q^2 + p^3}$ as a number, the usage of formula (1.0.3) still requires to understand the meaning of a 3rd root of these imaginary objects.

So, we need to understand *complex numbers* to find explicit solutions to equations like (1.0.1), even when these equations have three real solutions.

This is just an instance of the numerous problems in (real) analysis that can only be solved with the usage of complex analysis. We will see a few of these applications in this course.

1.1 The Field of Complex Numbers

This section is devoted to the rigorous definition of complex numbers and their operations.

Definition 1.1. *A* ***complex number*** *is an ordered pair $z = (a, b)$ of real numbers $a, b \in \mathbb{R}$. We define the addition '+' between two complex numbers (a, b) and (c, d) by*

$$(a, b) + (c, d) := (a + c, b + d).$$

So, this operation coincides with the usual sum of vectors in $\mathbb{R}^2$. However, we additionally define a product operation '$\cdot$' between complex numbers by

$$(a, b) \cdot (c, d) := (ac - bd, ad + bc).$$

This product is clearly commutative: $(a, b) \cdot (c, d) = (c, d) \cdot (a, b)$.

We denote by $\mathbb{C}$ the set of all complex numbers equipped with the operations '+' and '$\cdot$'.

We next define the real and imaginary parts of a complex number, as its orthogonal projection onto the x-axis and y-axis respectively.

Definition 1.2. *Given a complex number $z = (a, b)$ the* ***real part*** *of z is defined by $\mathrm{Re}(z) = a$ and the* ***imaginary part*** *of z is $\mathrm{Im}(z) = b$. The* ***pure imaginary*** *numbers are those z with $\mathrm{Re}(z) = 0$.*

The real and imaginary parts of sums and products of complex numbers are:

$$\begin{aligned}
&\mathrm{Re}(z+w) = \mathrm{Re}(z) + \mathrm{Re}(w), \quad \mathrm{Im}(z+w) = \mathrm{Im}(z) + \mathrm{Im}(w),\\
&\mathrm{Re}(zw) = \mathrm{Re}(z)\,\mathrm{Re}(w) - \mathrm{Im}(z)\,\mathrm{Im}(w), \quad \mathrm{Im}(zw) = \mathrm{Re}(z)\,\mathrm{Im}(w) + \mathrm{Im}(z)\,\mathrm{Re}(w).
\end{aligned}$$

Now we describe an easier way to express complex numbers and their operations.

Definition 1.3. *We define the* ***imaginary unit*** *of $\mathbb{C}$ as the complex number $i := (0, 1)$. Note that then*

$$i^2 := i \cdot i = (0,1) \cdot (0,1) = (-1, 0).$$

Now, given $z \in \mathbb{C}$, $z = (a, b)$, we can write z as $z = a(1, 0) + b(0, 1)$ for unique real numbers a, b. Identifying $a(1, 0)$ with a and $b(0, 1)$ with bi, the linear or polynomic expression of z is then

$$z = a + bi.$$

Using this notation, we can define the sum $+ : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and the product $\cdot : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ between complex numbers $z = a + bi$, $w = c + di$ as:

$$\begin{aligned}
z + w &= (a + bi) + (c + di) = (a + c) + (b + d)i\\
z \cdot w &= (a + bi) \cdot (c + di) = (ac - bd) + (ab + bc)i.
\end{aligned}$$

This definition for the addition and the product coincides with those in Definition 1.1.

It is worth noticing that, when regarded as vector spaces over $\mathbb{R}$, the spaces $\mathbb{R}^2$ and $\mathbb{C}$ are identical. In particular a (affine $\mathbb{R}$-)line L of $\mathbb{C}$ is simply

$$L = \{z = x + iy \in \mathbb{C} \,:\, Ax + By = D\}, \quad \text{with } A, B, D \in \mathbb{R},\ (A, B) \neq (0, 0). \tag{1.1.1}$$

Naturally, denoting by F-dim(V) the dimension of the vector/affine space V over the field F, one has $\mathbb{R}$-dim$(\mathbb{C}) = 2$ and $\{1, i\}$ is an $\mathbb{R}$-basis of $\mathbb{C}$. Any line $L \subset \mathbb{C}$ as in (1.1.1) satisfies $\mathbb{R}$-dim$(L) = 1$. But of course, if $\mathbb{C}$ is thought as a vector space over the field $\mathbb{C}$ of scalars (this is verified in Theorem 1.4), then $\mathbb{C}$-dim$(\mathbb{C}) = 1$, and any non-zero complex number provides a $\mathbb{C}$-basis of $\mathbb{C}$.

Theorem 1.4. *The set $\mathbb{C}$ with the operations '$+$' and '$\cdot$' is a field.*

Proof. The operations $+ : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $\cdot : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ are commutative, as we noted in Definition 1.1. We next verify the rest of the axioms of a field, namely, for all $z, w, \xi \in \mathbb{C}$:

(i) $z + (w + \xi) = (z + \xi) + w$;

(ii) $z \cdot (w \cdot \xi) = (z \cdot w) \cdot \xi$;

(iii) $z + 0 = z$, where 0 denotes $0 + 0i$ (Definition 1.3) or $(0, 0)$ (Definition 1.1);

(iv) $z \cdot 1 = z$, where 1 denotes $1 + 0i$ (Definition 1.3) or $(1, 0)$ (Definition 1.1);

(v) $z \cdot (w + \xi) = z \cdot w + z \cdot \xi$;

(vi) $z + (-z) = 0$;

(vii) If $z \neq 0$, there exists $z^{-1} \in \mathbb{C}$ with $z \cdot z^{-1} = 1$.

Properties purely concerning the addition (i), (iii), (vi) are immediate, since the sum in $\mathbb{C}$ is identical to adding the components of the vectors in $\mathbb{R}^2$. Properties (ii) and (v) are tedious but straightforward: if $z = z_1 + iz_2$, $w = w_1 + iw_2$ and $\xi = \xi_1 + i\xi_2$, where $z_1, z_2, w_1, w_2, \xi_1, \xi_2 \in \mathbb{R}$.[1] Then

$$\begin{aligned} z \cdot (w \cdot \xi) &= (z_1 + iz_2) \cdot ((w_1\xi_1 - w_2\xi_2) + i(w_1\xi_2 + w_2\xi_1)) \\ &= z_1(w_1\xi_1 - w_2\xi_2) - z_2(w_1\xi_2 + w_2\xi_1) + i\left(z_1(w_1\xi_2 + w_2\xi_1) + z_2(w_1\xi_1 - w_2\xi_2)\right). \end{aligned}$$

And

$$\begin{aligned} (z \cdot w) \cdot \xi &= ((z_1w_1 - z_2w_2) + i(z_1w_2 + z_2w_1))\,(\xi_1 + i\xi_2) \\ &= (z_1w_1 - z_2w_2)\xi_1 - (z_1w_2 + z_2w_1)\xi_2 + i((z_1w_1 - z_2w_2)\xi_2 + (z_1w_2 + z_2w_1)\xi_1), \end{aligned}$$

confirming that $z \cdot (w \cdot \xi) = (z \cdot w) \cdot \xi$. Also,

$$\begin{aligned} &z(w + \xi) = (z_1 + iz_2)(w_1 + \xi_1 + i(w_2 + \xi_2)) = z_1(w_1 + \xi_1) - z_2(w_2 + \xi_2) + i(z_1(w_2 + \xi_2) + z_2(w_1 + \xi_1)), \\ &zw + z\xi = z_1w_1 - z_2w_2 + i(z_1w_2 + z_2w_1) + z_1\xi_1 - z_2\xi_2 + i(z_1\xi_2 + z_2\xi_1), \end{aligned}$$

which proves (v).

Property (iv) is immediate to check: if $z = a + ib$, then $z \cdot 1 = (a + ib)(1 + i\,0) = a + ib$.

Finally, to prove (vii), let $z = a + ib$, with $(a, b) \neq (0, 0)$, and observe that the number

$$z^{-1} := \frac{a}{a^2 + b^2} + i\frac{(-b)}{a^2 + b^2},$$

satisfies $z \cdot z^{-1} = 1$. □

Let us gather some information that we learnt from the proof Theorem 1.4:

- The identity element for the sum is, obviously, the number $0 = 0 + i \cdot 0$.
- If $z = a+ib \in \mathbb{C}$, the inverse with respect to the sum operation is $-z := -a+(-b)i = -a-ib$. Aslo, in the sequel, by $z - w$ (for any two $z, w \in \mathbb{C}$) we understand $z + (-w)$.
- The identity element for the product is, obviously, the number $1 = 1 + i \cdot 0$.
- If $z = a + ib \in \mathbb{C} \setminus \{0\}$, the inverse with respect to the product operation is the number

$$\frac{a - ib}{a^2 + b^2}.$$

  We will denote the inverse of z by z^{-1} or $\frac{1}{z}$ or $1/z$. In the sequel, for $z, w \in \mathbb{C}$ with $w \neq 0$, $\frac{z}{w} = z/w$ will denote $z \cdot w^{-1}$.

Also, we will often denote products of numbers $z, w \in \mathbb{C}$ by zw, instead of $z \cdot w$.

We finish this section by defining the integer powers of a complex number in the natural way.

Definition 1.5. *Let $z \in \mathbb{C}$ and $n \in \mathbb{N}$. The nth power of z is a complex number z^n defined by*

$$z^n = \overbrace{z \cdots z}^{n \text{ times}}.$$

For $n = 0$, we define $z^n := 1$. And for $n \in \mathbb{Z}$, $n < 0$, and $z \neq 0$, we define $z^n := (z^{-1})^{-n} = \frac{1}{z^{-n}}$.

[1] We will usually use the letters z's, w's, ξ's to denote complex numbers. This is an exception, convenient for the proof.

For example, let us determine the powers of i. Using that $i^2 = -1$ and that $i^{-1} = 1/i = -i$, we can easily deduce that

$$i^{4n} = 1, \quad i^{4n+1} = i, \quad i^{4n+2} = -1, \quad i^{4n+3} = -i \quad \text{for all} \quad n \in \mathbb{Z}. \tag{1.1.2}$$

By of the conmutativity of the product, we have the Newton's binomial formula, and the *ciclotomic* formula for complex numbers:

$$(z+w)^n = \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}, \quad z, w \in \mathbb{C}, \, n \in \mathbb{N}, \tag{1.1.3}$$

$$z^n - w^n = (z-w) \sum_{k=0}^{n-1} z^{n-1-k} w^k, \quad z, w \in \mathbb{C}, \, n \in \mathbb{N}. \tag{1.1.4}$$

The proofs are identical to those of the corresponding identities for real numbers.

1.2 The Conjugate and the Modulus

Definition 1.6. *Let $z = a + ib \in \mathbb{C}$. The* ***complex conjugate*** *of z is the complex number*

$$\overline{z} := a - ib.$$

Also, the ***modulus*** *of z is the (nonnegative) real number given by*

$$|z| := \sqrt{a^2 + b^2}.$$

It is immediate from the definition of $\overline{z}$ that

$$\operatorname{Re}(z) = \frac{z + \overline{z}}{2} \quad \operatorname{Im}(z) = \frac{z - \overline{z}}{2i}. \tag{1.2.1}$$

The conjugate $\overline{z}$ of a complex number $z = a + ib$ is the reflection $(a, -b)$ of the point (a, b) about the x-axis (the real axis).

Also, if $z = a + bi$, the modulus $|z|$ of z coincides with the modulus in $\mathbb{R}^2$ of the vector (a, b). Therefore, $|z|$ represents the distance from z (as a point in the plane) to the origin. And, since $i^2 = -1$, we have $z\overline{z} = (a + ib)(a - ib) = a^2 + b^2 = |z|^2$. That is,

$$|z| = \sqrt{z\overline{z}}, \quad |z|^2 = z\overline{z}, \quad z^{-1} = \frac{1}{z} = \frac{\overline{z}}{|z|^2}, \quad \text{(the last one for } z \neq 0). \tag{1.2.2}$$

We now collect some elementary properties of the conjugate and the modulus. The *compatibility* of these operations with the product is particularly useful.

Proposition 1.7. *Let $z, w \in \mathbb{C}$. The following properties are satisfied.*

(i) *If* $\operatorname{Im}(z) = 0$, *then* $|z|$ *coincides with the absolute value of* $\operatorname{Re}(z) = z$.

(ii) $|z| = 0$ *if and only if* $z = 0$.

(iii) $\overline{\overline{z}} = z$ *and* $|z| = |\overline{z}|$.

(iv) $\overline{z + w} = \overline{z} + \overline{w}$.

(v) $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$ *and* $|z \cdot w| = |z||w|$.

(vi) *If* $w \neq 0$, *then*

$$\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}} \quad \text{and} \quad \left|\frac{z}{w}\right| = \frac{|z|}{|w|}.$$

(vii) $\max\{|\operatorname{Re}(z)|, |\operatorname{Im}(z)|\} \leq |z|$.

(viii) (The Triangle Inequality) $|z+w| \leq |z| + |w|$.

Proof. Properties (i)–(iv) are immediate from the definition of the modulus and conjugates. Writing $z = a + bi$, $w = c + di$, the first equality in (v) follows from

$$\overline{z \cdot w} = \overline{(ac - bd) + i(ad + bc)} = (ac - bd) - i(ad + bc) = (a - ib)(c - id) = \overline{z} \cdot \overline{w}.$$

For the second indentity, we use the definition of modulus, (1.2.2), and that $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$:

$$|z \cdot w|^2 = (zw)(\overline{zw}) = zw\overline{z}\,\overline{w} = (z\overline{z})(w\overline{w}) = |z|^2|w|^2.$$

To verify (vi), denote $\xi = z/w$, so that $\xi w = z$. By (v) this implies $\overline{\xi}\,\overline{w} = \overline{z}$ and $|\xi||w| = |z|$, and dividing (resp.) by $\overline{w}$ and $|w|$, we deduce the identities.

Property (vii) is an immediate consequence of the definition of modulus given in Definition 1.6.

As concerns (viii), observe that, thanks to (1.2.2),

$$|z + w|^2 = (z + w)(\overline{z + w}) = (z + w)(\overline{z} + \overline{w}) = z\overline{z} + w\overline{w} + z\overline{w} + \overline{z}w = |z|^2 + |w|^2 + 2\operatorname{Re}(z\overline{w}).$$

The last equality follows from $\overline{z\overline{w}} = \overline{z}w$ (thanks to property (v)). Using properties (vii), (v), and (iii) (in that order):

$$|\operatorname{Re}(z\overline{w})| \leq |z\overline{w}| = |z||w|.$$

Combining the previous chain of equalities and this inequality, we conclude

$$|z + w|^2 = |z|^2 + |w|^2 + 2\operatorname{Re}(z\overline{w}) \leq |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2.$$

□

1.3 Lines and Circles in the Complex Plane

As we observed in Section 1.1, lines L of $\mathbb{C}$ (1-dimensional affine subspaces of $\mathbb{C}$, as a vector space over $\mathbb{R}$) have the following general description:

$$L = \{x + iy \in \mathbb{C} \,:\, Ax + By = D\}, \quad \text{with } A, B, D \in \mathbb{R},\ (A, B) \neq (0, 0). \tag{1.3.1}$$

There is an alternate way to represent the equation of a line using complex conjugates. Indeed, if $z = x + iy \in L$, with L as in (1.3.1), then (1.2.1) gives

$$Ax + By = A\operatorname{Re}(z) + B\operatorname{Im}(z) = A\left(\frac{z + \overline{z}}{2}\right) + B\left(\frac{z - \overline{z}}{2i}\right) = \frac{A - Bi}{2}z + \frac{A + Bi}{2}\overline{z} = \xi z + \overline{\xi}\overline{z},$$

taking $\xi = \frac{A - Bi}{2}$. This shows that **any line L in the complex plane can be described as:**

$$L = \{z \in \mathbb{C} \,:\, \xi z + \overline{\xi}\,\overline{z} = D\}, \quad \text{with } \xi \in \mathbb{C} \setminus \{0\},\ D \in \mathbb{R}. \tag{1.3.2}$$

Observing that $\xi z + \overline{\xi}\overline{z} = 2\operatorname{Re}(\xi z)$, we can also express this as

$$L = \{z \in \mathbb{C} \,:\, \operatorname{Re}(\xi z) = D\}, \quad \text{with } \xi \in \mathbb{C} \setminus \{0\},\ D \in \mathbb{R}. \tag{1.3.3}$$

On the other hand, we can use the modulus to describe circles in the plane: the circle centered at $z_0 = x_0 + iy_0$ and with radius $r > 0$ is the set of all $z = x + iy \in \mathbb{C}$ determined by the equation

$$|z - z_0| = r, \quad \text{or, equivalently,} \quad (x - x_0)^2 + (y - y_0)^2 = r^2. \tag{1.3.4}$$

But also note that (1.2.2) yields

$$|z - z_0|^2 = (z - z_0)(\overline{z - z_0}) = (z - z_0)(\overline{z} - \overline{z_0}) = |z|^2 + |z_0|^2 - \overline{z_0}z - z_0\overline{z} = |z|^2 + |\xi|^2 + \xi z + \overline{\xi}\,\overline{z},$$

after the substitution $\xi = -\overline{z_0}$. This shows that

$$|z - z_0|^2 \iff |z|^2 + \xi z + \overline{\xi z} = r^2 - |\xi|^2.$$

Therefore, an **alternate description of circles** S **of** $\mathbb{C}$ is given by

$$S = \{z \in \mathbb{C} \,:\, |z|^2 + \xi z + \overline{\xi z} = D\}, \quad \text{with } \xi \in \mathbb{C},\, K \in \mathbb{R},\, K > -|\xi|^2. \tag{1.3.5}$$

Then S is the circle with center $-\overline{\xi}$ and radius $\sqrt{K + |\xi|^2}$. Note that also $\xi z + \overline{\xi z} = 2\,\mathrm{Re}(\xi z)$, from which we can give another formulation using only real parts.

1.4 Polar Coordinates representation. The Argument

Given a complex number $z \in \mathbb{C} \setminus \{0\}$, the number $\frac{z}{|z|}$ has modulus equal to 1, and so it is contained the unit circle of $\mathbb{R}^2$. Identifying $\frac{z}{|z|}$ with a vector (x, y), we thus have $x^2 + y^2 = 1$, and so $(x, y) = (\cos\theta, \sin\theta)$ for some angle $\theta \in \mathbb{R}$. Let us formalize this.

Theorem 1.8. *For any $z \in \mathbb{C} \setminus \{0\}$, there exists a unique $\alpha \in (-\pi, \pi]$ so that*

$$z = |z|\,(\cos\alpha + i \sin\alpha)\,. \tag{1.4.1}$$

Proof. Assuming we have proven the existence, let us verify the uniqueness of α. If $\alpha, \beta \in (-\pi, \pi]$ satisfy (1.4.1), then

$$z = |z|\,(\cos\alpha + i\sin\alpha) = |z|\,(\cos\beta + i\sin\beta)\,;$$

whereby $\cos\alpha = \cos\beta$ and $\sin\alpha = \sin\beta$. Because $|\alpha - \beta| < 2\pi$, this yields $\alpha = \beta$.

To prove the existence of $\alpha \in (-\pi, \pi]$ such that (1.4.1) holds, we write $z/|z| = x + iy$, where $x^2 + y^2 = 1$ due to the fact that $|z/|z|| = 1$. Also recall that the $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ is defined to be a continuous bijection between $\mathbb{R}$ and $(-\pi/2, \pi/2)$. We distinguish some cases.

Case 1: $x > 0$. Then it suffices to define $\alpha := \arctan\left(\frac{y}{x}\right) \in (-\pi/2, \pi/2)$, where clearly $x = \cos\alpha$, $y = \sin\alpha$.

Case 2: $x < 0, y \geq 0$. In this case $\beta := \arctan\left(\frac{y}{x}\right) \in (-\pi/2, 0]$ is not the desired angle, as $\cos\beta = -x$ and $\sin\beta = -y$. But instead we can take $\alpha := \beta + \pi \in (\pi/2, \pi]$, from which $\cos\alpha = x$ and $\sin\alpha = y$.

Case 3: $x < 0$, $y < 0$. Here again $\beta := \arctan\left(\frac{y}{x}\right) \in (0, \pi/2)$ gives $\cos\beta = -x$ and $\sin\beta = -y$. So, we take $\alpha := \beta - \pi \in (-\pi, -\pi/2)$ and $\cos\alpha = x$, $\sin\alpha = y$.

Case 4: $x = 0$ and $y > 0$. We define $\alpha = \pi/2$, and obviously $\cos\alpha = x$, $\sin\alpha = y$.

Case 5: $x = 0$ and $y < 0$. We define $\alpha = -\pi/2$, and we get $\cos\alpha = x$, $\sin\alpha = y$. □

Observe that if $z \in \mathbb{C} \setminus \{0\}$ and α is as in (1.4.1), then also

$$z = |z|\,(\cos(\alpha + 2k\pi) + i\sin(\alpha + 2k\pi))\,,$$

for all $k \in \mathbb{Z}$. This is due to the 2π-periodicity of the functions $\mathbb{R} \ni \theta \mapsto \cos(\theta)$, $\mathbb{R} \ni \theta \mapsto \sin(\theta)$. Theorem 1.8 and this small observation lead us to the following fundamental definition.

Definition 1.9. *Given $z \in \mathbb{C} \setminus \{0\}$, the **argument** of z is the **set** of real numbers*

$$\arg(z) := \{\alpha \in \mathbb{R} \,:\, z = |z|\,(\cos\alpha + i\sin\alpha)\}.$$

*And the **principal argument** of z is the **unique real number** $\mathrm{Arg}(z) \in (-\pi, \pi] \cap \arg(z)$.*

By virtue of Theorem 1.8, $\operatorname{Arg}(z)$ is well-defined, and, rephrasing the previous definition, $\operatorname{Arg}(z)$ is the element of $\arg(z)$ contained in the interval $(-\pi, \pi]$. Moreover, from the proof of Theorem 1.8 we learnt how to explicitly define $\operatorname{Arg}(z)$, for $z \neq 0$, in terms of $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$:

$$\operatorname{Arg}(z) = \operatorname{Arg}(x+iy) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } \ x>0, \\ \arctan\left(\frac{y}{x}\right)+\pi & \text{if } \ x<0,\ y\geq 0, \\ \arctan\left(\frac{y}{x}\right)-\pi & \text{if } \ x>0,\ y<0, \\ \frac{\pi}{2} & \text{if } \ x=0,\ y>0, \\ -\frac{\pi}{2} & \text{if } \ x=0,\ y<0. \end{cases} \tag{1.4.2}$$

Here $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ is the usual $\tan^{-1}$ bijection. In particular,

$$\begin{cases} \operatorname{Arg}(z) \in [0,\pi] & \text{if } \ \operatorname{Im}(z) \geq 0, \\ \operatorname{Arg}(z) \in (-\pi, 0) & \text{if } \ \operatorname{Im}(z) < 0. \end{cases} \tag{1.4.3}$$

For example,

$$\operatorname{Arg}(1+i) = \frac{\pi}{4}, \quad \operatorname{Arg}(-1+i) = \frac{3\pi}{4}, \quad \operatorname{Arg}(-1-i) = \frac{-3\pi}{4}, \quad \operatorname{Arg}(1-i) = \frac{-\pi}{4}.$$

In the following lemma we show that it is enough to find one value of the argument of z to obtain all of $\arg(z)$.

Lemma 1.10. *If* $z \in \mathbb{C} \setminus \{0\}$, *and* $\alpha \in \arg(z)$, *then*

$$\arg(z) = \{\alpha + 2\pi k \,:\, k \in \mathbb{Z}\}.$$

In particular, $\arg(z) = \{\operatorname{Arg}(z) + 2\pi k \,:\, k \in \mathbb{Z}\}$.

Proof. If $\theta \in \arg(z)$, then

$$|z| (\cos\theta + i \sin\theta) = z = |z| (\cos\alpha + i\sin\alpha),$$

and so $\cos\theta = \cos\alpha$ and $\sin\theta = \sin\alpha$, implying $\theta = \alpha + 2\pi k$ for some $k \in \mathbb{Z}$. Conversely, any number of the form $\alpha + 2\pi k$, $k \in \mathbb{Z}$, belongs to $\arg(z)$ by the observation subsequent to Theorem 1.8. □

Slightly abusing of terminology, Lemma 1.10 can be rewritten as $\arg(z) = \operatorname{Arg}(z) + 2\pi\mathbb{Z}$.

1.5 De Moivre's Formula and the Exponential Form

In the previous section, we saw how to express the (non-zero) complex numbers through the bijection

$$(0,\infty) \times (-\pi, \pi] \ni (r, \theta) \mapsto r(\cos\alpha + i\sin\alpha) \in \mathbb{C} \setminus \{0\},$$

whose inverse is the map $\mathbb{C} \setminus \{0\} \ni z \mapsto (|z|, \operatorname{Arg}(z)) \in (0,\infty) \times (-\pi,\pi]$. This polar coordinate representation turns out to be instrumental in computing products, powers, and roots of any complex number. One of the key ingredients is the following theorem due to De Moivre.

Theorem 1.11 (De Moivre)**.** *The following statements hold.*

(i) Let $\alpha, \beta \in \mathbb{R}$. *Then*

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta).$$

(ii) Let $n \in \mathbb{Z}$ and $\theta \in \mathbb{R}$. Then

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

(iii) Let $z \in \mathbb{C} \setminus \{0\}$, and $n \in \mathbb{Z}$, with $z = |z|(\cos\alpha + i\sin\alpha)$, for $\alpha \in \mathbb{R}$, then

$$z^n = |z|^n \left(\cos(n\theta) + i\sin(n\theta)\right).$$

Proof. To prove part (i) we note that the product equals

$$(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i\,(\cos\alpha\sin\beta + \sin\alpha\cos\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta),$$

after employing the well-known trigonometric formulae for the sum of two angles:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta, \quad \sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta.$$

Let us now prove part (ii), so fix $\theta \in \mathbb{R}$. The identity is true in the case $n = 0$, since $\cos(n\theta) = 1$ and $\sin(n\theta) = 0$. Let us verify the assertion for all $n \in \mathbb{N}$ by induction on n. The case $n = 1$ is trivial. Assume now that $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$, and for the $n+1$ power we write

$$\begin{aligned}(\cos\theta + i\sin\theta)^{n+1} &= (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)\left(\cos(n\theta) + i\sin(n\theta)\right)\\ &= \cos(\theta + n\theta) + i\sin(\theta + n\theta) = \cos((n+1)\theta) + i\sin((n+1)\theta),\end{aligned}$$

where we used statement (i) in the second last equality. So, (ii) holds for all $n \in \mathbb{N} \cup \{0\}$. Now, for $n < 0$, we use the result for the positive power $-n$ to get

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= \frac{1}{(\cos\theta + i\sin\theta)^{-n}} = \frac{1}{\cos(-n\theta) - i\sin(-n\theta)}\\ &= \frac{\cos(-n\theta) - i\sin(-n\theta)}{\cos^2(-n\theta) + \sin^2(-n\theta)} = \cos(n\theta) + i\sin(n\theta).\end{aligned}$$

Finally, part (iii) is immediate from (ii). □

Among other applications, Theorem 1.11 permits to describe the argument of the product of complex numbers.

Corollary 1.12. *Let $z, w \in \mathbb{C} \setminus \{0\}$. Then*

$$\arg(zw) = \arg(z) + \arg(w) := \{\alpha + \beta \,:\, \alpha \in \arg(z),\ \beta \in \arg(w)\}.$$

Proof. For any two angles $\alpha \in \arg(z)$ and $\beta \in \arg(w)$, we have $z = |z|(\cos\alpha + i\sin\alpha)$ and $w = |w|(\cos\beta + i\sin\beta)$. The product zw is then

$$zw = |z|(\cos\alpha + i\sin\alpha)|w|(\cos\beta + i\sin\beta) = |zw|\left(\cos(\alpha+\beta) + i\sin(\alpha+\beta)\right);$$

where we have invoked Theorem 1.11(i). According to Definition 1.9, this shows $\alpha + \beta \in \arg(zw)$. Therefore, by Lemma 1.10,

$$\arg(zw) = \{(\alpha+\beta) + 2k\pi \,:\, k \in \mathbb{Z}\} = \{\alpha + 2k\pi \,:\, k \in \mathbb{Z}\} + \{\beta + 2k\pi \,:\, k \in \mathbb{Z}\} = \arg(z) + \arg(w).$$

□

In the proof of Corollary 1.12 we saw the ideology behind the multiplication of complex numbers $z, w \in \mathbb{C} \setminus \{0\}$: if $\alpha \in \arg(z)$, $\beta \in \arg(z)$, then

$$zw = |z||w|\left(\cos(\alpha+\beta) + i\sin(\alpha+\beta)\right).$$

Roughly speaking: to multiply complex numbers, *we multiply the moduli and sum the arguments.*

Corollary 1.12 does not hold if we replace the argument arg with the principal argument Arg . For instance, if $z = w = -i$, then $\operatorname{Arg}(z) = \operatorname{Arg}(w) = -\frac{\pi}{2}$ but $\operatorname{Arg}(zw) = \operatorname{Arg}(-1) = \pi$, so $\operatorname{Arg}(zw) \neq \operatorname{Arg}(z) + \operatorname{Arg}(w)$.

Notation 1.13 (Exponential form of complex numbers)**.** For every $\theta \in \mathbb{R}$, we define

$$e^{i\theta} := \cos\theta + i\sin\theta. \tag{1.5.1}$$

At the moment, this notation is simply a shorthand for the complex trigonometric formula, and, a priori, not related to the Euler number e. However, in Section 2.4, we will define the complex exponential function $\mathbb{C} \ni z \mapsto e^z$, which agrees with the previous formula in the pure imaginary numbers and with the real exponential $\mathbb{R} \ni x \mapsto e^x$ in the real numbers.

De Moivre's Theorem 1.11 in this exponential form reads as

$$e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)}, \quad (e^{i\theta})^n = e^{in\theta}, \quad \alpha, \beta, \theta \in \mathbb{R},\, n \in \mathbb{Z}. \tag{1.5.2}$$

Moreover, it is immediate that

$$\mathrm{Re}(e^{i\theta}) = \cos\theta, \quad \mathrm{Im}(e^{i\theta}) = \sin\theta, \quad |e^{i\theta}| = 1, \quad \overline{e^{i\theta}} = e^{-i\theta}, \quad \frac{1}{e^{i\theta}} = e^{-i\theta}. \tag{1.5.3}$$

Now, for any $z \in \mathbb{C}\setminus\{0\}$, we learnt from Theorem 1.8 and Definition 1.9 that $z = |z|(\cos\alpha + i\sin\alpha)$, for any $\alpha \in \arg(z)$. Thus z can be written as follows, called the **exponential form** of z,

$$z = |z|e^{i\alpha}.$$

1.6 Roots of Complex Numbers

Definition 1.14 (nth root)**.** *Let $w \in \mathbb{C}$ and $n \in \mathbb{N}$. The n**th root of** w **is the set** consisting of all solutions of the equation $z^n = w$, that is,*

$$\langle \sqrt[n]{w} \rangle := \{z \in \mathbb{C} \,:\, z^n = w\}.$$

In the case $n = 2$, we typically use the simpler notation $\langle \sqrt{w} \rangle$ instead of $\langle \sqrt[2]{w} \rangle$.

Let us give a precise description of these nth roots.

Theorem 1.15 (nth roots of complex numbers)**.** *Let $w \in \mathbb{C} \setminus \{0\}$, and $n \in \mathbb{N}$. Then $\langle \sqrt[n]{w} \rangle$ contains precisely n (distinct) elements. Moreover, for any $\alpha \in \arg(w)$, we have*

$$\langle \sqrt[n]{w} \rangle = \left\{ \sqrt[n]{|w|} \left(\cos\left(\frac{\alpha + 2\pi j}{n}\right) + i\sin\left(\frac{\alpha + 2\pi j}{n}\right) \right) \,:\, j = 0, 1, \ldots, n-1 \right\}.$$

Proof. Fix some $\alpha \in \arg(w)$. A complex number $z \in \mathbb{C} \setminus \{0\}$ satisfies the equation $z^n = w$ (i.e. belongs to $\langle \sqrt[n]{w} \rangle$) if and only if

$$|z|^n \left(\cos(\mathrm{Arg}(z)) + i\sin(\mathrm{Arg}(z))\right)^n = |w| \left(\cos\alpha + i\sin\alpha\right).$$

By Theorem 1.11, this is equivalent to

$$|z|^n \left(\cos(n\,\mathrm{Arg}(z)) + i\sin(n\,\mathrm{Arg}(z))\right) = |w| \left(\cos\alpha + i\sin\alpha\right).$$

But this is in turn equivalent to the three equations

$$|z|^n = |w|, \quad \cos(n\,\mathrm{Arg}(z)) = \cos\alpha, \quad \sin(n\,\mathrm{Arg}(z)) = \sin\alpha,$$

that is,

$$|z| = \sqrt[n]{|w|}, \quad n\,\mathrm{Arg}(z) - \alpha \in 2\pi\mathbb{Z}.$$

Therefore $z \in \langle \sqrt[n]{w} \rangle$ if and only if $z = z_j := \sqrt[n]{|w|}\left(\cos\left(\frac{\alpha+2\pi j}{n}\right) + i\sin\left(\frac{\alpha+2\pi j}{n}\right)\right)$, with $j \in \mathbb{Z}$. But the Euclidean division says that $j = nm_j + r_j$, for $m_j, r_j \in \mathbb{Z}$ with $0 < r_j \leq n-1$, and the 2π-periodicity then implies $z_j = z_{r_j}$, from which $\langle \sqrt[n]{w} \rangle = \{z_0, z_1, \ldots, z_{n-1}\}$. And if $k, j \in \{0, \ldots, n-1\}$ are distinct, then $|(\alpha + 2k\pi)/n - (\alpha + 2j\pi)/n| < 2\pi$, leading to $z_k \neq z_j$.

□

In exponential form, Theorem 1.15 can be summarized as

$$\left\langle \sqrt[n]{re^{i\theta}} \right\rangle = \{r^{1/n}e^{i\frac{\theta+2\pi j}{n}} \,:\, j = 0, 1, \ldots, n-1\}, \quad \text{for all} \quad r > 0,\, \theta \in \mathbb{R}.$$

Definition 1.16 (Principal nth root)**.** *Let $w \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$. The **principal nth root** of w is defined by*

$$\sqrt[n]{w} := \sqrt[n]{|w|}\, e^{\frac{\mathrm{Arg}(w)}{n} i} = \sqrt[n]{|w|} \left(\cos\left(\frac{\mathrm{Arg}(w)}{n}\right) + i \sin\left(\frac{\mathrm{Arg}(w)}{n}\right)\right).$$

And when $w = 0$ and $n \in \mathbb{N}$, we simply define $\sqrt[n]{0} := 0$. As in Definition 1.14, in the case $n = 2$ we may denote $\sqrt[2]{w}$ by $\sqrt{w}$.

Observe that $\sqrt[n]{w}$ is a (complex) number, but the nth root $\langle \sqrt[n]{w} \rangle$ of w in Definition 1.14 is a set. For nonnegative real numbers this principal nth root coincides with the usual nth root real function: $[0, +\infty) \ni x \mapsto x^{1/n}$. However, unlike for real numbers, the principal root of a product is **not** necessarily the product of the principal roots, as shown by the example $z = w = -1$, $n = 2$:

$$1 = \sqrt{(-1) \cdot (-1)} \neq \sqrt{-1} \cdot \sqrt{-1} = i \cdot i = -1.$$

Nonetheless, still certain product formula holds if we consider the sets nth roots.

Proposition 1.17. *Let $n \in \mathbb{N}$, $w, z \in \mathbb{C}$. Then*

$$\langle \sqrt[n]{zw} \rangle = \{uv \,:\, u \in \langle \sqrt[n]{z} \rangle,\, v \in \langle \sqrt[n]{w} \rangle\} =: \langle \sqrt[n]{z} \rangle \cdot \langle \sqrt[n]{w} \rangle.$$

Proof. If $\xi \in \langle \sqrt[n]{z} \rangle \cdot \langle \sqrt[n]{w} \rangle$, then $\xi = \xi_1 \cdot \xi_2$, with $\xi_1 \in \langle \sqrt[n]{z} \rangle$ and $\xi_2 \in \langle \sqrt[n]{w} \rangle$. By definition of the set nth root, we have $\xi_1^n = z$ and $\xi_2^n = w$, implying that $\xi^n = zw$, and so $\xi \in \langle \sqrt[n]{zw} \rangle$.

Conversely, let $\xi \in \langle \sqrt[n]{zw} \rangle$. By Theorem 1.15, we can write

$$\xi = \sqrt[n]{|zw|} e^{i\left(\frac{\theta+2\pi j}{n}\right)}, \quad \text{for some} \quad \theta \in \arg(zw),\, j \in \{0, \ldots, n-1\}.$$

But Corollary 1.12 tells us that $\theta = \theta_1 + \theta_2$ with $\theta_1 \in \arg(z)$ and $\theta_2 \in \arg(w)$. Hence,

$$\xi = \sqrt[n]{|zw|} e^{i\left(\frac{\theta_1+\theta_2+2\pi j}{n}\right)} = \sqrt[n]{|z|} e^{i\frac{\theta_1}{n}} \sqrt[n]{|w|} e^{i\left(\frac{\theta_2+2\pi j}{n}\right)};$$

where clearly $\sqrt[n]{|z|} e^{i\frac{\theta_1}{n}} \in \langle \sqrt[n]{z} \rangle$ and $\sqrt[n]{|w|} e^{i\left(\frac{\theta_2+2\pi j}{n}\right)} \in \langle \sqrt[n]{w} \rangle$ by virtue of Theorem 1.15. ☐

Theorem 1.15 gave all the solutions $z \in \mathbb{C}$ to equations of the form $z^n = w$, or equivalently all the zeros of the polynomial function $\mathbb{C} \ni z \mapsto z^n - w$. A polynomial in $\mathbb{C}$ is a function $P : \mathbb{C} \to \mathbb{C}$ of the form

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n, \quad a_0, \ldots, a_n \in \mathbb{C},\, n \in \mathbb{N},$$

and the zeros or roots of P is the set $P^{-1}(0) := \{z \in \mathbb{C} \,:\, P(z) = 0\}$. This is a problem we will take up later in Section 4.4.4, where we will show that $\mathbb{C}$ is *algebraically closed*, meaning that every nonconstant polynomial has at least one root in $\mathbb{C}$. We will actually show that a polynomial of degree n hast exactly n roots in $\mathbb{C}$, counted with multiplicity.

1.7 The Extended Complex Plane

It is sometimes useful to add to $\mathbb{C}$ an external point or *point at infinity* (for $\mathbb{C}$), denoted by ∞.

Definition 1.18 (Extended Complex Plane)**.** *If ∞ denotes a point at infinity for $\mathbb{C}$, we define the **extended complex plane** by $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$.*

So far $\mathbb{C}_\infty$ is nothing but $\mathbb{C}$ along with an element (denoted by ∞) such that $\infty \notin \mathbb{C}$. However, it turns out that when $\mathbb{C}_\infty$ is equipped with an appropriate distance, we get a *topological metric space* that is *homeomorphic* (a bijection whose inverse and itself are continuous) to the unit sphere of $\mathbb{R}^3$:

$$\mathbb{S}^2 := \{(X, Y, X) \in \mathbb{R}^3 \,:\, X^2 + Y^2 + Z^2 = 1\}.$$

The way to establish this relation between $\mathbb{C}_\infty$ and $\mathbb{S}^2$ is via the *stereographic projection.*

Definition 1.19 (The Stereographic Projection)**.** *Denote by $N = (0, 0, 1) \in \mathbb{R}^3$, the* north pole*. The **Stereographic Projection** onto $\mathbb{C}$ is the mapping $\Pi : \mathbb{S}^2 \to \mathbb{C}_\infty$ given by*

$$\Pi(P) = \begin{cases} \text{the unique point } z \in L_{N,P} \cap \mathbb{C} & \text{if } P \in \mathbb{S}^2 \setminus \{N\} \\ \infty & \text{if } P = N. \end{cases} \tag{1.7.1}$$

Here $L_{N,P}$ denotes the affine line in $\mathbb{R}^3$ passing through N and P.

The mapping $\Pi : \mathbb{S}^2 \to \mathbb{C}_\infty$ in (1.7.1) is obviously well-defined, and we next determine the explicit formula for those points $z \in L_{N,P} \cap \mathbb{C}$ in terms of P.

Proposition 1.20. *The mapping $\Pi : \mathbb{S}^2 \to \mathbb{C}_\infty$ defined in* (1.7.1) *satisfies*

$$\Pi(X, Y, Z) = \frac{X + iY}{1 - Z} \equiv \left(\frac{X}{1 - Z}, \frac{Y}{1 - Z}\right) \quad \text{for all} \quad (X, Y, Z) \in \mathbb{S}^2 \setminus \{N\}. \tag{1.7.2}$$

Moreover, $\Pi : \mathbb{S}^2 \to \mathbb{C}_\infty$ is a bijection whose inverse $\Pi^{-1} : \mathbb{C}_\infty \to \mathbb{S}^2$ is given by

$$\Pi^{-1}(z) = \begin{cases} \dfrac{1}{|z|^2 + 1}\left(2\operatorname{Re}(z), 2\operatorname{Im}(z), |z|^2 - 1\right) & \text{if } z \in \mathbb{C} \\ N = (0, 0, 1) & \text{if } z = \infty. \end{cases} \tag{1.7.3}$$

Proof. If $P := (X, Y, Z) \in \mathbb{S}^2 \setminus \{N\}$, the line in $\mathbb{R}^2$ generated by P and N is the set

$$L_{N,P} = \{(0, 0, 1) + \lambda(X, Y, Z - 1) \,:\, \lambda \in \mathbb{R}\}.$$

Identifying $\mathbb{C} \simeq \mathbb{R}^2$, this line intersects $\mathbb{C}$ if and only if $\lambda(Z - 1) = 1$, from which we must have $\lambda = \frac{1}{1-Z}$. The corresponding point in $L_{N,P} \cap \mathbb{R}^2$ is therefore $(\frac{X}{1-Z}, \frac{Y}{1-Z})$. This proves (1.7.2).

To prove that $\Pi : \mathbb{S}^2 \to \mathbb{C}_\infty$ is a bijection whose inverse satisfies (1.7.3), given $z \in \mathbb{C}$ we find a unique $P \in \mathbb{S}^2 \setminus \{(0, 0, 1)\}$ such that $\Pi(P) = z$. Regarding $z \equiv (\operatorname{Re}(z), \operatorname{Im}(z), 0)$ as point of $\mathbb{R}^3$, the point $P = (X, Y, Z)$ must belong to the intersection of $\mathbb{S}^2 \setminus \{(0, 0, 1)\}$ with the line that passes through N and z :

$$\{(0, 0, 1) + \lambda(\operatorname{Re}(z), \operatorname{Im}(z), -1) \,:\, \lambda \in \mathbb{R}\};$$

The desired $\lambda \in \mathbb{R}$ must satisfy

$$\lambda^2 \operatorname{Re}(z)^2 + \lambda^2 \operatorname{Im}(z)^2 + (1 - \lambda)^2 = 1,$$

or equivalently $\lambda^2 \left(|z|^2 + 1\right) = 2\lambda$. The value $\lambda = 0$ corresponds to the point $(0, 0, 1)$ of the line, which we are not interested in. So the unique admissible solution to the equation is $\lambda = \frac{2}{|z|^2+1}$, and the point P satisfies

$$P = \frac{1}{|z|^2 + 1}\left(2\operatorname{Re}(z), 2\operatorname{Im}(z), |z|^2 - 1\right).$$

Thus we get a bijection $\Pi : \mathbb{S}^2 \setminus \{N\} \to \mathbb{C}$, which obviously extends to $\mathbb{S}^2 \to \mathbb{C}_\infty$ since $\Pi(N) = \infty$. Moreover (1.7.3) holds for all $z \in \mathbb{C}_\infty$. □

By Proposition 1.20, the stereographic projection Π defines a bijection between $\mathbb{S}^2$ and $\mathbb{C}_\infty$. In fact, we can use Π to define a distance function in $\mathbb{C}_\infty$, and so a topology in $\mathbb{C}_\infty$.

Definition 1.21 (Spherical Metric). *The **spherical or chordal metric** in $\mathbb{C}_\infty$ is the function $\rho : \mathbb{C}_\infty \times \mathbb{C}_\infty \to [0, +\infty)$ given by*

$$\rho(z,w) := \|\Pi^{-1}(z) - \Pi^{-1}(w)\| = \sqrt{(X-X')^2 + (Y-Y')^2 + (Z-Z')^2}, \tag{1.7.4}$$

whenever $z, w \in \mathbb{C}_\infty$, $\Pi^{-1}(z) = (X,Y,Z) \in \mathbb{S}^2$, $\Pi^{-1}(w) = (X',Y',Z') \in \mathbb{S}^2$.

Note that $\rho(z,w) \le \operatorname{diam}(\mathbb{S}^2) = 2$ for all $z, w \in \mathbb{C}_\infty$. Let us express $\rho(z,w)$ solely in terms of $z, w \in \mathbb{C}_\infty$.

Proposition 1.22. *The sperical metric $\rho : \mathbb{C}_\infty \times \mathbb{C}_\infty \to [0,+\infty)$ is a distance function and*

$$\rho(z,w) = \begin{cases} \dfrac{2|z-w|}{\sqrt{|z|^2+1}\sqrt{|w|^2+1}} & \text{if } z, w \in \mathbb{C} \\ \dfrac{2}{\sqrt{|z|^2+1}} & \text{if } z \in \mathbb{C},\ w = \infty \\ 0 & \text{if } z = w = \infty. \end{cases} \tag{1.7.5}$$

Proof. The fact that ρ is a distance is a consequence of (1.7.4) and the fact that $\|\cdot\|$ is a norm in $\mathbb{R}^3$; and the only (perhaps) non-trivial property to verify is that $\rho(z,w) = 0 \implies z = w$. But this is also very easy because $\rho(z,w) = 0$ implies that $\Pi^{-1}(z) = \Pi^{-1}(w)$, where Π^{-1} is injective by Proposition 1.20, and hence $z = w$.

To check formula (1.7.5), we start with points $z, w \in \mathbb{C}$, for which $\Pi^{-1}(z) = (X,Y,Z)$ and $\Pi^{-1}(w) = (X',Y',Z')$, with $X^2+Y^2+Z^2 = (X')^2+(Y')^2+(Z')^2 = 1$. Using first these identities, then formula (1.7.3), and making some computations (recall (1.2.1)) we get:

$$\begin{aligned}
\rho(z,w)^2 &= 2 - 2\left(XX' + YY' + ZZ'\right) \\
&= 2 - \frac{2}{(|z|^2+1)(|w|^2+1)}\left[4\operatorname{Re}(z)\operatorname{Re}(w) + 4\operatorname{Im}(z)\operatorname{Im}(w) + (|z|^2-1)(|w|^2-1)\right] \\
&= 2 - \frac{2}{(|z|^2+1)(|w|^2+1)}\left[(z+\overline{z})(w+\overline{w}) - (z-\overline{z})(w-\overline{w}) + (|z|^2-1)(|w|^2-1)\right] \\
&= 2 - \frac{2}{(|z|^2+1)(|w|^2+1)}\left[2(z\overline{w} + \overline{z}w) + (|z|^2-1)(|w|^2-1)\right] \\
&= \frac{2}{(|z|^2+1)(|w|^2+1)}\left[(|z|^2+1)(|w|^2+1) - (|z|^2-1)(|w|^2-1) - 2\left(z\overline{w} + \overline{z}w\right)\right] \\
&= \frac{2}{(|z|^2+1)(|w|^2+1)}\left[2(|z|^2+|w|^2) - 2\left(z\overline{w} + \overline{z}w\right)\right] = \frac{4|z-w|^2}{(|z|^2+1)(|w|^2+1)}.
\end{aligned}$$

Thus we have (1.7.5) in the case $z, w \in \mathbb{C}$. Now, if $z \in \mathbb{C}$ and $w = \infty$, then $\Pi^{-1}(w) = N = (0,0,1)$, and $\pi^{-1}(z) = (X,Y,Z)$ with $X^2+Y^2+Z^2 = 1$ and $Z = (|z|^2-1)/(|z|^2+1)$ by (1.7.3). So by definition of $\rho(z,w)$:

$$\rho(z,w)^2 = X^2 + Y^2 + (Z-1)^2 = 2 - 2Z = 2 - \frac{2(|z|^2-1)}{|z|^2+1} = \frac{4}{|z|^2+1},$$

and get conclude (1.7.5) also in this case.

□

Definition 1.23 (Riemann Sphere). *We define the **Riemann sphere** as the set $\mathbb{C}_\infty$ equipped with the spherical metric $\rho : \mathbb{C}_\infty \to \mathbb{R}$ in (1.7.4).*

The sterographic projection defined an *homeomorphism* between $(\mathbb{C}_\infty, \rho)$ and $(\mathbb{S}^2, \|\cdot\|)$; where ρ is the spherical metric (Definition 1.21) and $\|\cdot\|$ is the Euclidean norm. This is the reason why $\mathbb{C}_\infty$ is called (the Riemann) sphere. In particular $(\mathbb{C}_\infty, \rho)$ is a *compact* space.

1.8 Exercises

Exercise 1.1. *Find the modulus, the arguments, the principal arguments, and the exponential forms of the following numbers.*

(a) $z = \sqrt{3} + i$.

(b) $z = \frac{1}{\sqrt{3}+i}$.

(c) $z = (-\sqrt{3} + i)^5$

(d) *The* 7*th roots of* $z = -\sqrt{3} - i$.

Exercise 1.2. *Express the following numbers in the form* $a + ib$*, with* $a, b \in \mathbb{R}$.

(a) $z = \frac{1+i}{3-2i}$

(b) $z = \frac{(1+2i)^2}{(1-i)^3}$.

(c) *All the* 3*rd roots of* $z = 1 + i$.

(d) *All the* 2*nd roots of* $z = \frac{1}{1-i}$.

Exercise 1.3. *Describe all the elements of the following sets.*

(a) $A = \{z \in \mathbb{C} \,:\, |z|^2 = z^2\}$.

(b) $B = \{z \in \mathbb{C} \,:\, z^2 = \overline{z}\}$.

(c) $C = \{z \in \mathbb{C} \,:\, z = (\overline{z})^2\}$.

(d) $D = \{z \in \mathbb{C} \,:\, z^2 = (\overline{z})^2\}$.

(e) $E = \{z \in \mathbb{C} \,:\, \overline{z} = \frac{1}{z}\}$.

(f) $F = \{z \in \mathbb{C} \,:\, \frac{1}{z} = -z\}$.

Exercise 1.4. *Find all the solutions* $z \in \mathbb{C}$ *of the following equations.*

(a) $z^2 + iz + 1 = 0$.

(b) $z^2 + 2iz - 1 = 0$.

(c) $z^3 - \sum_{k=0}^{100} i^k = 0$.

(d) $z^4 + z^2 + 1 = 0$.

(e) $z^4 + 81 = 0$.

(f) $(1 + z)^5 = (1 - z)^5$.

(g) $z^6 + 1 = i\sqrt{3}$.

Exercise 1.5. *Show that* $\mathbb{C}$ *does not admit a total order relation*[2] $\succ$ *satisfying the following rules (for all* $z_1, z_2, z_3 \in \mathbb{C}$*):*

- $z_1 \succ z_2 \implies z_1 + z_3 \succ z_2 + z_3$.
- $z_3 \succ 0,\ z_1 \succ z_2 \implies z_1 z_3 \succ z_2 z_3$.

[2] This means that for all $z, w, \xi \in \mathbb{C}$ we have (i) $z \succ z$; (ii) $z \succ w$ and $w \succ z$ implies $w = z$; (iii) $z \succ w$ and $w \succ \xi$ implies $z \succ \xi$; (iv) either $z \succ w$ or $w \succ z$.

Hint: Assume that such an order relation exists. Then either $0 \succ i$ *or* $i \succ 0$*. Arrive at a contradiction in both cases.*

Exercise 1.6. *Prove the following statements, for* $z, w \in \mathbb{C}$:

(a) $|z+w| = |z| + |w|$ *if and only if either* $w = 0$ *or* $z/w \in \mathbb{R}$ *with* $z/w \geq 0$.

(b) $|z-w| \geq \big||z| - |w|\big|$, *with equality if and only if either* $w = 0$ *or* $z/w \in \mathbb{R}$ *with* $z/w \geq 0$.

(c) $|z+w| = |z-w|$ *if and only if either* $w = 0$ *or* z/w *is pure imaginary.*

(e) *The Parallelogram Law:* $|z+w|^2 + |z-w|^2 = 2\left(|z|^2 + |w|^2\right)$.

(f) $\big|\mathrm{Re}(z)\big| + \big|\mathrm{Im}(z)\big| \leq \sqrt{2}\,|z|$.

Exercise 1.7. *For each* $w \in \mathbb{C}$, $n \in \mathbb{N}$, *find* $M(w) := \max\{|z^n + w| \,:\, z \in \mathbb{C},\, |z| \leq 1\}$ *and a corresponding maximizer.*

Hint: Find a trivial upper bound for $M(w)$ *with the triangle inequality, and then consider when this triangle inequality becomes equality; Exercise 1.6.*

Exercise 1.8. *Find* $\sup\{\mathrm{Re}(iz^3) \,:\, z \in \mathbb{C},\, |z| < 2\}$.

Exercise 1.9. *Prove Lagrange's Identity: for complex numbers* $z_1, \ldots, z_n$, $w_1, \ldots, w_n$:

$$\left|\sum_{k=1}^{n} z_k w_k\right|^2 = \left(\sum_{k=1}^{n} |z_k|^2\right)\left(\sum_{k=1}^{n} |w_k|^2\right) - \sum_{1\leq k<j\leq n} |z_k\overline{w_j} - z_j\overline{w_k}|^2.$$

Deduce the Cauchy-Schwarz inequality:

$$\left|\sum_{k=1}^{n} z_k w_k\right|^2 \leq \left(\sum_{k=1}^{n} |z_k|^2\right)\left(\sum_{k=1}^{n} |w_k|^2\right).$$

Hint: Argue by induction on n.

Exercise 1.10. *Show that if* $z \in \mathbb{C} \setminus \{1\}$, *then*

$$1 + z + z^2 + \cdots + z^{n-1} = \frac{1 - z^n}{1 - z}.$$

Deduce that if $z \in \mathbb{C} \setminus \{1\}$ *is an* n*-root of* 1, *then* $1 + z + z^2 + \cdots + z^{n-1} = 0$.

Exercise 1.11. *Denote by* $w_0, \ldots, w_{n-1}$ *all the* n*th roots of* 1, *for* $n \geq 2$. *Show that*

(a) $\prod_{k=0}^{n-1}(z - w_k) = z^n - 1$ *for all* $z \in \mathbb{C}$.

(b) $\sum_{k=0}^{n-1} w_k = 0$.

(c) $\prod_{k=0}^{n-1} w_k = (-1)^{n-1}$.

(d) $\sum_{k=0}^{n-1} w_k^j = \begin{cases} 0, & \text{if } 1 \leq j \leq n-1 \\ n, & \text{if } j = n. \end{cases}$

Hint: In (a), you can first prove that if z_0 *is a root of a polynomial* P *with* $\deg(P) = n$, *then* $P(z) = (z - z_0)Q(z)$, $z \in \mathbb{C}$, *for some polynomial* Q *with* $\deg(Q) \leq n-1$. *By induction, you can decompose the polynomial* $z^n - 1$. *But, please do not use the Fundamental Theorem of Algebra.*

Exercise 1.12. *Use De Moivre's formula in combination with Newton's binomial formula to express* $\sin(5\theta)$ *and* $\cos(5\theta)$ *as a polynomial expression of* $\sin(\theta)$ *and* $\cos(\theta)$, *for* $\theta \in \mathbb{R}$.

Exercise 1.13. *Prove that for $n \geq 2$,*

$$\sum_{k=1}^{n-1} \cos\left(\frac{2k\pi}{n}\right) = -1 \quad \text{and} \quad \sum_{k=1}^{n-1} \sin\left(\frac{2k\pi}{n}\right) = 0.$$

Exercise 1.14. *Prove that*

$$\sum_{k=1}^{n} e^{ik\theta} = \frac{\sin(\frac{n\theta}{2})}{\sin(\frac{\theta}{2})} e^{i\frac{(n+1)\theta}{2}}.$$

Use this formula to deduce

(a) $\sum_{k=1}^{n} \cos(k\theta) = \frac{\sin(\frac{n\theta}{2})\cos(\frac{(n+1)\theta}{2})}{\sin(\frac{\theta}{2})}$.

(b) $\sum_{k=1}^{n} \sin(k\theta) = \frac{\sin(\frac{n\theta}{2})\sin(\frac{(n+1)\theta}{2})}{\sin(\frac{\theta}{2})}$.

Hint: Use Exercise 1.10 to find a formula for the sum in terms of exponentials.

Exercise 1.15. *Show that, for $n > 2$,*

$$\prod_{k=1}^{n-1} \sin\left(\frac{\pi k}{n}\right) = \frac{n}{2^{n-1}}.$$

Hint: Describe the nonzero roots $\{z_k\}_k$ of the polynomial $(1-z)^n - 1$ in terms of the nth roots of unity, and find the modulus of z_k. Then, Exercise 1.11(a) can be helpful.

Exercise 1.16. *Prove that if $z \in \mathbb{C} \setminus \{0\}$, then the points $0, z$, and $1/\overline{z}$ are align in the plane.*

Exercise 1.17. *Prove that if $z \in \mathbb{C} \setminus \{1\}$ with $|z| = 1$, then $z + \frac{1}{z}$ is a real number (meaning that $\mathrm{Im}(z + \frac{1}{z}) = 0$), and that $\frac{1+z}{1-z}$ is pure imaginary (meaning that $\mathrm{Re}\left(\frac{1+z}{1-z}\right) = 0$).*

Exercise 1.18. *Let $z_1, z_2, z_3 \in \mathbb{C}$ be three distinct points. Show that the following statements are equivalent.*

(a) $\dfrac{z_2 - z_1}{z_3 - z_1} = \dfrac{z_1 - z_3}{z_2 - z_3}$.

(b) $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_1z_3 + z_2z_3$.

(c) $\{z_1, z_2, z_3\}$ *are the vertices of an equilateral triangle.*

Hint: The equivalence (a) $\Longleftrightarrow$ (b) is a computation. To prove that (c) is equivalent to the others ((a) or (b), choose your favorite), prove it first in the case $z_3 = 0$.

Exercise 1.19. *Let $w \in \mathbb{C}$ with $|w| < 1$ and $z \in \mathbb{C}$ so that $\overline{w}z \neq 1$. Show that*

$$\left|\frac{z - w}{1 - \overline{w}z}\right| \leq 1 \iff |z| \leq 1.$$

Exercise 1.20. *Let $z, w \in \mathbb{C}$ so that $\overline{w}z \neq 1$. Show that*

(a) If $|z| < 1$ and $|w| < 1$, then

$$\left|\frac{z - w}{1 - \overline{w}z}\right| < 1.$$

(b) If $|z| = 1$ or $|w| = 1$, then

$$\left|\frac{z - w}{1 - \overline{w}z}\right| = 1.$$

Exercise 1.21. *Consider the function* $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ *given by* $f(z) = 1/z$*. Prove that:*

(a) If L *line of* $\mathbb{C}$ *so that* $0 \in L$*, then* $f(L \setminus \{0\}) = L' \setminus \{0\}$ *for a line* $L' \subset \mathbb{C}$*.*

(b) If L *is line of* $\mathbb{C}$ *so that* $0 \notin L$*, then* $f(L) = S \setminus \{0\}$ *for a circle* S *of* $\mathbb{C}$ *with* $0 \in S$*.*

Suggestion: In (b), the formulas for circles and lines from Section 1.3 can help you.

Exercise 1.22. *Let* $\Pi : \mathbb{S}^2 \to \mathbb{C}_\infty$ *be the stereographic projection,* $N = (0, 0, 1)$ *the north pole, and* ∞ *the point at infinity for* $\mathbb{C}$*. Prove that:*

(a) If $S \subset \mathbb{S}^2$ *is a circle with* $N \in S$*, then* $\Pi(S) = L \cup \{\infty\}$*; where* L *is a line of* $\mathbb{C}$*.*

(b) If $S \subset \mathbb{S}^2$ *is a circle with* $N \notin S$*, then* $\Pi(S)$ *is a circle of* $\mathbb{C}$*.*

Clarification: By a circle $S \subset \mathbb{S}^2$ *we mean the intersection of* $\mathbb{S}^2$ *with a plane of* $\mathbb{R}^3$ *that is not tangent to* $\mathbb{S}^2$ *(if that plane is tangent,* S *is merely a singleton, and* $\Pi(S)$ *is a singleton too, which is a trivial circle of* $\mathbb{C}$*.)*

Hint: In (b), first explain why we can write $S = \{(X, Y, Z) \in \mathbb{S}^2 : A_0X + B_0Y + C_0Z = D_0\}$*, for* $(A_0, B_0, C_0) \in \mathbb{S}^2$*,* $D_0 \in (-1, 1)$*, and* $D_0 \neq C_0$*. Write the equation of a general circle* S' *of* $\mathbb{C}$ *as in* (1.3.5)*, and determine the parameters* ξ *and* K *(in terms of* A_0, B_0, C_0, D_0*) so that* $\Pi^{-1}(S') \subset S$*.*

Chapter 2

Complex Functions

2.1 Topology, convergence, and continuity in $\mathbb{C}$.

There is a natural way to define a distance between two points $z, w \in \mathbb{C}$, using the modulus function $|\cdot| : \mathbb{C} \to [0, +\infty)$. That is,

$$d(z, w) := |z - w|.$$

In particular, the function $d : \mathbb{C} \times \mathbb{C} \to [0, +\infty)$ satisfies the axioms of a metric:

- $d(z, w) = 0$ if and only if $z = w$.
- $d(z, w) = d(w, z)$ for all $z, w \in \mathbb{C}$.
- $d(z, w) \leq d(z, \xi) + d(\xi, w)$, for all $z, w, \xi \in \mathbb{C}$. This is a consequence of Proposition 1.7(viii).

Thus $(\mathbb{C}, |\cdot|)$ is a metric space, whose distance coincides with the Euclidean distance in the plane $\mathbb{R}^2$.

2.1.1 Open and Closed disks and sets

We next define the corresponding metric *balls*, which we call *disks* in the complex setting.

Definition 2.1 (Open and closed disks). *Given $z \in \mathbb{C}$ and $r > 0$, the* ***open disk centered at*** *z* ***with radius*** *r is*

$$D(z, r) := \{w \in \mathbb{C} \,:\, |w - z| < r\}.$$

The corresponding ***closed disk centered at*** *z* ***with radius*** *r is*

$$\overline{D}(z, r) := \{w \in \mathbb{C} \,:\, |w - z| \leq r\}$$

Then the ***circle centered at*** *z* ***with radius*** *r is the set*

$$S(z, r) := \overline{D}(z, r) \setminus D(z, r) = \{w \in \mathbb{C} \,:\, |w - z| = r\}.$$

We use the disks to define the fundamental class of sets.

Definition 2.2 (Open and closed sets). *A subset U of $\mathbb{C}$ is* ***open*** *if for every $z \in U$ there exists $r > 0$ so that $D(z, r) \subset U$. Also, we say that a set $F \subset \mathbb{C}$ is* ***closed*** *if $\mathbb{C} \setminus F$ is open.*

Trivial examples of open sets are $U = \emptyset$ and $U = \mathbb{C}$. These two sets are also closed according to Definition 2.2. But the main non-trivial examples of open and closed are precisely the open and closed disks.

Proposition 2.3. *For every $z \in \mathbb{C}$ and $r > 0$, the set $D(z, r)$ is open and the set $\overline{D}(z, r)$ is closed.*

Proof. To check that $D(z,r)$ is open, take $w \in D(z,r)$ and let us prove that $D(w,\varepsilon) \subset D(z,r)$, if $0 < \varepsilon < r - |w-z|$ (notice that $|w-z| < r$, as $w \in D(z,r)$). Indeed, if $\xi \in D(w,\varepsilon)$, the triangle inequality gives

$$|\xi - z| \le |\xi - w| + |w-z| < \varepsilon + |w-z| < r - |w-z| + |w-z| = r,$$

showing that $\xi \in D(z,r)$.

Now, to verify that $\overline{D}(z,r)$ is closed, we need to check that $U := \mathbb{C} \setminus \overline{D}(z,r)$ is open. Thus, let $w \in U$, and $0 < \varepsilon < |w-z| - r$. The open disk $D(w,\varepsilon)$ is contained in U, because if $\xi \in D(w,\varepsilon)$, then

$$|\xi - z| \ge |w-z| - |\xi - w| > |w-z| - \varepsilon > |w-z| - (|w-z| - r) = r,$$

implying $\xi \in \mathbb{C} \setminus \overline{D}(z,r) = U$. □

Arbitrary unions of open sets are open, and finite intersection of open sets are open as well. The same holds for closed sets swapping union and intersection.

Proposition 2.4. *Let $\{U_i\}_{i\in\mathcal{I}}$ be a family of open subsets of $\mathbb{C}$, and let $\{F_j\}_{j\in\mathcal{J}}$ be a family of closed subsets of $\mathbb{C}$. The following holds.*

(i) $\bigcup_{i\in\mathcal{I}} U_i$ is an open set.

(ii) If $\mathcal{I}$ is finite, then also $\bigcap_{i\in\mathcal{I}} U_i$ is open.

(iii) $\bigcap_{j\in\mathcal{J}} F_j$ is a closed set.

(iv) If $\mathcal{J}$ is finite, then also $\bigcup_{j\in\mathcal{J}} F_j$ is closed.

Proof.

(i) This is immediate from the definition of open sets.

(ii) If $\mathcal{I} = \{i_1, \dots, i_n\}$ and $z \in \bigcap_{i\in\mathcal{I}} U_i$, then there are radii $r_1, \dots, r_n$ that make each disk $D(z, r_i)$ be contained in U_i. If $r = \min\{r_1, \dots, r_n\}$, the disk $D(z,r)$ is contained in all the U_i simultaneously.

(iii) Write

$$\mathbb{C} \setminus \bigcap_{j\in\mathcal{J}} F_j = \bigcup_{j\in\mathcal{J}} \mathbb{C} \setminus F_j;$$

where each $\mathbb{C} \setminus F_j$ is open, since F_j is closed. By (i), we derive that $\mathbb{C} \setminus \bigcap_{j\in\mathcal{J}} F_j$ is open, ergo $\bigcap_{j\in\mathcal{J}} F_j$ is closed.

(iv) Using (ii), the proof follows from taking the pertinent complements on $\mathbb{C}$, as we did in (iii). □

Proposition 2.4 shows that, for example, singletons $\{z\}$, $z \in \mathbb{C}$, are closed sets, as they can be written as $\{z\} = \bigcap_{\varepsilon>0} \overline{D}(z,\varepsilon)$. Consequenly, aslo by Proposition 2.4, $D(z,\varepsilon) \setminus \{w\}$ is an open set for any $z, w \in \mathbb{C}$, $\varepsilon > 0$, as it is the intersection of the two open sets $D(z,\varepsilon)$ and $\mathbb{C} \setminus \{w\}$. Also, we can use Proposition 2.4 to deduce that each circle $\partial D(z,r)$ is closed, as the intersection of the closed sets $\overline{D}(z,r)$ and $\mathbb{C} \setminus D(z,r)$.

2.1.2 The interior, the closure, and the boundary

We continue defining more key topological concepts.

Definition 2.5 (Interior, Accumulation, Closure, Boundary)**.** *Let $A \subset \mathbb{C}$ be a subset, and $z \in \mathbb{C}$.*

- *We say that z is an **interior point** of A if there exists $r > 0$ so that $D(z,r) \subset A$. We define the **interior** of A, denoted by $\operatorname{int}(A)$, and the set consisiting of all interior points of A.*
- *We say that z is an **accumulation point** of A if, for every $\varepsilon > 0$, we have $A \cap (D(z,\varepsilon) \setminus \{z\}) \neq \emptyset$. The set of all accumulation points of A will be denoted by A'.*

- *We define the **closure** of A as the set*
$$\overline{A} := \bigcap\{F \subset \mathbb{C} \,:\, A \subset F \text{ and } F \text{ is closed}\}.$$
We can also refer to $\overline{A}$ as the smallest closed set containing A.

- *The **boundary** of A is the set*
$$\partial A := \overline{A} \setminus \operatorname{int}(A).$$

To get acquainted with some of these concepts, we propose showing that:

- The closure $\overline{D(z,r)}$ of an open disk $D(z,r)$ is precisely the corresponding closed disk $\overline{D}(z,r)$.
- The interior $\operatorname{int}(D(z,r))$ of a closed disk $\overline{D}(z,r)$ is precisely the corresponding open disk $D(z,r)$.
- The boundaries $\partial D(z,r)$ and $\partial \overline{D}(z,r)$ are both equal to the corresponding circle $S(z,r)$.

Let us collect some basic remarks and properties concerning the elements from Definition 2.5. Some of them will offer alternate definitions for the concepts of interior, closure, and boundary.

Proposition 2.6. *Let $A, B, C \subset \mathbb{C}$ be arbitrary subsets. The following statements are true.*

(i) *A is open if and only if $\operatorname{int}(A) = A$. Also, if $B \subset C$, then $\operatorname{int}(B) \subset \operatorname{int}(C)$.*

(ii) *The interior $\operatorname{int}(A)$ of A is an open set contained in A satisfying*
$$\operatorname{int}(A) = \bigcup\{U \subset \mathbb{C} \,:\, U \subset A \text{ and } U \text{ is open}\} = \bigcup\{D(z,r) \,:\, D(z,r) \subset A,\ z \in \mathbb{C},\ r > 0\}.$$
In particular, if U is an open set containing A, then $U \subset \operatorname{int}(A)$.

(iii) *If $B \subset C$, then $B' \subset C'$ and $\overline{B} \subset \overline{C}$.*

(iv) *A' is always a closed set.*

(v) *The closure $\overline{A}$ of A is a closed superset of A satisfying $\overline{A} = A \cup A'$. Consequently, the closure admits the following description:*
$$\overline{A} = \{z \in \mathbb{C} \,:\, D(z,\varepsilon) \cap A \neq \emptyset \text{ for every } \varepsilon > 0\}. \tag{2.1.1}$$
Also, we have the following characterizations of "closedness":
$$A \text{ is closed} \iff A = \overline{A} \iff A' \subset A.$$

(vi) *$\overline{\mathbb{C} \setminus A} = \mathbb{C} \setminus \operatorname{int}(A)$ and $\operatorname{int}(\mathbb{C} \setminus A) = \mathbb{C} \setminus \overline{A}$.*

(vii) *The boundary ∂A of A is a closed subset of $\overline{A}$, and $\partial A = \overline{A} \cap \overline{\mathbb{C} \setminus A}$.*

Proof.

(i) If A is open, for every $z \in A$ there is $r > 0$ so that $D(z,r) \subset A$, which means that $z \in \operatorname{int}(A)$, according to Definition 2.5. The implication "$A = \operatorname{int}(A) \implies A$ is open" is obvious. It is also immediate that $B \subset C \implies \operatorname{int}(B) \subset \operatorname{int}(C)$.

(ii) If $D(z,r)$ is an open disk contained in A, then, by (i) and Proposition 2.3, we have
$$D(z,r) = \operatorname{int}(D(z,r)) \subset \operatorname{int}(A).$$

Thus the union of the disks is contained in $\operatorname{int}(A)$. And if $z \in \operatorname{int}(A)$, then $D(z,r) \subset A$ for some $r > 0$, and thus we deduce $\operatorname{int}(A) = \bigcup\{D(z,r) \,:\, D(z,r) \subset A,\ z \in \mathbb{C},\ r > 0\}$. In particular, this shows that $\operatorname{int}(A)$ is open, e.g. by Proposition 2.4. Using again that the open disks are open and that $\operatorname{int}(A)$ is open, we deduce the middle identity of (ii).

(iii) This is immediate from Definition 2.5.

(iv) Let $z \in \mathbb{C} \setminus A'$. There exists $\varepsilon > 0$ so that $(D(z,\varepsilon) \setminus \{z\}) \cap A = \emptyset$. Let us show that $D(z,\varepsilon) \subset \mathbb{C} \setminus A'$, which will imply that $\mathbb{C} \setminus A'$ is open. Indeed, since we already know that $z \in \mathbb{C} \setminus A'$, it suffices to show $D(z,\varepsilon) \setminus \{z\} \subset \mathbb{C} \setminus A'$. But $D(z,\varepsilon) \setminus \{z\}$ is an open set (see the comment subsequent to Proposition 2.4), so for any $w \in D(z,\varepsilon) \setminus \{z\}$ we can find $\delta > 0$ with $D(w,\delta) \subset D(z,\varepsilon) \setminus \{z\}$. Because $(D(z,\varepsilon) \setminus \{z\}) \cap A = \emptyset$, we have $D(w,\delta) \cap A = \emptyset$ as well, showing that $w \in \mathbb{C} \setminus A'$.

(v) $\overline{A}$ is closed as intersection of closed sets; see Proposition 2.4. Let us verify the identity $\overline{A} = A \cup A'$. Let $z \notin A \cup A'$. Then there exists, by definition of A', a radius $\varepsilon > 0$ so that $(D(z,\varepsilon) \setminus \{z\}) \cap A = \emptyset$. But also $z \notin A$, so we actually have $D(z,\varepsilon) \cap A = \emptyset$, implying $A \subset \mathbb{C} \setminus D(z,\varepsilon)$, where $\mathbb{C} \setminus D(z,\varepsilon)$ is closed and does not contain z. Hence $z \notin \overline{A}$, and this shows that $\overline{A} \subset A \cup A'$. For the reverse inclusion, assume $z \notin \overline{A}$, which implies the existence of $F \subset \mathbb{C}$ closed with $A \subset F$ and $z \notin F$. The complement of F is open, so there exists $\varepsilon > 0$ with $D(z,\varepsilon) \cap F = \emptyset$. Since $A \subset F$, this clearly shows that $z \notin A'$.

The expression for $\overline{A}$ is immediate from the identity $\overline{A} = A \cup A'$.

We now prove the equivalences. If A is closed, then $A = \overline{A}$, by the definition of closure.

Now, if $\overline{A} = A$, and we use the already proven identity $\overline{A} = A \cup A'$, we get $A' \subset A$.

Finally, if $A' \subset A$ and $z \in \mathbb{C} \setminus A$, then z also do not belong to A', and there is $\varepsilon > 0$ with $(D(z,\varepsilon) \setminus \{z\}) \cap A = \emptyset$. But since $z \notin A$, this yields $D(z,\varepsilon) \cap A = \emptyset$, and so $D(z,\varepsilon) \subset \mathbb{C} \setminus A$. We have shown that $\mathbb{C} \setminus A$ is open, ergo, A is closed.

(vi) It suffices to apply the formula from (ii) for interiors and that U is open iff $\mathbb{C} \setminus U$ is closed.

(vii) This follows from the definition of boundary (Definition 2.5) and (vi).

□

2.1.3 Convergence of Sequences. The Bolzano-Weierstrass Theorem

Definition 2.7 (Convergence of sequences)**.** *Let* $\{z_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ *be a sequence, and* $z_0 \in \mathbb{C}$*. We say that* $\{z_n\}_{n\in\mathbb{N}}$ ***converges to*** z_0*, and denote it by* $z_0 = \lim_{n\to\infty} z_n$*, if for every* $\varepsilon > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that* $z_n \in D(z_0,\varepsilon)$ *for all* $n \geq n_0$*. We will often denote the property* $z_0 = \lim_{n\to\infty} z_n$ *simply by* $z_n \to z_0$*.*

The limits of sequences satisfy the following basic properties.

Proposition 2.8. *Let* $\{z_n\}_{n\in\mathbb{N}}$*,* $\{w_n\}_{n\in\mathbb{N}}$ *be sequences in* $\mathbb{C}$*, and* $z_0, w_0 \in \mathbb{C}$*. The following holds.*

(i) $\lim_{n\to\infty} z_n = z_0$ *if and only if* $\mathrm{Re}(z_0) = \lim_{n\to\infty} \mathrm{Re}(z_n)$ *and* $\mathrm{Im}(z_0) = \lim_{n\to\infty} \mathrm{Im}(z_n)$*; understanding these limits as convergence of real numbers.*

Consequently, $\lim_{n\to\infty} z_n = z_0$ *if and only if* $\lim_{n\to\infty} \overline{z_n} = \overline{z_0}$*.*

Also, if $\lim_{n\to\infty} z_n = z_0$*, then* $\lim_{n\to\infty} |z_n| = |z_0|$*.*

Assume from now on that $\lim_{n\to\infty} z_n = z_0$ *and* $\lim_{n\to\infty} w_n = w_0$*. Then:*

(ii) $\lim_{n\to\infty}(z_n + w_n) = z_0 + w_0$*.*

(iii) $\lim_{n\to\infty} z_n w_n = z_0 w_0$*.*

(iv) If $w_0 \neq 0$*, then* $\lim_{n\to\infty} z_n/w_n = z_0/w_0$*.*

Proof.

(i) For the first equivalence, use Proposition 1.7 and the definition of modulus to deduce:

$$|z_n - z_0| < \varepsilon \implies \max\{|\mathrm{Re}(z_n) - \mathrm{Re}(z_0)|, |\mathrm{Im}(z_n) - \mathrm{Im}(z_0)|\} < \varepsilon \implies |z_n - z_0| < \sqrt{2}\,\varepsilon.$$

By Definition 2.7, the above shows that $z_n \to z_0$ if and only if both $\mathrm{Re}(z_n) \to \mathrm{Re}(z_0)$ and $\mathrm{Im}(z_n) \to \mathrm{Im}(z_0)$.

The rest of (i) follows from what we have already proven for the real and imaginary parts.

(ii) This is very easy.

(iii) Using that $\operatorname{Re}(uv) = \operatorname{Re}(u)\operatorname{Re}(v) - \operatorname{Im}(u)\operatorname{Im}(v)$, $\operatorname{Im}(uv) = \operatorname{Re}(u)\operatorname{Im}(v) + \operatorname{Im}(u)\operatorname{Re}(v)$, for all $u, v \in \mathbb{C}$, together with (i), the proof of (iii) is straightforward.

(iv) Thanks to (iii), it is enough to show that $1/w_n \to 1/w_0$, or equivalently, $\overline{w_n}/|w_n|^2 \to \overline{w_0}/|w_0|^2$. But this is easily seen, because, by virtue of (i), $w_n \to w_n$ implies both $|w_n|^2 \to |w_0|^2$ and $\overline{w_n} \to \overline{w_0}$. □

We can use sequences to give useful criteria for closures and accumulation points of sets.

Proposition 2.9. *Let $A \subset \mathbb{C}$ be a set, and $z_0 \in \mathbb{C}$ a point. The following properties are true.*

(i) $z_0 \in A'$ if and only if there is a sequence $\{z_n\}_{n\in\mathbb{N}} \subset A \setminus \{z_0\}$ so that $\lim_n z_n = z_0$.

(ii) $z_0 \in \overline{A}$ if and only if there is a sequence $\{z_n\}_{n\in\mathbb{N}} \subset A$ so that $\lim_n z_n = z_0$.

Consequently, A is closed if and only if for every $\{z_n\}_{n\in\mathbb{N}} \subset A$ convergent to $z_0 \in \mathbb{C}$, one has $z_0 \in A$.

Proof.

(i) If $z_0 \in A'$, then $(D(z_0, 1/n) \setminus \{z_0\}) \cap A \neq \emptyset$ for each $n \in \mathbb{N}$. Taking z_n as any point $z_n \in D(z_0, 1/n) \setminus \{z_0\} \cap A$, one has $\lim_{n\to\infty} z_n = z_0$ (as $0 < |z_n - z_0| < 1/n$), and $\{z_n\}_{n\in\mathbb{N}} \subset A$.

For the reverse implication, let $\{z_n\}_{n\in\mathbb{N}} \subset A \setminus \{z_0\}$ with $\lim_n z_n = z_0$, and let $\varepsilon > 0$. Since $z_n \to z_0$, we have $z_n \in D(z_0, \varepsilon)$ for some n (actually for all n from certain n_0 on). Thus, the intersection $D(z_0, \varepsilon) \setminus \{z_0\} \cap A$ is nonempty.

(ii) Using the description (2.1.1) for closure points, the proof is almost identical to that of (i). □

Definition 2.10. *Given two sets $A, B \subset \mathbb{C}$ we say that B **is dense in** A if $A \subset \overline{B}$.*

In particular, B is dense in $\mathbb{C}$ if $\overline{B} = \mathbb{C}$.

According to Proposition 2.9, if B is dense in A, then for every $z \in A$ we can find a sequence $\{z_n\}_n \subset B$ so that $z_n \to z$.

Definition 2.11. *We say that a sequence $\{z_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ is **bounded** if $\sup\{|z_n| : n \in \mathbb{N}\} < \infty$, that is, there exists $M > 0$ so that $|z_n| \leq M$ for all $n \in \mathbb{N}$.*

*More generally, we say that a set $A \subset \mathbb{C}$ is **bounded** if $\sup\{|z| : z \in A\} < \infty$.*

As in the real line $\mathbb{R}$, bounded sequences in $\mathbb{C}$ admit convergent subsequences.

Theorem 2.12 (Bolzano-Weierstrass in $\mathbb{C}$)**.** *If $\{z_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ is a bounded sequence, then there exists a subsequence $\{z_{n_k}\}_{k\in\mathbb{N}}$ of $\{z_n\}_{n\in\mathbb{N}}$ convergent to some $z_0 \in \mathbb{C}$.*

Proof. Recall that $\max\{|\operatorname{Re}(z_n)|, |\operatorname{Im}(z_n)|\} \leq |z_n|$, e.g., by Proposition 1.7. Thus, both $\{\operatorname{Re}(z_n)\}_{n\in\mathbb{N}}$ and $\{\operatorname{Im}(z_n)\}_{n\in\mathbb{N}}$ are bounded sequences of real numbers. By Bolzano-Weierstrass theorem in $\mathbb{R}$, there is a subsequence $\{\operatorname{Re}(z_{n_k})\}_{k\in\mathbb{N}}$ of $\{\operatorname{Re}(z_n)\}_{n\in\mathbb{N}}$ convergent to some $x_0 \in \mathbb{R}$. Now, the subsequence $\{\operatorname{Im}(z_{n_k})\}_{k\in\mathbb{N}}$ of $\{\operatorname{Im}(z_n)\}_{n\in\mathbb{N}}$ is also bounded, and thus, again by Bolzano-Weierstras, there is a subsequence $\{\operatorname{Im}(z_{n_{k_j}})\}_{j\in\mathbb{N}}$ of $\{\operatorname{Im}(z_{n_k})\}_{k\in\mathbb{N}}$ convergent to some $y_0 \in \mathbb{R}$. The subsequence $\{\operatorname{Re}(z_{n_{k_j}})\}_{j\in\mathbb{N}}$ converges to x_0 (because it is a subsequence of $\{\operatorname{Re}(z_{n_k})\}_{k\in\mathbb{N}}$). Defining $z_0 := x_0 + iy_0$ and applying Proposition 2.8(i), the sequence $\{z_{n_{k_j}} = \operatorname{Re}(z_{n_{k_j}}) + i\operatorname{Im}(z_{n_{k_j}})\}_{j\in\mathbb{N}}$ converges to z_0. □

2.1.4 Compactness. The Heine-Borel Theorem

We now turn out attention to a special (and crucial) class of sets: the compact sets. Althought the formal definition seems a bit abstract, in metric spaces (and specially in $\mathbb{C}$) there are several alternate characterizations that are easier to use in practice.

Definition 2.13. *Let $K \subset \mathbb{C}$. We say that K is a* ***compact*** *set if given any collection $\{U_i\}_{i\in\mathcal{I}}$ of open subsets of $\mathbb{C}$ so that $K \subset \bigcup_{i\in\mathcal{I}} U_i$, there exists a* ***finite*** *family of indices $\mathcal{F} \subset \mathcal{I}$ so that $K \subset \bigcup_{i\in\mathcal{F}} U_i$.*

Compactness can be rephrased in the following manner: a set K is compact if from any *open covering* of K one can find a *finite sub-covering* of K.

Definition 2.13 is the formal definition of compactness that is given in the setting of topological spaces, a class of spaces that is much more general (and leading to beautiful and more abstract phenomena) than the class of metric spaces.

In metric spaces, compactness is equivalent to *sequential compactness.* Moreover, in $(\mathbb{C}, |\cdot|)$, compact sets are simply the bounded and closed sets.

Theorem 2.14 (Heine-Borel theorem in $\mathbb{C}$)**.** *Let $K \subset \mathbb{C}$ be a subset. The following statements are equivalent.*

(i) K is compact.

(ii) K is closed and bounded.

(iii) Every sequence $\{z_k\}_{k\in\mathbb{N}} \subset K$ has a subsequence $\{z_{n_k}\}_{k\in\mathbb{N}}$ convergent to some $z_0 \in K$.

Proof. We will assume throughout the proof that $K \neq \emptyset$.

$(i) \implies (ii)$: Assume that K is compact. To check that K is bounded, take a point $z \in K$ and consider the trivial covering of K given by the collection $\{D(z, j)\}_{j\in\mathbb{N}}$. By the compactness of K, there exists a finite collection of those disks, say $D(z, j_1), \dots, D(z, j_N)$ so that $K \subset \bigcup_{i=1}^{N} D(z, j_i)$. Then obviously, K is contained in $D(z, R)$, with $R := \max\{j_1, \dots, j_N\}$, showing that K is bounded.

To verify that K is closed, let $z \in \mathbb{C} \setminus K$, and notice, for every $w \in K$, one has $D(z, r_w) \cap D(w, r_w) = \emptyset$ for $r_w := |z - w|/2$ (by virtue of the triangle inequality for the modulus). Obviously, $K \subset \bigcup_{w\in K} D(w, z_w)$ and so the compactness of K gives a finite set F of K so that $K \subset \bigcup_{w\in F} D(w, z_w)$. Now, define the set

$$U := \bigcup_{w\in F} D(z, r_w).$$

By Proposition 2.4, U is open (as a finite intersection of open sets). And it is clear that $z \in U$ and $U \cap \left(\bigcup_{w\in F} D(w, z_w)\right) = \emptyset$. So, also $U \cap K = \emptyset$, and therefore $z \in U \subset \mathbb{C} \setminus K$; which shows that $\mathbb{C} \setminus K$ is open.

$(ii) \implies (iii)$: This follows by combining Theorem 2.12 and Proposition 2.9.

$(iii) \implies (i)$: Let $\{U_i\}_{i\in\mathcal{I}}$ a collection open sets whose union contains K.

We claim that there exists $\varepsilon > 0$ so that, for every $z \in K$, the disk $D(z, \varepsilon)$ is entirely contained in one U_{i_z}, $i_z \in \mathcal{I}$. Indeed, otherwise we can find, for each $n \in \mathbb{N}$, a point $z_n \in K$ with $D(z_n, 1/n) \not\subset U_i$ for all $i \in \mathcal{I}$. By the assumption, $\{z_n\}_n$ has a subsequent $\{z_{n_k}\}_k$ convergent to some $z_0 \in K$. Note that z_0 must be contained in some U_{i_0}, (because the union of the U_i's cover K) and in fact $D(z_0, \delta) \subset U_{i_0}$ for some $\delta > 0$, as U_{i_0} is open. But it is clear that, for k large enough, the disk $D(z_{n_k}, 1/n_k)$ is contained in $\subset D(z_0, \delta)$, and so is contained in U_{i_0}, a contradiction. So, our claim is proven.

Next, we can find a finite set $\mathcal{F} \subset \mathcal{I}$ such that $K \subset \bigcup_{z\in\mathcal{F}} D(z, \varepsilon)$. Indeed, suppose, for the sake of contradiction, that such a finite set does not exist. Then we can find points $z_1 \in K$, $z_2 \in K \setminus D(z_1, \varepsilon)$, $\dots$, $z_n \in K \setminus \bigcup_{j=1}^{n-1} D(z_j, \varepsilon)$, thus forming a sequence $\{z_n\}_{n\in\mathbb{N}}$ so that $d(z_n, z_m) \geq \varepsilon$ for all n, m with $n \neq m$. This is a contradiction because $\{z_n\}_{n\in\mathbb{N}}$ must have a convergent subsequence. So, there exists the desired finite set.

Using the previous claims, we have that $K \subset \bigcup_{z\in\mathcal{F}} D(z,\varepsilon) \subset \bigcup_{z\in\mathcal{F}} U_{i_z}$, showing that K is contained in the union of finitely many U_i's. $\square$

Alternatively, compact sets in metric spaces can also be characterized as *complete and totally bounded* sets, but we will not cover that criteria in these notes.

We also include the following intersection property for nested compact subsets.

Lemma 2.15. *Let $\{K_n\}_{n\geq 1}$ be a sequence of nonempty compact subsets of $\mathbb{C}$ such that $K_{n+1} \subset K_n$ for all $n \in \mathbb{N}$, and so that $\lim_{n\to\infty} \operatorname{diam}(K_n) = 0$. Then there exists a unique point $z_0 \in \mathbb{C}$ with*

$$z_0 \in \bigcap_{n=0}^{\infty} K_n.$$

Proof. Let $z_n \in K_n$ for every $n \in \mathbb{N}$. Given $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ so that $\operatorname{diam}(K_n) \leq \varepsilon$ for all $n \geq n_0$. Thus, if $m \geq n \geq n_0$, then $K_m \subset K_n$, and so

$$|z_n - z_m| \leq \operatorname{diam}(K_n) \leq \varepsilon.$$

This shows that $\{z_n\}_n$ is a Cauchy sequence, and so it converges to a point $w_0 \in \mathbb{C}$, by Exercise 2.2. Also note that for every $n \in \mathbb{N}$, we have that $\{z_m\}_{m\geq n} \subset K_n$, and since K_n is closed (by Theorem 2.14, we get that $z_0 \in K_n$ as well. Thus $z_0 \in \bigcap_{n=0}^{\infty} K_n$. To show that z_0 is the unique such point, suppose that $w_0 \in \bigcap_{n=0}^{\infty} K_n$ as well. Then

$$|w_0 - z_0| \leq \operatorname{diam}(K_n), \quad \text{for all} \quad n \in \mathbb{N}.$$

Because $\lim_{n\to\infty} \operatorname{diam}(K_n) = 0$, we deduce that $w_0 = z_0$. $\square$

2.1.5 Limits and Continuity of functions

Our next objective is studying the continuity of functions with complex source and range. We first define the limits of functions.

Definition 2.16 (Limits of functions)**.** *Let $A \subset \mathbb{C}$, $f : A \to \mathbb{C}$ a function, and $z_0 \in \overline{A}$. We say that $w \in \mathbb{C}$ is the **limit of f as z converges to** z_0, denoted by $\lim_{z\to z_0} f(z) = w$, if for every $\varepsilon > 0$ there exists $\delta > 0$ so that*

$$z \in A \setminus \{z_0\},\ |z - z_0| < \delta \implies |f(z) - w| < \varepsilon.$$

This is equivalent to saying that $f\left((D(z_0,\delta) \setminus \{z_0\}) \cap A\right) \subset D(w,\varepsilon)$.

Also, we will say that $\lim_{z\to z_0} f(z) = +\infty$ if for every $M > 0$ there exists $\delta > 0$ so that

$$z \in A \setminus \{z_0\},\ |z - z_0| < \delta \implies |f(z)| > M.$$

In practice, it is often easier to use the following sequential characterization of limits.

Proposition 2.17. *Let $A \subset \mathbb{C}$, $f : A \to \mathbb{C}$ a function, $z_0 \in \overline{A}$, and $w \in \mathbb{C}$. The following holds.*

(i) $\lim_{z\to z_0} f(z) = w$ *if and only if for every sequence $\{z_n\}_n \subset A \setminus \{z_0\}$ with $\lim_{n\to\infty} z_n = z_0$, one has $\lim_{n\to\infty} f(z_n) = w$.*

(ii) $\lim_{z\to z_0} f(z) = +\infty$ *if and only if for every sequence $\{z_n\}_n \subset A \setminus \{z_0\}$ with $\lim_{n\to\infty} z_n = z_0$, one has $\lim_{n\to\infty} |f(z_n)| = +\infty$*

(iii) The limit $\lim_{z\to z_0} f(z)$ *exists if and only if the two limits* $\lim_{z\to z_0} \mathrm{Re}(f)(z)$ *and* $\lim_{z\to z_0} \mathrm{Im}(f)(z)$ *exist. Moreover, in such case,*

$$\lim_{z\to z_0} f(z) = \lim_{z\to z_0} \mathrm{Re}(f)(z) + i \lim_{z\to z_0} \mathrm{Im}(f)(z).$$

Here we are denoting the real-valued functions $\mathrm{Re}(f) : A \to \mathbb{R}$, $\mathrm{Re}(f)(z) = \mathrm{Re}(f(z))$ *and* $\mathrm{Im}(f) : A \to \mathbb{R}$, $\mathrm{Im}(f)(z) = \mathrm{Im}(f(z))$, *for all* $z \in A$.

Proof.

(i) Assume $\lim_{z\to z_0} f(z) = w$, and let $\{z_n\}_n \subset A \setminus \{z_0\}$ with $z_n \to z_0$. For every $\varepsilon > 0$, let $\delta > 0$ be so that $0 < |z - z_0| < \delta$, $z \in A$ implies $|f(z) - w| < \varepsilon$. Since $z_n \to z_0$, we can find $n_0 \in \mathbb{N}$ (depending on δ) so that $0 < |z_n - z_0| < \delta$, whenever $n \geq n_0$. Therefore, $|f(z_n) - w| < \varepsilon$ for all $n \geq n_0$.

Conversely, assume the statement for sequences holds, and suppose, for the sake of contradiction that $\lim_{z\to z_0} f(z) \neq w$. This means that there is $\varepsilon > 0$ so that for no choice of δ one has $f((D(z_0, \delta) \setminus \{z_0\}) \cap A) \subset D(w, \varepsilon)$. Thus, for each n, we can find some $z_n \in (D(z_0, 1/n) \setminus \{z_0\}) \cap A$ and $|f(z_n) - w| \geq \varepsilon$. Clearly $\lim_{n\to\infty} z_n = z_0$ and, by the assumption, we have $\lim_{n\to\infty} f(z_n) = w$, a contradiction.

(ii) The proof is very similar to that of (i).

(iii) This is immediate from statement (i) and Proposition 2.8.

□

Definition 2.18 (Continuous functions)**.** *Let* $A \subset \mathbb{C}$, $f : A \to \mathbb{C}$ *and* $z_0 \in A$. *We say that* f ***is continuous at*** z_0 *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *so that*

$$z \in A, |z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon.$$

This is equivalent to $f((D(z_0, \delta) \cap A) \subset D(f(z_0), \varepsilon)$. *In other words,* f *is continuous as* z_0 *if* $\lim_{z\to z_0,\, z\in A} f(z) = f(z_0)$.

And if $f : A \to \mathbb{C}$ *is continuous at every* $z \in A$, *we say that* f *is* ***continuous in*** A.

Remark 2.19. A function $f : A \to \mathbb{C}$, with $A \subset \mathbb{C}$, is continuous at $z_0 \in A$ if and only if the functions $\mathrm{Re}(f) : A \to \mathbb{R}$ and $\mathrm{Im}(f) : A \to \mathbb{R}$ are continuous at z_0. This is a simple consequence of Proposition 2.17(iii).

We have several ways to characterize continuity of functions.

Proposition 2.20. *Let* $A \subset \mathbb{C}$, $f : A \to \mathbb{C}$ *and* $z_0 \in A$. *The following statements are equivalent.*

(i) $\lim_{n\to\infty} f(z_n) = f(z_0)$ *for each sequence* $\{z_n\}_{n\in\mathbb{N}} \subset A$ *with* $\lim_{n\to\infty} z_n = z_0$.

(ii) f *is continuous at* z_0.

Concerning continuity of f *in all of* A, *the following statements are equivalent:*

(i)' f *is continuous in* A.

(ii)' For every open $U \subset \mathbb{C}$, *we have* $f^{-1}(U) = V \cap A$ *for some* $V \subset \mathbb{C}$ *open.*

(iii)' For every closed $F \subset \mathbb{C}$, *we have* $f^{-1}(F) = E \cap A$ *for some* $E \subset \mathbb{C}$ *closed.*

Proof. The first equivalences $(i) \iff (ii)$ are immediate from from Proposition 2.17. Let us verify the equivalences between (i)', (ii)' and (iii)'.

$(i)' \implies (ii)'$: If $z \in f^{-1}(U)$, then $f(z) \in U$ and since U is open, we can find $\varepsilon_z > 0$ so that $D(f(z), \varepsilon_z) \subset U$. Due to the continuity of f on z, there exists $\delta_z > 0$ so that $f(A \cap D(z, \delta_z)) \subset D(f(z), \varepsilon_z)$. Therefore $A \cap D(z, \delta_z) \subset f^{-1}(U)$, which shows that

$$f^{-1}(U) = \bigcup_{z \in f^{-1}(U)} (A \cap D(z, \delta_z)) = A \cap \left(\bigcup_{z \in f^{-1}(U)} D(z, \delta_z) \right).$$

Defining $V := \bigcup_{z \in f^{-1}(U)} D(z, \delta_z)$, we prove the assertion.

$(ii)' \implies (iii)'$: Because $\mathbb{C} \setminus F$ is open, we can write $A \setminus f^{-1}(F) = f^{-1}(\mathbb{C} \setminus F) = A \cap V$ for some open set $V \subset \mathbb{C}$. Thus

$$f^{-1}(F) = A \setminus (A \cap V) = A \cap (\mathbb{C} \setminus V),$$

where $\mathbb{C} \setminus V$ is closed.

$(iii)' \implies (i)'$: Suppose, for the sake of contradiction, that there is $z_0 \in A$ so that f is not continuous at z_0. Then, by the (already prove) equivalence $(i) \implies (ii)$, we can find a sequence $\{z_n\}_n \subset A$ converging to z_0 and so that $f(z_n) \not\to f(z_0)$. After possibly passing to a subsequence, this means that there is $\varepsilon > 0$ so that $|f(z_n) - f(z_0)| \geq \varepsilon$ for each $n \in \mathbb{N}$. Now, by (iii)', we have that $f^{-1}(\mathbb{C} \setminus D(f(z_0), \varepsilon)) = E \cap A$ for some closed set $E \subset \mathbb{C}$. Clearly $f(z_n) \in \mathbb{C} \setminus D(f(z_0), \varepsilon)$, so $\{z_n\}_n \subset f^{-1}(\mathbb{C} \setminus D(f(z_0), \varepsilon))$, implying that $\{z_n\}_n \subset E$. But E is closed and $z_n \to z_0$, so $z_0 \in E$, according to Proposition 2.9. Therefore, $z_0 \in E \cap A$ and hence $z_0 \in f^{-1}(\mathbb{C} \setminus D(f(z_0), \varepsilon))$, which is of course a contradiction.

□

Proposition 2.20 for $A \subset \mathbb{C}$ an open set implies that f is continuous on A iff $f^{-1}(V)$ is an open subset of A. For A arbitrary, the sets of the form $A \cap V$ with $V \subset C$ open (resp. $A \cap E$ with $E \subset \mathbb{C}$ closed) are called *open relative to* A (resp. *closed relative to* A). One has to be very carefully when studying the continuity of a function over a set, as it strongly depends on the set of definiton of the function. For instance, the function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = 1$ when $\mathrm{Re}(z) \neq 0$ and $f(z) = 0$ when $\mathrm{Re}(z) = 0$ is continuous on the open set $\mathbb{C} \setminus \{\mathrm{Re}(z) = 0\}$, but is not continuous at any point of the set $A = \{z \in \mathbb{C} : \mathrm{Re}(z) = 0\}$. However, the restriction of f to A, gives a *new* function $f_{|A} : A \to \mathbb{C}$ that is identically 0 on A, and thus continuous on all of A.

Let us now see expected properties concerning operations with limits and continuity.

Proposition 2.21. *Let $A \subset \mathbb{C}$, $z_0 \in \overline{A}$, $f : A \to \mathbb{C}$, $g : \mathbb{C} \to \mathbb{C}$ so that the limits $\lim_{z \to z_0} f(z)$ and $\lim_{z \to z_0} g(z)$ exist. Then*

(i) There exists $\lim_{z \to z_0} (f(z) + g(z)) = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z)$.

(ii) There exists $\lim_{z \to z_0} (f(z) \cdot g(z)) = \lim_{z \to z_0} f(z) \cdot \lim_{z \to z_0} g(z)$

(iii) If $\lim_{z \to z_0} g(z) \neq 0$ and there exists $r > 0$ so that $g(z) \neq 0$ for every $z \in D(z_0, r) \cap A$, then there exists

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)}.$$

If we further have $z_0 \in A$, and f, g continuous at z_0, and another function $h : g(A) \to \mathbb{C}$ continuous at $g(z_0)$, then

(i)' $f + g$ is continuous at z_0.

(ii)' $f \cdot g$ is continuous at z_0.

(iii)' $h \circ g$ is continuous at z_0.

(iv)' If $g(z_0) \neq 0$, then f/g is continuous at z_0.

Proof. The part concerning the limits is immediate from Propositions 2.8 and 2.17.

For the second part, (i)' and (ii)' follow from statements (i) and (ii). Statement (iv)' also is a consequence of (iii), because $g(z_0)$ actually implies that $g(z) \neq 0$ for all $z \in D(z_0, r) \cap A$ and some $r > 0$, due to the continuity of g at z_0. Finally, to prove (iii)' we can imitate the proof of the corresponding result for functions $\mathbb{R} \to \mathbb{R}$. □

Example 2.22. Using Remark 2.19 and 2.21, one can easily verify that the following functions are continuous in the indicated sets:

- The function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \overline{z}$ is continuous in $\mathbb{C}$.
- The functions $f, g : \mathbb{C} \to \mathbb{C}$ given by $f(z) = |z|$ and $g(z) = |z|^2$ are continuous in $\mathbb{C}$.
- Any polynomial $P : \mathbb{C} \to \mathbb{C}$, $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is continuous in $\mathbb{C}$.
- The principal argument $\operatorname{Arg} : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is continuous in $\mathbb{C} \setminus (-\infty, 0]$; see Definition 1.9. However, it is not continuous at any point $z \in (-\infty, 0]$.
- For $n \geq 2$, the principal nth root $\sqrt[n]{\cdot} : \mathbb{C}\setminus\{0\} \to \mathbb{C}$ is continuous in $\mathbb{C}\setminus(-\infty, 0)$; see Definition 1.16. However, it is not continuous at any point $z \in (-\infty, 0)$.

Only the principal argument and the nth root cases are non-trivial, but bearing in mind the explicit formula (1.4.2) and the continuity of $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$, we easily get the continuity of Arg in $\mathbb{C} \setminus (-\infty, 0]$. To see that Arg is discontinuous at every $a \in (-\infty, 0]$, consider sequences $z_n = a + \frac{i}{n}$ and $w_n = a + \frac{i}{n}$, for all $n \in \mathbb{N}$. Of course $z_n, w_n \to a$, but formula (1.4.2) says that $\lim_{n\to\infty} \operatorname{Arg}(z_n) = \pi$ and $\lim_{n\to\infty} \operatorname{Arg}(w_n) = -\pi$, showing that $\lim_{z\to a} \operatorname{Arg}(z)$ does not even exist.

Now, recall that $\sqrt[n]{z} := \sqrt[n]{|z|}e^{i\frac{\operatorname{Arg}(z)}{n}}$ for all $z \in \mathbb{C} \setminus \{0\}$ and $\sqrt[n]{0} := 0$, according to Definition 1.16. The continuity of Arg in $\mathbb{C}\setminus\{0\}$ gives the continuity of $z \mapsto \sqrt[n]{z}$ in $\mathbb{C}\setminus(-\infty, 0]$. Also, because $|\sqrt[n]{z}| = \sqrt[n]{|z|}$, clearly $\lim_{z\to 0} |\sqrt[n]{z}| = 0$, showing that also $z \mapsto \sqrt[n]{z}$ is continuous at $z_0 = 0$. Now, for every $a \in (-\infty, 0)$, consider the sequence

$$z_k := -ae^{(-\pi+\frac{1}{k})i}, \quad k \in \mathbb{N}.$$

We have that $\lim_{k\to\infty} z_k = -ae^{-\pi i} = a$. However, $\operatorname{Arg}(a) = -\pi$ and $\operatorname{Arg}(z_k) = -\pi + \frac{1}{k}$, which implies

$$\sqrt[n]{a} = \sqrt[n]{|a|}e^{i\frac{\pi}{n}}, \quad \sqrt[n]{z_k} = \sqrt[n]{|z_k|}e^{i\frac{-\pi+\frac{1}{k}}{n}} \longrightarrow \sqrt[n]{|a|}e^{-i\frac{\pi}{n}} \neq \sqrt[n]{|a|}e^{i\frac{\pi}{n}}.$$

This shows that $z \mapsto \sqrt[n]{z}$ is discontinuous at a.

Definition 2.23 (Uniform continuity). *Let $A \subset \mathbb{C}$, and $f : A \to \mathbb{C}$. We say that f **is uniformly continuous on** A if for every $\varepsilon > 0$ there exists $\delta > 0$ so that*

$$z, w \in A, \ |z - w| < \delta \implies |f(z) - f(w)| < \varepsilon.$$

Equivalently, $f(D(z, \delta) \cap A) \subset D(f(z), \varepsilon)$ for every $z \in A$.

Note the crucial difference with the mere continuity (Definition 2.18), where the number $\delta > 0$ depends on the point $z \in A$. We can characterize uniform continuity via sequences.

Proposition 2.24. *Let $A \subset \mathbb{C}$, and $f : A \to \mathbb{C}$. The following statements are equivalent.*

(i) f is uniformly continuous on A.

(ii) For every couple of sequences $\{z_n\}_n, \{w_n\}_n \subset A$ *one has*

$$\lim_n |z_n - w_n| = 0 \implies \lim_n |f(z_n) - f(w_n)| = 0.$$

Proof.

$(i) \implies (ii)$: Assume that $\lim_n |z_n - w_n| = 0$ and let $\varepsilon > 0$, and let $\delta > 0$ the number associated with ε as in Definition 2.23. If $n_0 \in \mathbb{N}$ is such that $|z_n - w_n| < \delta$ whenever $n \geq n_0$, then $|f(z_n) - f(w_n)| < \varepsilon$ for $n \geq n_0$ as well.

$(ii) \implies (i)$: If f is not uniformly continuous on A, then there exists $\varepsilon > 0$ and points $z_n, w_n \in A$ with $|z_n - w_n| < 1/n$ and yet $|f(z_n) - f(w_n)| \geq \varepsilon$ for all $n \in \mathbb{N}$. This contradicts (i). ☐

Proposition 2.25. *Let* $K \subset \mathbb{C}$ *be a compact set, and* $f : K \to \mathbb{C}$ *a continuous function. Then,*

(i) $f(K)$ *is compact.*

(ii) There exist $\sup\{|f(z)| : z \in K\}$ *and* $\inf\{|f(z)| : z \in K\}$ *and are attained in* K.

(iii) f *is uniformly continuous on* K.

Proof. To prove (i), we can use the original Definition 2.13 of compactness. Let $\{U_i\}_{i\in\mathcal{I}}$ a collection of open subsets of $\mathbb{C}$ so that $f(K) \subset \bigcup_{i\in\mathcal{I}} U_i$. Then $K \subset \bigcup_{i\in\mathcal{I}} f^{-1}(U_i)$, and, since f is continuous in K, we can use Proposition 2.20 to obtain open sets $V_i \subset \mathbb{C}$ so that $f^{-1}(U_i) = V_i \cap K$, for all $i \in \mathcal{I}$, and hence

$$K \subset \bigcup_{i\in\mathcal{I}} f^{-1}(U_i) \subset \bigcup_{i\in\mathcal{I}} (K \cap V_i) \subset \bigcup_{i\in\mathcal{I}} V_i.$$

Because K is compact, there exists $\mathcal{F} \subset \mathcal{I}$ finite for which $K \subset \bigcup_{i\in\mathcal{F}} V_i$. This implies

$$K \subset \bigcup_{i\in\mathcal{F}} (K \cap V_i) = \bigcup_{i\in\mathcal{F}} f^{-1}(U_i).$$

We conclude that $f(K) \subset \bigcup_{i\in\mathcal{F}} U_i$.

Onto property (ii), we have that $f(K)$ is bounded, and so the supremum exists. The infimum exists in any case due to the bound $|f(z)| \geq 0$. Denote by S and I the supremum and infimum respectively. Then S is attained on K because for any sequence $(z_k)_k \subset K$ with $\lim_k |f(z_k)| = S$ one can find a subsequent $\{z_{n_k}\}_k$ convergent to $z_0 \in K$; thanks to Theorem 2.14. By the continuity of $|f| : \mathbb{C} \to \mathbb{R}$ (given by $z \mapsto |f(z)|$) one has that $S = \lim_k |f(z_{n_k})| = |f(z_0)|$. An identical argument shows that I is attained in K.

Let us now prove (iii). Suppose, seeking a contradiction, that f is not uniformly continuous. By Proposition 2.24, there exists $\varepsilon > 0$ and sequences $\{z_n\}_n$, $\{w_n\}_n \subset K$ such that $|z_n - w_n| \to 0$, and $|f(z_n) - f(w_n)| \geq \varepsilon$ for all $n \in \mathbb{N}$. Because K is compact, by Theorem 2.14 we can find subsequences $\{z_{n_k}\}_k$, $\{w_{n_k}\}_k$ convergent to $z \in K$ and $w \in K$ respectively. But the fact that $|z_{n_k} - w_{n_k}| \to 0$ implies that $z = w$. By the continuity of f in K and Proposition 2.20, we have that $f(z_{n_k}) \to f(z)$ and $f(w_{n_k}) \to f(z)$. Therefore $|f(z_{n_k}) - f(w_{n_k})| \to 0$, contradicting that $|f(z_{n_k}) - f(w_{n_k})| \geq \varepsilon$ for all k. ☐

2.1.6 Connected Sets and Domains

Definition 2.26 (Connected sets, Domains, and Path Connected sets)**.** *Let* $A \subset \mathbb{C}$ *be a subset.*

- *We say that a couple of open sets* $U, V \subset \mathbb{C}$ *is a* ***separation of*** A *if*

  1. $U \cap V \cap A = \emptyset$.
  2. $A \cap U \neq \emptyset \neq A \cap V$.
  3. $A \subset U \cup V$.

- *We say that A is* ***connected*** *if there exists* ***no*** *separation of A.*
- *We say that A is a* ***domain*** *if A is open and connected.*
- *We say that A is* ***path-connected*** *if for any two points $z, w \in w$ there exists a continuous mapping $\gamma : [a,b] \to A$, with $a < b$, $a, b \in \mathbb{R}$, so that $\gamma(a) = z$ and $\gamma(b) = w$.*

The definition of connectedness for a set A looks a bit technical but it essentially says that A **cannot** be *decomposed* as a non-trivial and disjoint union of two opens relative to A. Conditions $U \cap V = \emptyset$, $A \cap U \neq \emptyset \neq A \cap V$ merely express the non-triviality of the separations. Path-connectedness is a bit more intuitive: roughly speaking it says that any two points in A can be joined by a continuous path within the set A. Here is another perspective to connectedness.

Proposition 2.27. *If $A \subset \mathbb{C}$, the following statements are equivalent.*

(i) A is connected.

(ii) Every set E with $\emptyset \neq E \subset A$ that can be written as $E = U \cap A = F \cap A$, for some $U \subset \mathbb{C}$ open and $F \subset \mathbb{C}$ closed, must satisfy $E = A$.

Proof. Let us begin with $(i) \implies (ii)$. Assume that A is connected, and for the sake of contradiction, that E is a set as in (ii) and still $E \subsetneq A$. Putting $V := \mathbb{C} \setminus F$, we have

$$\emptyset \neq A \setminus E = (\mathbb{C} \setminus F) \cap A = V \cap A.$$

The set V, being the complement of a closed set, is open. It is immediately checked that $\{U, V\}$ form a separation of A, a contradiction.

To show $(ii) \implies (i)$, assume (ii) and suppose, seeking a contradiction, that A has a separation $\{U, V\}$. If we define $E := U \cap A$ and $F = \mathbb{C} \setminus V$, the properties of the separation $U \cap V \cap A = \emptyset$ and $A \subset U \cup V$ show that $E = F \cap A$ as well. Because $U \cap A \neq \emptyset \neq V \cap A$, we have that $E \neq \emptyset$ and $A \setminus E \neq \emptyset$, contradiction (ii). □

Proposition 2.27 is often through which we show that a function f on a domain Ω satisfies certain pointwise property, say $\mathcal{P}$, at all points of Ω. If f satisfies $\mathcal{P}$ at some $z_0 \in \Omega$, we define

$$E := \{z \in \Omega \,:\, f \text{ satisfies property } \mathcal{P} \text{ at the point } z\}.$$

Since $z_0 \in E$, and $\Omega \subset \mathbb{C}$ is already open (and connected), if we manage to prove that E is open, and that $E = F \cap \Omega$ for some closed set $F \subset \mathbb{C}$, Proposition 2.27 tells us that then $E = \Omega$, implying that f satisfies property $\mathcal{P}$ at all $z \in \Omega$.

Let us implement this idea to obtain a useful property involving *locally constant* functions.

Proposition 2.28. *Let $\Omega \subset \mathbb{C}$ be a domain, and $f : \Omega \to \mathbb{C}$ continuous such that f is locally constant, meaning that for every $z \in \Omega$ there exists $\varepsilon > 0$ with $D(z, \varepsilon) \subset \Omega$ and f constant on $D(z, \varepsilon)$. Then f is constant in Ω.*

In particular, if $\Omega \subset \mathbb{C}$ is a domain, $f : \Omega \to \mathbb{C}$ is continuous, and $f(\Omega)' = \emptyset$, then f is constant.

Proof. Fix a point $z_0 \in \Omega$, and define the set

$$E = \{z \in \Omega \,:\, f(z) = f(z_0)\}.$$

Obviously $z_0 \in E \subset \Omega$. Given $z \in E$, there is $\varepsilon > 0$ with $D(z, \varepsilon) \subset \Omega$ and $f(w) = f(z) = f(z_0)$ for all $w \in D(z, \varepsilon)$. This shows that $D(z, \varepsilon) \subset E$, and so E is open. Also, since f is continuous, $f^{-1}(\{f(z_0)\})$ can be written as $f^{-1}(\{f(z_0)\}) = F \cap \Omega$ for a closed subset $F \subset \mathbb{C}$, by virtue of Proposition 2.20. And then clearly $E = f^{-1}(\{f(z_0)\}) = F \cap \Omega$. By Proposition 2.27 and the connectedness of Ω, we get that $E = \Omega$, showing that f is constant in Ω.

For the second part, if $f(\Omega)' = \emptyset$, then for every $z \in \Omega$ we can find $\varepsilon > 0$ so that $f(\Omega) \cap (D(f(z), \varepsilon) \setminus \{f(z)\}) = \emptyset$; see the Definition 2.5 of accumulation point. By the continuity of f, there exists $\delta > 0$ so that

$$f(D(z, \delta)) \subset D(f(z), \varepsilon);$$

which, by the previous observation, leads us to $f(D(z, \delta)) \setminus \{f(z)\} = \emptyset$. Thus f is constant on $D(z, \delta)$. By the first part of the present proposition, we conclude that f is constant in Ω.

□

All path-connected sets are connected. However, there of connected sets in $\mathbb{C}$ that are not path-connected. A prototypical example is the graph of the function $(0, 1] \ni x \mapsto \sin(1/x)$ together with the origin, that is,

$$A = \{(x, \sin(1/x)) \,:\, x \in (0, 1]\} \cup \{(0, 0)\}.$$

However, for open sets, the notions of connectedness and path-connectedness are identical.

Proposition 2.29. *Let $A \subset \mathbb{C}$ and $f : A \to \mathbb{C}$ be continuous. The following hold.*

(i) If A is connected (resp. path-connected), then $f(A)$ is connected (resp. path-connected).

(ii) If A is path-connected, then A is connected.

(iii) If A is a domain, that is, open and connected, then A is path-connected.

Proof.

(i) We begin with the statement concerning connectedness. Suppose that $f(A)$ is not connected. Then $f(A)$ admits a separation into open sets $U, V \subset \mathbb{C}$ as in Definition 2.26. By Proposition 2.20, $f^{-1}(U) = W_1 \cap A$ and $f^{-1}(V) = W_2 \cap A$ for open sets $W_1, W_2 \subset \mathbb{C}$. It is straightforward to check that W_1, W_2 provide a separation of A, implying that A is not connected.

Now, if A is path-connected, and $u, v \in f(A)$ are two points, let $z, w \in A$ with $f(z) = u$ and $f(w) = v$. Let $\gamma : [a, b] \to A$ a continuous function with $\gamma(a) = z$ and $\gamma(b) = w$. The mapping $g := f \circ \gamma : [a, b] \to f(A)$ defines a continuous function with $g(a) = u$, $g(b) = v$.

(ii) Assume that A has a separation U, V. In particular we can find points $z \in U \cap A$ and $w \in U \cap V$. By the path-connectedness of A, there is $\gamma : [a, b] \to A$ continuous with $\gamma(a) = z$ and $\gamma(b) = w$. From Proposition 2.20, $\gamma^{-1}(U) = I \cap [a, b]$ and $\gamma^{-1}(V) = J \cap [a, b]$ for open sets $I, J \subset \mathbb{R}$. This contradicts that $[a, b]$ is a connected set of $\mathbb{R}$.

(iii) Fix a point $z_0 \in A$, and define

$$A_{z_0} := \{z \in A \,:\, \text{there exists } \gamma : [a, b] \to A \text{ continuous with } \gamma(a) = z_0,\ \gamma(b) = z\}.$$

Clearly, $z_0 \in A_{z_0}$ because we can take γ as the path constantly equal to z_0. We now show that A_{z_0} is open. Indeed, given $z \in A_{z_0}$, let γ_1 be joining z_0 and z as in the definition of A_{z_0}. Because A is open, $D(z, \varepsilon) \subset A$ for some $\varepsilon > 0$. Now, let $w \in D(z, \varepsilon)$, and let $\gamma_2 : [b, b+1] \to D(z, \varepsilon)$ be the path defined by $\gamma_2(t) = (1 + b - t)z + (t - b)w$, for $t \in [b, b+1]$. Note that γ_2 is just the segment line that joins z and w, and clearly $\gamma_2(t) \in D(z, \varepsilon)$ for all $t \in [b, b+1]$. Concatenating γ_1 and γ_2, we obtain a new path $\gamma : [a, b+1] \to A$ that is continuous and joins z_0 and w. This shows that $D(z, \varepsilon) \subset A_{z_0}$, and hence A_{z_0} is open.

Now we show that $A \setminus A_{z_0}$ is open too. Indeed, if $z \in A \setminus A_{z_0}$ then $D(z, \varepsilon) \subset A$ for some $\varepsilon > 0$ because A is open. If there exists $w \in D(z, \varepsilon) \cap A_{z_0}$, then there is a continuous path γ in A joining w to z_0. But again the concatenation of γ with the segment that joins w to z (this segment is contained in $D(z, \varepsilon)$ and so in A), we obtain a continuous path in A joining z to z_0, a contradiction because $z \notin A_{z_0}$. We have shown that $D(z, \varepsilon) \subset A \setminus A_{z_0}$, whence $A \setminus A_{z_0}$ is open.

Defining $F = A \setminus A_{z_0}$, we have that $F \subset \mathbb{C}$ is closed and $A_{z_0} = F \cap A$. Since A is connected, Proposition 2.27, we obtain $A_{z_0} = A$, which shows that any $z \in A$ can be joined to z_0 by a continuous path in A. Now, if $z, w \in A$, let $\gamma_1, \gamma_2 : [c,d] \to A$ continuous with $\gamma_1(a) = z_0$, $\gamma_1(b) = z$, $\gamma_2(a) = z_0$ and $\gamma_2(b) = w$. The path $\widetilde{\gamma_1} : [a,b] \to A$ defined by $\widetilde{\gamma_1}(t) = \gamma_1(b+a-t)$ is continuous and satisfies $\widetilde{\gamma_1}(a) = z$, $\widetilde{\gamma_1}(b) = z_0$. We thus concatenate $\widetilde{\gamma_1}$ and γ_2 to obtain a new continuous path γ on A joining z and w. □

Example 2.30. A very special case of path-connected sets are the convex sets. We say that $A \subset \mathbb{C}$ is **convex** if for any two points $z, w \in A$, the segment joining z, w, i.e., $\{(1-t)z + tw : t \in [0,1]\}$, is entirely contained in A.

2.2 Complex differentiability

In Section 2.1.5, we learnt that continuity of functions in $\mathbb{C}$ is essentially the same as continuity of functions in $\mathbb{R}^2 \to \mathbb{R}^2$. For instance, we showed in Proposition 2.20, that $f : \mathbb{C} \to \mathbb{C}$ is continuous at a point $z_0 \in \mathbb{C}$ iff $\mathrm{Re}(f) : \mathbb{C} \to \mathbb{R}$ and $\mathrm{Im}(f) : \mathbb{C} \to \mathbb{R}$ are continuous at z_0. In coordinates, it is the same as saying that $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous at $(x_0, y_0) \in \mathbb{R}^2$ iff $f_1 : \mathbb{R}^2 \to \mathbb{R}$ and $f_2 : \mathbb{R}^2 \to \mathbb{R}$ are continuous at (x_0, y_0).

This section is devoted to the complex version of differentiability. We remind that a function $g : \Omega \to \mathbb{R}^m$ defined on an open set $\Omega \subset \mathbb{R}^n$ is said to be differentiable at $(x_0, y_0) \in \Omega$ provided there exists a linear map $Dg(x_0, y_0) : \mathbb{R}^n \to \mathbb{R}^m$, called the *differential of g at (x_0, y_0)*, such that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\|g(x,y) - g(x_0,y_0) - Dg(x_0,y_0)(x-x_0, y-y_0)\|}{\|(x-x_0, y-y_0)\|} = 0, \tag{2.2.1}$$

where we denote by $\|\cdot\|$ the Euclidean norm both in $\mathbb{R}^n$ and $\mathbb{R}^m$. In the case $m = 1$, the differential $Dg(x_0, y_0)$ is identified with a vector $\nabla g(x_0, y_0)$, called the gradient of g at (x_0, y_0).

So, if $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at (x_0, y_0), and we write f in components $f(x,y) = (f_1(x,y), f_2(x,y))$, the differential has associated matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x}(x_0,y_0) & \frac{\partial f_1}{\partial y}(x_0,y_0) \\ \frac{\partial f_2}{\partial x}(x_0,y_0) & \frac{\partial f_2}{\partial y}(x_0,y_0) \end{pmatrix};$$

where $\frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y}, \frac{\partial f_2}{\partial x}, \frac{\partial f_2}{\partial y}$ are the *partial derivatives* of f at (x_0, y_0), namely,

$$\frac{\partial f_1}{\partial x}(x_0,y_0) := \lim_{t\to 0} \frac{f_1(x_0+t, y_0) - f_1(x_0,y_0)}{t}, \quad \frac{\partial f_1}{\partial y}(x_0,y_0) := \lim_{t\to 0} \frac{f_1(x_0, y_0+t) - f_1(x_0,y_0)}{t} \tag{2.2.2}$$

$$\frac{\partial f_2}{\partial x}(x_0,y_0) := \lim_{t\to 0} \frac{f_2(x_0+t, y_0) - f_2(x_0,y_0)}{t}, \quad \frac{\partial f_2}{\partial y}(x_0,y_0) := \lim_{t\to 0} \frac{f_2(x_0, y_0+t) - f_2(x_0,y_0)}{t}. \tag{2.2.3}$$

We remind that the existence of these partial derivatives does not guarantee the differentiability (and not even the continuity) of f at (x_0, y_0).

It is easy to see that the differentiability of f at (x_0, y_0) is equivalent to both $f_1, f_2 : \Omega \to \mathbb{R}$ being differentiable (in the sense of (2.2.1)) at the same point. However, unlike for continuity, the complex differentiability of $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ at $z_0 = x_0 + iy_0$ is strictly stronger than saying that both $\mathrm{Re}(f) : \Omega \to \mathbb{C}$, $\mathrm{Im}(f) : \Omega \to \mathbb{R}$ should be real-differentiable at (x_0, y_0). Complex differentiability additionally implies that the differential $Df(x_0, y_0)$ must have associated matrix of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

We will take care of this in the forthcoming subsection.

2.2.1 Differentiable and Holomorphic functions

The definition of derivative for a complex function is the natural complex version of real functions.

Definition 2.31 (Complex differentiability)**.** *Let $\Omega \subset \mathbb{C}$ an open set, $f : \Omega \to \mathbb{C}$ a function, and $z_0 \in \Omega$. We say that f is* ***(complex) differentiable at*** z_0 *if the following limit exists (meaning belonging to $\mathbb{C}$):*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

In such case, we denote this limit by $f'(z_0)$, and call it the derivative of f at z_0.

And if f is differentiable at every point of Ω, we then say that f is ***holomorphic*** *in Ω.*

We denote by $\mathcal{H}(\Omega)$ the collection of all holomorphic functions in Ω.

For functions $f : \Omega \to \mathbb{C}$, by differentiability we will always understand **complex** differentiability in the sense of Definition 2.31.

Example 2.32. Naturally, the function $f : \mathbb{C} \to \mathbb{C}$ given $f(z) = z$ is holomorphic in $\mathbb{C}$, and $f'(z) = 1$ for all $z \in \mathbb{C}$. A constant function $f(z) = w$, for all $z \in \mathbb{C}$, is also holomorphic with $f'(z) = 0$ for each $z \in \mathbb{C}$.

The function $f : \mathbb{C}\setminus\{0\} \to \mathbb{C}$ given by $f(z) = 1/z$ is holomorphic in $\mathbb{C}\setminus\{0\}$, with $f'(z) = -1/z^2$ for all $z \neq 0$. Indeed, if $z \neq 0$,

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z} = \lim_{w \to z} \frac{1/w - 1/z}{w - z} = \lim_{w \to z} \frac{-1}{wz} = \frac{-1}{z^2}.$$

Functions that are not differentiable at any point of $\mathbb{C}$ are for example $z \mapsto \overline{z}$, $z \mapsto \mathrm{Re}(z)$, $z \mapsto \mathrm{Im}(z)$. The reason is that for any $z \in \mathbb{C}$, for any $w \in \mathbb{C} \setminus \{0\}$, one has

$$\frac{\overline{w} - \overline{z}}{w - z} = \frac{\overline{w - z}}{w - z},$$

whose limit as $w \to z$ does not exist, because for $w = z + t$ with $t \in \mathbb{R}$, the above fraction is identically 1, while for $w = z + it$, with $t \in \mathbb{R}$, the fraction equals -1. For the same reason, $z \mapsto \mathrm{Re}(z)$, $z \mapsto \mathrm{Im}(z)$ are nowhere differentiable $\mathbb{C}$. Notice that these three functions are of class $C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ and even $\mathbb{R}$-linear.

Remark 2.33. If $f : \Omega \to \mathbb{C}$ is differentiable at $z_0 \in \Omega$, then f is continuous at z_0. In fact, f is *Lipschitz at* z_0, meaning that there exist $M > 0$ and $r > 0$ so that $|f(z) - f(z_0)| \leq M|z - z_0|$ for all $z \in D(z_0, r)$.

Proof. Indeed, taking $\varepsilon = 1$ in Definition 2.31, we find some $r > 0$ so that $D(z_0, r) \subset \Omega$, and, if $z \in D(z_0, r) \setminus \{z_0\}$, then

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| \leq 1.$$

The triangle inequality gives, for all $z \in D(z_0, r)$:

$$|f(z) - f(z_0)| \leq \big(1 + |f'(z_0)|\big)\, |z - z_0| = M|z - z_0|;$$

where $M = 1 + |f'(z_0)|$. In particular, this implies that f is continuous at z_0. □

Proposition 2.34. *If $\Omega \subset \mathbb{C}$ is open and $f, g : \Omega \to \mathbb{C}$ are differentiable at $z_0 \in \Omega$, then:*

(i) $f + g$ is differentiable at z_0, and $(f + g)'(z_0) = f'(z_0) + g'(z_0)$.

(ii) fg is differentiable at z_0, and $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$.

(iii) If $g(z_0) \neq 0$, then f/g (defined on a disk $D(z_0, r) \subset \Omega$) is differentiable at z_0, with

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - g'(z_0)f'(z_0)}{g(z_0)^2}.$$

(iv) (The Chain Rule). *If $U \subset \mathbb{C}$ is open with $g(z_0) \in U$, $g : \Omega \to U$, and $h : U \to \mathbb{C}$ is differentiable at z_0, then $h \circ g : \Omega \to \mathbb{C}$ is differentiable at z_0 and*

$$(h \circ g)'(z_0) = h'(g(z_0))g'(z_0).$$

Proof. We can reproduce the arguments of the proofs of the corresponding results for real functions of one variable.

(i) This is very easy, using simply the definition of complex differentiability.

(ii) It suffices to write

$$\frac{f(z)g(z) - f(z_0)g(z_0)}{z - z_0} = g(z)\left(\frac{f(z) - f(z_0)}{z - z_0}\right) + f(z_0)\left(\frac{g(z) - g(z_0)}{z - z_0}\right),$$

and bearing in mind that $\lim_{z\to z_0} g(z) = g(z_0)$ (Remark 2.33), the previous expression converges to $f'(z_0)g(z_0) + g'(z_0)f'(z_0)$ as $z \to z_0$.

(iii) By (ii), it is enough to prove it for $f(z) = 1$ for all $z \in \Omega$. Now, denote $h = 1/g$, which is well defined on a disk $D(z_0, r)$ because $g(z_0) \neq 0$ and g is continuous at z_0. Then $1 = hg$ on $D(z_0, r)$, and differentiating at z_0 using (ii), we get $0 = h'(z_0)g(z_0) + h(z_0)g'(z_0)$. Therefore

$$\left(\frac{1}{g}\right)'(z_0) = h'(z_0) = \frac{-h(z_0)g'(z_0)}{g(z_0)} = \frac{-g'(z_0)}{g(z_0)^2}.$$

(iv) Given $\varepsilon > 0$, the differentiability of g at z_0 gives $\delta > 0$ so that $D(z_0, \delta) \subset \Omega$ and

$$\frac{|g(z) - g(z_0) - g'(z_0)(z - z_0)|}{|z - z_0|} \leq \frac{\varepsilon}{2(1 + |h'(g(z_0))|)}, \quad \text{for all } z \in D(z_0, \delta) \setminus \{z_0\}. \tag{2.2.4}$$

Also, by Remark 2.33, we can find $M > 0$ and $r > 0$ (depending only on g and z_0) for which

$$|g(z) - g(z_0)| \leq M|z - z_0| \quad \text{for all } z \in D(z_0, r) \subset \Omega. \tag{2.2.5}$$

And the differentiability of h at $g(z_0)$ gives some $\eta > 0$ such that

$$|h(w) - h(g(z_0)) - h'(g(z_0))(w - g(z_0))| \leq \frac{\varepsilon}{2M}|w - g(z_0)| \quad \text{for all } w \in D(g(z_0), \eta) \setminus \{g(z_0)\}. \tag{2.2.6}$$

So, define $\delta^* := \min\{\delta, r, \frac{\eta}{M}\}$ and let $z \in D(z_0, \delta^*)$. We simultaneously have (2.2.4), (2.2.5) and (2.2.6) with $w = g(z)$ (as, by (2.2.6), $|g(z) - g(z_0)| \leq M|z - z_0| < \eta$). Therefore, we can use all these estimates to conclude

$$\begin{aligned}
&\frac{|h(g(z)) - g(h(z_0)) - h'(g(z_0))g'(z_0)(z - z_0)|}{|z - z_0|} \\
&\leq \frac{|h(g(z)) - g(h(z_0)) - h'(g(z_0))(g(z) - g(z_0))|}{|z - z_0|} + \frac{|h'(g(z_0))\,(g(z) - g(z_0) - g'(z_0)(z - z_0))\,|}{|z - z_0|} \\
&\leq \frac{\varepsilon}{2M}\frac{|g(z) - g(z_0)|}{|z - z_0|} + |h'(g(z_0))|\frac{\varepsilon}{2(1 + |h'(g(z_0))|)} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

□

For example, a consequence of Proposition 2.34 is that every function of the form $f(z) = z^n$, with $n \in \mathbb{N}$, is holomorphic in $\mathbb{C}$, with $f'(z) = nz^{n-1}$. This, in combination with the Chain Rule, shows that for $n \in \mathbb{Z}$, $n < 0$, the function $f(z) = z^n$ is holomorphic in $\mathbb{C} \setminus \{0\}$ with derivative $f'(z) = nz^{n-1}$.

Also, every polynomial $P : \mathbb{C} \to \mathbb{C}$, $P(z) = a_0 + \cdots + a_n z^n$, $n \in \mathbb{N}$, is holomorphic in $\mathbb{C}$. Moreoer, all *rational functions* $f = P/Q$ with P, Q polynomials, are holomorphic in the open set $\{z \in \mathbb{C} : Q(z) \neq 0\}$.

2.2.2 The Cauchy-Riemann Equations and some consequences

The following theorem characterizes complex differentiability of a function and is a key component in complex analysis.

Theorem 2.35 (The Cauchy-Riemann Equations)**.** *Let $\Omega \subset \mathbb{C}$ an open set, $f : \Omega \to \mathbb{C}$ a function, and $z_0 = x_0 + iy_0 \in \Omega$. Denote $u = \mathrm{Re}(f) : \Omega \to \mathbb{R}$ and $v = \mathrm{Im}(f) : \Omega \to \mathbb{R}$. Then, f is differentiable at z_0 if and only if*

(a) both u and v are differentiable at (x_0, y_0) and

(b) the partial derivatives of u, v satisfy the Cauchy–Riemann equations:

$$(\mathrm{CR}) \equiv \begin{cases} \dfrac{\partial u}{\partial x}(x_0, y_0) = \dfrac{\partial v}{\partial y}(x_0, y_0) \\ \dfrac{\partial u}{\partial y}(x_0, y_0) = -\dfrac{\partial v}{\partial x}(x_0, y_0). \end{cases} \tag{2.2.7}$$

Moreover, in such case we have

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0). \tag{2.2.8}$$

Proof. We will prove it by following a chain of equivalences. Looking at Definition 2.31, f being differentiable at z_0 is equivalent to the existence of $L \in \mathbb{C}$ (which will be $f'(z_0)$) so that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0) - L(z - z_0)}{z - z_0} = 0, \quad \text{or, equivalently,} \quad \lim_{z \to z_0} \frac{f(z) - f(z_0) - L(z - z_0)}{|z - z_0|} = 0. \tag{2.2.9}$$

By Proposition 2.17, that the limit in (2.2.9) equals 0 is equivalent to saying that the limit of the real and imaginary parts exist and equal 0 as well. Let us find the real and imaginary parts. Writing $z = x + iy$, and using that $f = u + iv$, the numerator is

$$u(x, y) + iv(x, y) - u(x_0, y_0) - iv(x_0, y_0) - (\mathrm{Re}(L) + i\,\mathrm{Im}(L))\,((x - x_0) + i(y - y_0)),$$

whose real and imaginary parts are respectively

$$\begin{aligned} &u(x, y) - u(x_0, y_0) - (\mathrm{Re}(L)(x - x_0) - \mathrm{Im}(L)(y - y_0))\,, \\ &v(x, y) - v(x_0, y_0) - (\mathrm{Im}(L)(x - x_0) + \mathrm{Re}(L)(y - y_0))\,. \end{aligned}$$

Hence, after writing $|z - z_0| = \|(x, y) - (x_0, y_0)\|$, (2.2.9) is equivalent to the existence of $L \in \mathbb{C}$ so that

$$\begin{aligned} &\lim_{(x,y)\to(x_0,y_0)} \frac{u(x, y) - u(x_0, y_0) - (\mathrm{Re}(L)(x - x_0) - \mathrm{Im}(L)(y - y_0))}{\|(x, y) - (x_0, y_0)\|} = 0, \text{ and} \\ &\lim_{(x,y)\to(x_0,y_0)} \frac{v(x, y) - v(x_0, y_0) - (\mathrm{Im}(L)(x - x_0) + \mathrm{Re}(L)(y - y_0))}{\|(x, y) - (x_0, y_0)\|} = 0, \text{ for some } L \in \mathbb{C}. \end{aligned}$$

The vectors $(\mathrm{Re}(L), -\mathrm{Im}(L))$ and $(\mathrm{Im}(L), \mathrm{Re}(L))$ define linear mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$. So, the two last equations are equivalent to saying that "there exists some $L \in \mathbb{C}$ for which

1. $u : \Omega \to \mathbb{R}$ is differentiable at (x_0, y_0) and

$$\nabla u(x_0, y_0) := \left(\frac{\partial u}{\partial x}(x_0, y_0), \frac{\partial u}{\partial y}(x_0, y_0)\right) = (\mathrm{Re}(L), -\mathrm{Im}(L)). \tag{2.2.10}$$

2. $v : \Omega \to \mathbb{R}$ is differentiable at (x_0, y_0) and

$$\nabla v(x_0, y_0) := \left(\frac{\partial v}{\partial x}(x_0, y_0), \frac{\partial v}{\partial y}(x_0, y_0)\right) = (\mathrm{Im}(L), \mathrm{Re}(L))." \tag{2.2.11}$$

But this is obviously the same as saying that $u, v : \Omega \to \mathbb{R}$ are differentiable at (x_0, y_0) and

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \quad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0).$$

We have shown the equivalence of the assertion. Now, observe that if any of the previous equivalent conditions are satisfied, then $L = f'(z_0)$ in our notation, and from (2.2.10)-(2.2.11), we get $\mathrm{Re}(f'(z_0)) = \frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$ and $\mathrm{Im}(f'(z_0)) = \frac{\partial v}{\partial y}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0)$. We derive

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0).$$

□

For instance, the Cauchy-Riemann equations (2.2.7) offer another argument to justify why $f(z) = \overline{z}$ is not differentiable at any $z \in \mathbb{C}$. Indeed, writing $f(x + iy) = x - iy$, we have that $u(x, y) := \mathrm{Re}(f)(x, y) = x$ and $v(x, y) := \mathrm{Im}(f)(x, y) = -y$. So $u, v : \mathbb{R}^2 \to \mathbb{R}$ are real-differentiable. However,

$$\frac{\partial u}{\partial x}(x, y) = 1, \quad \frac{\partial u}{\partial y}(x, y) = 0, \quad \frac{\partial v}{\partial x}(x, y) = 0, \quad \frac{\partial v}{\partial y}(x, y) = -1.$$

So clearly $\frac{\partial u}{\partial x}(x, y) \neq \frac{\partial v}{\partial y}(x, y)$, and Theorem 2.35 says that f is *not* differentiable at any $z = x + iy \in \mathbb{C}$.

It is important to notice that the real and imaginary parts $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ of a function $f : \Omega \to \mathbb{C}$ may have partial derivatives that satisfy the Cauchy–Riemann equations at a point z_0, and yet f is not even continuous at z_0. It is therefore really necessary the hypothesis in Theorem 2.35 that the functions $\mathrm{Re}(f) : \Omega \to \mathbb{R}$, $\mathrm{Im}(f) : \Omega \to \mathbb{R}$ are real differentiable at z_0. We will illustrate this by means of Exercise 2.24, using the *complex exponential function.*

Corollary 2.36. *Let $\Omega \subset \mathbb{C}$ be open, and $f : \Omega \to \mathbb{C}$ a function so that the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial x}$ exist at every point of Ω, where $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$. If $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial x} : \Omega \to \mathbb{R}$ are continuous at some $z_0 \in \Omega$, and satisfy the Cauchy-Riemann equations* (2.2.7) *at z_0, then f is differentiable at z_0.*

Proof. Since the partial derivatives of u and v exist in all of Ω and are continuous at $z_0 = x_0 + iy_0$, then $u, v : \Omega \to \mathbb{R}$ are differentiable at (x_0, y_0). Together with the assumption that the partial derivatives of u and v satisfy the Cauchy-Riemann equations (2.2.7), Theorem 2.35 says that f is differentiable at z_0. □

Corollary 2.37. *Let $\Omega \subset \mathbb{C}$ be open and connected, and $f \in \mathcal{H}(\Omega)$ with $f'(z) = 0$ for all $z \in \Omega$. Then f is constant.*

Proof. For each $z = x + iy \in U$, formulas (2.2.8) and (2.2.7) imply

$$\frac{\partial \,\mathrm{Re}(f)}{\partial x}(x, y) = \frac{\partial \,\mathrm{Re}(f)}{\partial y}(x, y) = \frac{\partial \,\mathrm{Im}(f)}{\partial x}(x, y) = \frac{\partial \,\mathrm{Im}(f)}{\partial y}(x, y) = 0.$$

Thus $\nabla \mathrm{Re}(f)$ and $\nabla \mathrm{Im}(f)$ are null on Ω. From differential calculus in $\mathbb{R}^2$, the connectedness of Ω leads to $\mathrm{Re}(f), \mathrm{Im}(f)$ being constant in Ω, and thus f is constant in Ω as well. □

Remark 2.38. Let $\Omega \subset \mathbb{C}$ an open set, $f : \Omega \to \mathbb{C}$ a function (complex) differentiable at $z_0 = x_0 + iy_0 \in \Omega$. As as $\mathbb{R}^2$-mapping $f : \Omega \to \mathbb{R}^2$, by Theorem 2.35, is (real) differentiable at (x_0, y_0) with differential map $Df(x_0, y_0) : \mathbb{R}^2 \to \mathbb{R}^2$ determined by the matrix

$$\begin{pmatrix} \frac{\partial \,\mathrm{Re}(f)}{\partial x}(x_0, y_0) & \frac{\partial \,\mathrm{Re}(f)}{\partial y}(x_0, y_0) \\ \frac{\partial \,\mathrm{Im}(f)}{\partial x}(x_0, y_0) & \frac{\partial \,\mathrm{Im}(f)}{\partial y}(x_0, y_0) \end{pmatrix} = \begin{pmatrix} \frac{\partial \,\mathrm{Re}(f)}{\partial x}(x_0, y_0) & -\frac{\partial \,\mathrm{Im}(f)}{\partial x}(x_0, y_0) \\ \frac{\partial \,\mathrm{Im}(f)}{\partial x}(x_0, y_0) & \frac{\partial \,\mathrm{Re}(f)}{\partial x}(x_0, y_0) \end{pmatrix}. \tag{2.2.12}$$

In particular, $Df(x_0, y_0)$ is given by an antisymmetric matrix. Moreover, computing the determinant and looking at formula (2.2.8) we conclude

$$|f'(z_0)|^2 = \left(\frac{\partial \operatorname{Re}(f)}{\partial x}(x_0, y_0)\right)^2 + \left(\frac{\partial \operatorname{Im}(f)}{\partial x}(x_0, y_0)\right)^2 = \det(Df(x_0, y_0)). \tag{2.2.13}$$

2.2.3 The Inverse Function Theorem for Holomorphic maps

The identity (2.2.8) will help us to prove an inverse function theorem for holomorphic functions.

Theorem 2.39 (Inverse Function Theorem). *Let $\Omega \subset \mathbb{C}$ an open set, $f : \Omega \to \mathbb{C}$ holomorphic in Ω with $f' : \Omega \to \mathbb{C}$ continuous* [1]*, and $z_0 \in \Omega$ so that $f'(z_0) \neq 0$. Then there exists an open set $U \subset \Omega$ with $z_0 \in U$ such that $V := f(U)$ is open, the restriction of $f_{|U} : U \to f(U)$ is a bijection, and its inverse $f^{-1} : V \to U$ is holomorphic in V, with*

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))} \quad \textit{for all} \quad w \in V. \tag{2.2.14}$$

Proof. Since $f : \Omega \to \mathbb{C}$ is holomorphic and f' is continuous, by Theorem 2.35 and (2.2.8) we have that $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ (regarded as a function $\mathbb{R}^2 \to \mathbb{R}^2$) is real differentiable, with its partial derivatives continuous in Ω. Thus, f is of class $C^1(\Omega, \mathbb{R}^2)$. Write $z_0 = x_0 + iy_0$. By (2.2.13) and the assumption $f'(z_0) \neq 0$, we have that $\det(Df(x_0, y_0)) \neq 0$, meaning that $Df(x_0, y_0)$ is invertible. By the Inverse Function Theorem in $\mathbb{R}^n$, there exists an open subset U of Ω containing z_0, with $f(U)$ open and $f_{|U} : U \to f(U)$ is a bijection whose inverse $f^{-1} : f(U) \to U$ is also of class $C^1(f(U), \mathbb{R}^2)$. Moreover, $Df(x, y)$ is invertible for every $(x, y) \in U$ and the differential of f^{-1} at $w \in f(U)$ satisfies

$$D(f^{-1})(w) = \left(Df(f^{-1}(w))\right)^{-1}. \tag{2.2.15}$$

Let us check that $f^{-1} : f(U) \to \mathbb{C}$ is holomorphic in $f(U)$ and prove (2.2.14). Let $w \in f(U)$ and $z \in U$ with $f(z) = w$. By (2.2.15), $Df(z)$ is invertible, and by (2.2.13) we get $f'(z) \neq 0$. Thus, considering the limit of the inverse, given $\varepsilon > 0$ there exists $\delta >$ so that $0 < |u - z| < \delta$, $u \in U$, implies

$$\left| \frac{u - z}{f(u) - f(z)} - \frac{1}{f'(z)} \right| < \varepsilon. \tag{2.2.16}$$

Now, by the continuity of f^{-1} on $f(U)$, there exists $\eta > 0$ so that $|\xi - w| < \eta$, $\xi \in f(U)$, implies $|f^{-1}(\xi) - f^{-1}(w)| < \delta$. We can thus apply (2.2.16) with $f^{-1}(\xi)$ in place of u to obtain

$$\left| \frac{f^{-1}(\xi) - f^{-1}(w)}{\xi - w} - \frac{1}{f'(z)} \right| < \varepsilon.$$

□

The argument we used in the proof of Theorem 2.39 to calculate the derivative of an inverse function will be reproduced when establishing the definition of holomorphic roots, logarithmic and power functions; see Subsection 2.4.3.

2.3 Conformal and Harmonic maps

There are two fundamental classes of real mappings that are closely related to holomorphic functions: the conformal mappings, and the harmonic functions.

We begin by study the conformal maps, for which we first need to understand the differentiation of curves in the complex plane.

[1] We will see later that holomorphic functions are of class C^∞, and so the assumption that f' is continuous can be done away with.

Definition 2.40. *If $a, b \in \mathbb{R}$ with $a < b$, we say that a curve $\gamma : (a,b) \to \mathbb{C}$* ***is differentiable at*** *$t_0 \in (a,b)$ if its real and imaginary parts $\mathrm{Re}(\gamma), \mathrm{Im}(\gamma) : (a,b) \to \mathbb{R}$ are differentiable at t_0. And in this case, we define*

$$\gamma'(t_0) = \mathrm{Re}(\gamma)'(t_0) + i\,\mathrm{Im}(\gamma)'(t_0) \equiv \left(\mathrm{Re}(\gamma)'(t_0),\, \mathrm{Im}(\gamma)'(t_0)\right).$$

Similarly, we say that γ is of class $C^1((a,b))$, provided $\mathrm{Re}(\gamma), \mathrm{Im}(\gamma) : (a,b) \to \mathbb{R}$ are of class $C^1((a,b))$.

We remind that if $\Omega \subset \mathbb{R}^n$ is open, a function $h : \Omega \to \mathbb{R}^m$ is of class $C^k(\Omega)$ if h has partial derivatives up to order k at every point of Ω, and those partial derivatives are continuous in Ω.

Lemma 2.41. *Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \to \mathbb{C}$ differentiable at $z_0 \in \Omega$ and $\gamma : (-\varepsilon, \varepsilon) \to \Omega$ is differentiable at 0 with $\gamma(0) = z_0$. Then $f \circ \gamma : (-\varepsilon, \varepsilon) \to \mathbb{C}$ is differentiable at $z_0 = x_0 + iy_0$, with*

$$(f \circ \gamma)'(0) = f'(z_0) \cdot \gamma'(0) = Df(x_0, y_0)\gamma'(0);$$

where the first product is between complex numbers, and the second as a matrix and a vector.

Proof. If $z_0 = x_0 + iy_0$, by Theorem 2.35, $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ is real differentiable at (x_0, y_0). Regarding $\gamma : (-\varepsilon, \varepsilon) \to \Omega$ as a real-valued curve, we can apply the Chain Rule for differentiable real-valued maps to deduce that $f \circ \gamma$ is differentiable at 0 with

$$(f \circ \gamma)'(0) = Df(\gamma(0))\gamma'(0) = Df(x_0, y_0)\gamma'(0).$$

But the Cauchy-Riemann equations (2.2.7) for f at z_0 imply that

$$\begin{aligned} Df(z_0)\gamma'(0) &= \begin{pmatrix} \frac{\partial \mathrm{Re}(f)}{\partial x}(x_0, y_0) & -\frac{\partial \mathrm{Im}(f)}{\partial x}(x_0, y_0) \\ \frac{\partial \mathrm{Im}(f)}{\partial x}(x_0, y_0) & \frac{\partial \mathrm{Re}(f)}{\partial x}(x_0, y_0) \end{pmatrix} \cdot \begin{pmatrix} \mathrm{Re}(\gamma)'(0) \\ \mathrm{Im}(\gamma)'(0) \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{Re}(\gamma)'(0)\frac{\partial \mathrm{Re}(f)}{\partial x}(x_0, y_0) - \mathrm{Im}(\gamma)'(0)\frac{\partial \mathrm{Im}(f)}{\partial x}(x_0, y_0) \\ \mathrm{Re}(\gamma)'(0)\frac{\partial \mathrm{Im}(f)}{\partial x}(x_0, y_0) + \mathrm{Im}(\gamma)'(0)\frac{\partial \mathrm{Re}(f)}{\partial x}(x_0, y_0). \end{pmatrix} \end{aligned}$$

Recall that $f'(z_0) = \frac{\partial \mathrm{Re}(f)}{\partial x}(x_0, y_0) + i\frac{\partial \mathrm{Im}(f)}{\partial x}(x_0, y_0)$ by (2.2.8), and so the components of the last vector are respectively the real and imaginary part of the complex product $f'(z_0) \cdot \gamma'(0)$. □

We define conformal maps as those real differentiable maps that *preserves angles and orientation.*

Definition 2.42 (Conformal Map)**.** *Let $\Omega \subset \mathbb{R}^2$ be open, $z_0 \in \Omega$, and $f : \Omega \to \mathbb{R}^2$ real-differentiable at z_0. We say that f is* ***orientation-preserving*** *at z_0 if $\det(Df(z_0)) > 0$.*

Also, we say that f is ***angle-preserving*** *at z_0 if for any two C^1 curves $\gamma_1, \gamma_2 : (-\varepsilon, \varepsilon) \to \Omega$ with $\gamma_1(0) = \gamma_2(0) = z_0$ and $\gamma_1'(0) \neq 0 \neq \gamma_2'(0)$, one has that $(f \circ \gamma_1)'(0) \neq 0 \neq (f \circ \gamma_2)'(0)$ and*

$$\frac{\langle (f \circ \gamma_1)'(0), (f \circ \gamma_2)'(0) \rangle}{|(f \circ \gamma_1)'(0)||(f \circ \gamma_2)'(0)|} = \frac{\langle \gamma_1'(0), \gamma_2'(0) \rangle}{|\gamma_1'(0)||\gamma_2'(0)|}.$$

Here $\langle u, v \rangle$ denotes the dot product between $u, v \in \mathbb{R}^2$, namely, if $u = (u_1, u_2)$ and $v = (v_1, v_2)$, then $\langle u, v \rangle = u_1v_1 + u_2v_2$.

Finally, we say that f is ***conformal at*** *z_0 if f is both angle-preserving and orientation-preserving. And if f is conformal at each $z_0 \in \Omega$, we then say that f is* ***conformal in*** *Ω.*

Note that if $u, v \in \mathbb{R}^2$ are non-zero vectors, we can write

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\|v\|};$$

where $\theta \in [0, \pi]$ is the *angle between* u and v, and $\|u\|, \|v\|$ are the Euclidean norms of u and v. The angle-preserving along with orientation-preserving can alternatively be described using the argument $\mathrm{Arg} : \mathbb{C} \setminus \{0\} \to (-\pi, \pi]$. Namely, if f, Ω, z_0 are as in Definition 2.42, then f is both angle and orientation-preserving if for any two curves γ_1, γ_2 as in Definition 2.42, one has $(f \circ \gamma_1)'(0) \neq 0 \neq (f \circ \gamma_2)'(0)$ and

$$\mathrm{Arg}\left((f \circ \gamma_1)'(0) \cdot \overline{(f \circ \gamma_2)'(0)}\right) = \mathrm{Arg}\left(\gamma_1'(0) \cdot \overline{\gamma_2'(0)}\right).$$

We next show that conformal mappings are precisely the holomorphic functions with non-zero derivatives.

Theorem 2.43. *Let $\Omega \subset \mathbb{C}$ be open, $z_0 \in \Omega$, and $f : \Omega \to \mathbb{R}^2$ a function. Then, the following are equivalent.*

(i) f is (complex) differentiable at z_0 with $f'(z_0) \neq 0$.

(ii) f is conformal at z_0.

Proof.

$(i) \implies (ii)$. Then f is real-differentiable with $Df(z_0)$ having associated matrix of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, for $a, b \in \mathbb{R}$, and $f'(z_0) = a + bi$. Since $f'(z_0) \neq 0$, we have that a or b are non-zero, and so $\det(Df(z_0)) = a^2 + b^2 > 0$. Thus f is orientation-preserving at z_0. To check that f is angle-preserving, let γ_1 and γ_2 be as in Definition 2.42. Since $f'(z_0) \neq 0$, Lemma 2.41 implies that also

$$(f \circ \gamma_j)'(0) = f'(z_0)\gamma_j'(0) \neq 0,$$

for $j = 1, 2$. Now, if $A := \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ is the matrix above, then clearly $A^t A = (a^2 + b^2)I$, where I denotes the identity map. Thus, using Lemma 2.41, we get

$$\begin{aligned}\langle (f \circ \gamma_1)'(0), (f \circ \gamma_2)'(0)\rangle &= \langle A \cdot \gamma_1'(0), A \cdot \gamma_2'(0)\rangle = (A \cdot \gamma_1'(0))^t (A \cdot \gamma_2'(0)) \\ &= \left(\gamma_1'(0)\right)^t A^t A \gamma_2'(0) = (a^2 + b^2)\langle \gamma_1'(0), \gamma_2'(0)\rangle = |f'(z_0)|^2 \langle \gamma_1'(0), \gamma_2'(0)\rangle;\end{aligned}$$

where we used formula (2.2.8) in the last identity. And again using Lemma 2.41, we have that $(f \circ \gamma_j)'(0) = f'(z_0)\gamma_j'(0)$, from which we deduce

$$\frac{\langle (f \circ \gamma_1)'(0), (f \circ \gamma_2)'(0)\rangle}{|(f \circ \gamma_1)'(0)||(f \circ \gamma_2)'(0)|} = \frac{|f'(z_0)|^2 \langle \gamma_1'(0), \gamma_2'(0)\rangle}{|f'(z_0)|^2 |\gamma_1'(0)||\gamma_2'(0)|} = \frac{\langle \gamma_1'(0), \gamma_2'(0)\rangle}{|\gamma_1'(0)||\gamma_2'(0)|}.$$

$(ii) \implies (i)$. Assume, without loss of generality, that $z_0 = 0$. Let $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ be the matrix associated with $Df(0)$, which has positive determinant, as f is orientation-preserving. Define, for each $\theta \in \mathbb{R}$, the curve $\gamma_\theta(t) = t(\cos\theta, \sin\theta) = te^{i\theta}$, for all $t \in \mathbb{R}$. Clearly $\gamma_\theta'(0) \neq 0$, and, employing the Chain Rule for real-differentiable functions, we get

$$(f \circ \gamma_\theta)'(0) = Df(0)\gamma_\theta'(0) \neq 0,$$

for all $\theta \in \mathbb{R}$. Now, since f is angle-preserving, we have

$$\frac{\langle Df(0)\gamma_0'(0), Df(0)\gamma_\theta'(0)\rangle}{|Df(0)\gamma_0'(0)||Df(0)\gamma_\theta'(0)|} = \frac{\langle \gamma_0'(0), \gamma_\theta'(0)\rangle}{|\gamma_0'(0)||\gamma_\theta'(0)|},$$

which means that

$$\frac{\langle (a,b),(a\cos\theta+c\sin\theta, b\cos\theta+d\sin\theta)\rangle}{\sqrt{a^2+b^2}\sqrt{(a\cos\theta+c\sin\theta)^2+(b\cos\theta+d\sin\theta)^2}}=\cos\theta.$$

Computing the terms above, we get that

$$\begin{aligned}(a^2+b^2)\cos\theta+(ac+bd)\sin\theta=\\ =\cos\theta\sqrt{a^2+b^2}\sqrt{(a^2+b^2)\cos^2\theta+(c^2+d^2)\sin^2\theta+2\sin\theta\cos\theta(ac+bd)},\end{aligned} \tag{2.3.1}$$

for all $\theta\in\mathbb{R}$. Letting $\theta=\pi/2$ in (2.3.1) implies $ac+bd=0$. So, for all $\theta\in\mathbb{R}$, (2.3.1) becomes

$$(a^2+b^2)\cos\theta=\cos\theta\sqrt{a^2+b^2}\sqrt{(a^2+b^2)\cos^2\theta+(c^2+d^2)\sin^2\theta}. \tag{2.3.2}$$

Now we take $\theta=\pi/4$ in (2.3.2) and use that $\cos^2\theta=\sin^2\theta=\frac{1}{2}$ to derive:

$$a^2+b^2=\frac{1}{2}(a^2+b^2)+\frac{1}{2}(c^2+d^2),\quad \text{and so}\quad a^2+b^2=c^2+d^2.$$

We have deduced the relations $a^2+b^2=c^2+d^2$ and $ac+bd=0$. But note that this implies $a^2+2aci-c^2=d^2-2bdi-b^2$, which in turn yields

$$(a+ci)^2=(d-bi)^2.$$

Thus $(a+ci+d-ib)(a+ci-d+ib)=0$, so we get that either $a=d$ and $c=-b$, or $a=-d$ and $b=c$. But the latter gives the matrix $\begin{pmatrix}a & b\\ b & -a\end{pmatrix}$, whose determinant is $-a^2-b^2>0$, a contradiction. Therefore, we must have $a=d$ and $c=-b$, and so the partial derivatives of f at z_0 satisfy the Cauchy-Riemann equations; see (2.2.7). By Theorem 2.35, f is complex differentiable at z_0. Moreover, in our notation, and by virtue of (2.2.8), we deduce

$$f'(z_0)=a+ib\neq 0,$$

as $a^2+b^2=\det(Df(z_0))>0$. □

Now we consider the class of harmonic functions. We begin by defining those that are real-valued, essentially as those functions that satisfy the Laplace Equation.

Definition 2.44 (Real Harmonic Function). *Let $\Omega\subset\mathbb{R}^2$ be open and $u:\Omega\to\mathbb{R}$ a function of class $C^2(\Omega)$. We say that u is **harmonic** if u satisfies the Laplace Equation:*

$$\Delta u:=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\quad on\quad \Omega. \tag{2.3.3}$$

By saying that $u\in C^2(\Omega)$ we of course mean that u has partial derivatives up to order two, and are continuous in Ω.

For example, the function $f:\mathbb{R}^2\to\mathbb{R}^2$ defined by $f(x,y)=x^2-y^2$ is harmonic, but $g:\mathbb{R}^2\to\mathbb{R}^2$, $g(x,y)=x^2+y^2$ is not.

It turns out that the Cauchy Riemann-Equations imply that the real and imaginary parts of holomorphic functions are harmonic.

Proposition 2.45. *Let $\Omega\subset\mathbb{C}$ be open and $f:\Omega\to\mathbb{C}$ holomorphic such that* $\mathrm{Re}(f),\mathrm{Im}(f):\Omega\to\mathbb{R}^2$ *are of class $C^2(\Omega)$ [2]. Then* $\mathrm{Re}(f)$ *and* $\mathrm{Im}(f)$ *are harmonic in Ω.*

[2] We will see in Chapter 4 that this assumption is redundant, as holomorphic maps are of class C^∞.

Proof. By the Schwarz's theorem on the mixed partial derivatives, one has, at every $(x, y) \in \Omega$,

$$\frac{\partial}{\partial x}\left(\frac{\partial \operatorname{Im}(f)}{\partial y}\right)(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial \operatorname{Im}(f)}{\partial x}\right)(x, y).$$

But applying the Cauchy-Riemman equations (2.2.7) to the partial derivatives between brackets, we get

$$\frac{\partial^2 \operatorname{Re}(f)}{\partial x^2}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial \operatorname{Re}(f)}{\partial x}\right)(x, y) = -\frac{\partial}{\partial y}\left(\frac{\partial \operatorname{Re}(f)}{\partial y}\right)(x, y) = -\frac{\partial^2 \operatorname{Re}(f)}{\partial y^2}(x, y).$$

Thus $\operatorname{Re}(f)$ satisfies (2.3.3). Similarly, we get that $\operatorname{Im}(f)$ is harmonic. □

Proposition 2.45 motives the following definition.

Definition 2.46. *Let $\Omega \subset \mathbb{C}$ be open and let $u : \Omega \to \mathbb{R}$ be a harmonic function. We say that $v : \Omega \to \mathbb{R}^2$ is **a harmonic conjugate of** u if the function $u + iv$ is holomorphic in Ω.*

If Ω is open and connected, the harmonic conjugates are unique up to an additive constant; see Exercise 2.20. It is possible to show that on domains Ω that are *simply connected*, every harmonic function has a harmonic conjugate.

One can also define harmonicity for complex-valued functions, as those functions whose real and imaginary parts are (real) harmonic.

Definition 2.47 (Complex Harmonic Function)**.** *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ a function of class $C^2(\Omega)$. We say that f is **harmonic** if both $\operatorname{Re}(f), \operatorname{Im}(f) : \Omega \to \mathbb{R}$ are harmonic in Ω; in the sense of Definition 2.44.*

Looking at Proposition 2.45 and Definition 2.47, we deduce that, for $f : \Omega \to \mathbb{C}$ with $\Omega \subset \mathbb{C}$ open and $f \in C^2(\Omega)$, then

$$f \text{ is holomorphic} \implies f \text{ is harmonic.}$$

The converse is clearly not true, for instance, the function $f(z) = \operatorname{Re}(z)$, $z \in \mathbb{C}$, defines a harmonic function that is not (complex) differentiable at any point.

2.4 Elementary functions

This section is devoted to defining the complex analogous of elementary real functions such as exponentials, trigonometric functions, logarithms and power functions, and examine some of their properties. In particular, we see how these functions are holomorphic in appropriate domains, using either the Cauchy-Riemann equations (2.2.7), or some of the ideas from the Inverse Function Theorem 2.39.

2.4.1 The Complex Exponential Function

Let us now get back to formula (1.5.1) in Section 1.1, where we defined an exponential function in the axis $i\mathbb{R}$ by setting $e^{i\theta} = \cos\theta + i\sin\theta$ for all $\theta \in \mathbb{R}$, and examined some of its properties; see (1.5.2) and (1.5.3). We extend this function to the whole complex plane, in a very natural way.

Definition 2.48 (Complex exponential function)**.** *We define the **complex exponential function** $\mathbb{C} \ni z \mapsto e^z$ by the formula*

$$\mathbb{C} \ni z = x + iy \mapsto e^z := e^x \cdot e^{iy} = e^x (\cos y + i \sin y).$$

In other words, $e^z := e^{\operatorname{Re}(z)} e^{i \operatorname{Im}(z)}$.

This function coincides with the real exponential $x \mapsto e^x$ when $z \in \mathbb{R}$. Also, note that

$$|e^z| = e^{\mathrm{Re}(z)}, \quad z \in \mathbb{C}. \tag{2.4.1}$$

More fundamental properties are collected in the following theorem.

Theorem 2.49. *The complex exponential function has the following properties.*

(i) $z \mapsto e^z$ *is holomorphic in* $\mathbb{C}$ *with derivative equal to* e^z *for all* $z \in \mathbb{C}$.

(ii) $e^{z+w} = e^z e^w$ *for all* $z, w \in \mathbb{C}$.

(iii) $e^z \neq 0$ *and* $(e^z)^{-1} = e^{-z}$ *for all* $z \in \mathbb{C}$.

(iv) For each $n \in \mathbb{Z}$ *and* $z \in \mathbb{C}$, *we have* $(e^z)^n = e^{nz}$.

(v) The mapping $z \mapsto e^z$, $\mathbb{C} \to \mathbb{C}\setminus\{0\}$ *is surjective. Moreover, for each* $w \in \mathbb{C}\setminus\{0\}$, *the solutions* $z \in \mathbb{C}$ *of the equation* $e^z = w$ *are*

$$\log|w| + i\arg(w) := \{\log|w| + i\theta \,:\, \theta \in \arg(w)\} = \{\log|w| + i\,(\mathrm{Arg}(w) + 2k\pi) \,:\, k \in \mathbb{Z}\}$$

(vi) $e^z = e^w$ *if and only if* $z - w = 2k\pi i$ *for some* $k \in \mathbb{Z}$. *In particular,* $e^z = 1$ *if and only if* $z = 2k\pi i$ *for some* $k \in \mathbb{Z}$.

(vii) For each $a \in \mathbb{R}$, *define* $S_a := \{z \in \mathbb{C} \,:\, \mathrm{Im}(z) \in (a-\pi, a+\pi]\}$. *The mapping* $z \mapsto e^z$ *is a bijection* $S_a \to \mathbb{C}\setminus\{0\}$.

Proof.

(i) The real and imaginary parts of the exponential function are respectively $\mathbb{R}^2 \ni (x,y) \mapsto u(x,y) = e^x \cos y$ and $\mathbb{R}^2 \ni (x,y) \mapsto v(x,y) = e^x \sin y$. These functions are differentiable in $\mathbb{R}^2$, and their partial derivatives are

$$\frac{\partial u}{\partial x}(x,y) = e^x \cos y, \quad \frac{\partial u}{\partial y}(x,y) = -e^x \sin y, \quad \frac{\partial v}{\partial x}(x,y) = e^x \sin y, \quad \frac{\partial v}{\partial y}(x,y) = e^x \cos y.$$

The Cauchy Riemann equations (2.2.7) are satisfied for u, v, and hence $z \mapsto e^z \in \mathcal{H}(\mathbb{C})$.

(ii) Write $z = a + ib$ and $w = c + id$, for $a, b, c, d \in \mathbb{R}$. We use Definition 2.48 and formula (1.5.2):

$$e^{z+w} = e^{(a+c)+i(b+d)} = e^{a+c}e^{i(b+d)} = e^{a+c}e^{ib}e^{id} = e^a e^c e^{ib} e^{id} = (e^a e^{ib})(e^c e^{id}) = e^z e^w.$$

(iii) Since either $\cos y \neq 0$ or $\sin y \neq 0$ for all $y \in \mathbb{R}$, it is clear that $e^z \neq 0$ for all $z \in \mathbb{C}$. About the inverse, we use (ii) to write

$$e^z e^{-z} = e^0 = 1 \implies (e^z)^{-1} = e^{-z}.$$

(iv) It is a consequence of (ii) and (iii).

(v) If $w \in \mathbb{C}\setminus\{0\}$, by Theorem 1.8, we can write $w = |w|e^{i\theta} = e^{\log|w|}e^{i\theta}$ for all $\theta \in \arg(w)$. Writing $z = x + iy$ we have that $e^z = w$ if and only if

$$e^x \cos y = e^{\log|w|}\cos\theta, \quad e^x \sin y = e^{\log|w|}\sin\theta.$$

Thus $e^z = w$ if and only if $z = \log|w| + i\theta$ for $\theta \in \arg(w)$. The last identity is just Lemma 1.10.

(vi) $e^z = 1$ if and only if $e^x(\cos y + i\sin y) = 1$. This is equivalent to $\sin y = 0$ and $e^x \cos y = 1$, in turn equivalent to $x = 0$, $y \in 2\pi\mathbb{Z}$. This shows that $e^z = 1$ if and only if $z \in 2\pi i\mathbb{Z}$.

Now, $e^z = e^w$ if and only if $e^{z-w} = 1$ (by (ii) and (iii)). By what we have just proved $z - w \in 2\pi i\mathbb{Z}$.

(vii) The injectivity follows from (vi), because $z, w \in S_a$ and $z - w \in 2\pi i\mathbb{Z}$ imply $z = w$. And for the surjectivity, given $w \in \mathbb{C}\setminus\{0\}$, we saw in (v) that $e^z = w$ if $z = \log|w| + i\alpha$ for every $\alpha \in \arg(w)$. Choosing $\alpha \in \arg(w)$ so that $\alpha \in (a-\pi, a+\pi)$, we get that $z^* := \log|w| + i\alpha \in S_a$ and $e^{z^*} = w$. ☐

2.4.2 Complex Trigonometric and Hyperbolic Functions

We continue defining extensions of the trigonometric and hyperbolic real functions based on the complex exponential.

Definition 2.50 (Trigonometric complex functions)**.** *The **complex cosine** and **sine** are the functions $\mathbb{C} \ni z \mapsto \cos z, \sin z$ defined by*

$$\cos z := \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z := \frac{e^{iz} - e^{-iz}}{2i}. \tag{2.4.2}$$

*We also define the **tangent** $z \mapsto \tan z$, the **cosecant** $z \mapsto \csc z$, the **secant** $z \mapsto \sec z$, and the **cotangent** $z \mapsto \cot z$ by*

$$\tan z := \frac{\sin z}{\cos z}, \quad \csc z := \frac{1}{\sin z}, \quad \sec z := \frac{1}{\cos z}, \quad \cot z := \frac{\cos z}{\sin z}.$$

These functions are of course defined at those points where the denominators are nonzero.

For real numbers $\theta \in \mathbb{R}$, the functions $\cos z$ and $\sin z$ agree with their real analogous, since by expression (1.2.1) and property (1.5.3), we have

$$\frac{e^{i\theta} + e^{-i\theta}}{2} = \operatorname{Re}(e^{i\theta}) = \cos\theta \quad \text{and} \quad \frac{e^{i\theta} - e^{-i\theta}}{2i} = \operatorname{Im}(e^{i\theta}) = \sin\theta.$$

Proposition 2.51. *The functions $z \mapsto \cos z, \sin z$ satisfy the following properties.*

(i) $z \mapsto \cos z, \sin z$ *are holomorphic in* $\mathbb{C}$ *with* $(\cos z)' = -\sin z$ *and* $(\sin z)' = \cos z$ *for all* $z \in \mathbb{C}$.

(ii) $z \mapsto \cos z, \sin z$, $\mathbb{C} \mapsto \mathbb{C}$ *are surjective.*

(iii) $\cos(z) = \cos(-z)$, $\sin(-z) = -\sin z$, *and* $\cos z = \sin(z + \frac{\pi}{2})$ *for all* $z \in \mathbb{C}$.

(iv) $\cos^2 z + \sin^2 z = 1$ *for all* $z \in \mathbb{C}$.

(v) *For all* $z, w \in \mathbb{C}$, *we have*

$$\sin(z + w) = \sin z \cos w + \cos z \sin w, \quad \cos(z + w) = \cos z \cos w - \sin z \sin w.$$

Proof.

(i) By the definition (2.4.2), this follows from the fact that $z \mapsto e^z$ is holomorphic in $\mathbb{C}$.

(ii) Let $w \in \mathbb{C} \setminus \{0\}$. The equality $\cos z = w$ is equivalent to $\xi + \xi^{-1} = 2w$, for $\xi = e^{iz}$. By Theorem 2.49, the exponential is surjective onto $\mathbb{C} \setminus \{0\}$, so proving the existence of z so that $\cos z = w$ is equivalent to proving the existence of $\xi \in \mathbb{C} \setminus \{0\}$ such that $\xi + \xi^{-1} = 2w$. But this equation is the same as $\xi^2 - 2w\xi + 1 = 0$, which naturally has solutions on $\xi \in \mathbb{C} \setminus \{0\}$. And when $w = 0$, we have $\cos(\pi/2) = w$.

The proof of the surjectivity of $\sin z$ is almost identical.

(iii) The first two identities are immediate from (2.4.2). For the third one, use that $e^{i\frac{\pi}{2}} = i$, $e^{-i\frac{\pi}{2}} = -i$.

(iv) and (v) They are easily verified, using the properties of the exponential; see Theorem 2.49. □

An obvious warning is that, unlike for real numbers, the estimates $|\cos z|, |\sin z| \leq 1$ are **not** true in general.

Definition 2.52 (Hyperbolic functions)**.** *The* ***hyperbolic cosine*** *and* ***sine*** *are the functions* $\mathbb{C} \ni z \mapsto \cosh z, \sinh z$ *defined by*

$$\cosh z := \frac{e^z + e^{-z}}{2}, \quad \sinh z := \frac{e^z - e^{-z}}{2}. \tag{2.4.3}$$

We can also define the ***hyperbolic tangent*** $z \mapsto \tanh z$ *as*

$$\tanh z := \frac{\sinh z}{\cosh z}, \quad z \in \{w \in \mathbb{C} : \cos w \neq 0\}.$$

Clearly $\cosh(iz) = \cos z$ and $\sinh(iz) = i \sin z$. The functions $z \mapsto \cosh z, \sinh z$ are holomorphic in $\mathbb{C}$, with $(\cosh z)' = \sinh z$ and $(\sinh z)' = \cosh z$. We will derive further relationships in the exercises; Section 2.5.

2.4.3 Holomorphic Roots, Logarithms, and Power Functions

Defining the complex versions of the nth roots, the logarithmic and the power functions is more delicate than a simple formula, especially if we want those functions to be holomorphic. In order to get these functions defined with holomorphicity in as many points as possible, instead of considering a single function, we need to consider *branches* of these functions. The key step is to understand the structure of the *branches* of the argument.

Definition 2.53. *A* ***branch of the argument*** *in a set* $E \subset \mathbb{C} \setminus \{0\}$ *is any continuous function* $\alpha : E \to \mathbb{R}$ *with* $\alpha(z) \in \arg(z)$ *for all* $z \in E$.

According to Example 2.22, the principal argument $\operatorname{Arg} : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{R}$ *is a branch of the argument in* $E = \mathbb{C} \setminus (-\infty, 0]$.

It turns out that two branches of the argument in the same domain differ by an integer multiple of 2π. Also, for every half-line from the origin, we can find a branch of the argument which is continuous on the complement of that half-line. This is shown in the following proposition.

Proposition 2.54. *Let* $\Omega \subset \mathbb{C} \setminus \{0\}$ *be a domain, and* $\alpha_1, \alpha_2 : \Omega \to \mathbb{R}$ *be two branches of the argument. Then there exists* $k \in \mathbb{Z}$ *so that* $\alpha_1(z) = \alpha_2(z) + 2k\pi$ *for all* $z \in \Omega$.

Besides, for every $v \in \mathbb{R}^2 \setminus \{(0,0)\}$, *for the half-line* $\ell_v = \{\lambda v : \lambda \geq 0\}$, *there is a branch* $\alpha_v : \mathbb{C} \setminus \ell_v \to \mathbb{R}$ *of the argument in the domain* $\mathbb{C} \setminus \ell_v$.

Proof. For each $w \in \Omega$, we can apply Lemma 1.10 to obtain $\alpha_1(w) - \alpha_2(w) = 2\pi n(w)$, for some $n(w) \in \mathbb{Z}$. The continuity of $\alpha_1 - \alpha_2$ in Ω implies that for every $w \in \Omega$ there is $\varepsilon > 0$ such that $D(w, \varepsilon) \subset \Omega$ and

$$\alpha_1(z) - \alpha_2(z) = 2\pi n(w), \quad \text{for all} \quad z \in D(w, \varepsilon).$$

Therefore, $\alpha_1 - \alpha_2$ is continuous and *locally constant in* the domain Ω. By Proposition 2.28, we get that $\alpha_1 - \alpha_2$ is equal to a constant of the form $2\pi n$, with $n \in \mathbb{Z}$. This proves the first part.

And if $v \in \mathbb{R}^2$, $\|v\| = 1$, we write that $v = e^{i\theta}$ for $\theta = \operatorname{Arg}(v) \in (-\pi, \pi]$. If $\theta = \pi$, then $\ell_v = (-\infty, 0]$ and we simply set $\alpha_v := \operatorname{Arg}$. And if $\theta \in (-\pi, \pi)$, we define $\alpha_v : \mathbb{C} \setminus \ell_v \to \mathbb{R}$ by

$$\alpha_v(z) = \begin{cases} \operatorname{Arg}(z) & \text{if } \operatorname{Arg}(z) \in (-\pi, \theta] \\ \operatorname{Arg}(z) - 2\pi & \text{if } \operatorname{Arg}(z) \in (\theta, \pi]. \end{cases}$$

Since Arg is continuous in $\mathbb{C} \setminus \left(\operatorname{Arg}^{-1}(\{\pi\}) \cup \{0\}\right)$, it is clear that α_v is continuous in $\mathbb{C} \setminus \left(\operatorname{Arg}^{-1}(\{\theta\}) \cup \{0\}\right)$, and naturally $\ell_v = \{0\} \cup \operatorname{Arg}^{-1}(\{\theta\})$. □

We next define the branches of nth roots as continuous right-inverses of the function $z \mapsto z^n$.

Definition 2.55. *If* $n \in \mathbb{N}$, *with* $n \geq 2$, *and* $E \subset \mathbb{C}$ *is a set, a* ***branch of the*** n***th root*** *in* E *is a continuous function* $h : E \to \mathbb{C}$ *so that* $(h(w))^n = w$ *for all* $w \in E$. *That is,* $h(w) \in \langle \sqrt[n]{w} \rangle$ *for all* $w \in E$; *see Definition 1.14.*

According to Example 2.22, the Principal n*th root* $\sqrt[n]{\cdot} : \mathbb{C} \setminus (-\infty, 0) \to \mathbb{C}$ *is a branch of the* n*th root in the set* $\mathbb{C} \setminus (-\infty, 0)$.

Some observations on the branches of nth roots are in order.

Remark 2.56. If $\Omega \subset \mathbb{C}$ is a domain so that there is a branch $\alpha : \Omega \to \mathbb{R}$ of the argument, $n \geq 2$, and $h_1, h_2 : \Omega \to \mathbb{C}$ are two branches of the nth rooth in Ω, then there exists an nth rooth of unity $\xi \in \langle \sqrt[n]{1} \rangle$ so that $h_1(w) = \xi \cdot h_2(w)$ for all $w \in \Omega$.

Indeed, by Theorem 1.15 and because $\alpha(z) \in \arg(z)$ for all $z \in \Omega$, we can write

$$h_1(w) = \sqrt[n]{|w|} e^{\frac{i\alpha(w)}{n}} e^{\frac{2\pi k_1(w)}{n} i}, \quad h_2(w) = \sqrt[n]{|w|} e^{\frac{i\alpha(w)}{n}} e^{\frac{2\pi k_2(w)}{n} i}$$

for all $w \in \Omega \setminus \{0\}$ and for functions $k_1, k_2 : \Omega \to \mathbb{Z}$. Because the functions $\Omega \ni w \mapsto h_1(w), h_2(w), \sqrt[n]{|w|}, e^{\frac{i\alpha(w)}{n}}$ are continuous, so are the functions

$$\Omega \ni z \mapsto \varphi_1(w) := e^{\frac{2\pi k_1(w)}{n} i}, \ \varphi_2(w) := e^{\frac{2\pi k_2(w)}{n} i}.$$

But note that $\varphi_1, \varphi_1 : \Omega \to \langle \sqrt[n]{1} \rangle$ take values on the finite set $\langle \sqrt[n]{1} \rangle$. By Proposition 2.28, we get that φ_1 and φ_2 are constantly equal to roots of unity $\xi_1, \xi_2 \in \langle \sqrt[n]{1} \rangle$. Letting $\xi = \xi_2/\xi_1$, we conclude that $\xi \in \langle \sqrt[n]{1} \rangle$ and

$$h_1(w) = \xi \cdot h_2(w), \quad w \in \Omega.$$

Proposition 2.54 also says that there is a branch $h_v : \mathbb{C} \setminus \ell_v \to \mathbb{R}$ of the nth root in the half-line ℓ_v, for all $v \in \mathbb{R}^2$.

Using some arguments from the proof of the Inverse Function Theorem 2.39, we can prove a general lemma about holomorphicity of *branches of the inverse* of a given function f.

Lemma 2.57. *Let $U, V \subset \mathbb{C}$ be open sets, $f : U \to \mathbb{C}$ a function, and $h : V \to \mathbb{C}$ be a continuous function with $h(V) \subset U$ and $f(h(w)) = w$ for all $w \in V$. Then, if $w_0 \in V$ is so that f is differentiable at $h(w_0)$ with $f'(h(w_0)) \neq 0$, then h is differentiable at w_0, and*

$$h'(w_0) = \frac{1}{f'(h(w_0))}.$$

Proof. Define $z_0 := h(w_0) \in h(V) \subset U$. By assumption $f'(z_0) \neq 0$, and so, employing Exercise 2.9 and the Definition 2.31 of complex derivative, we find $\delta > 0$ so that $D(z_0, \delta) \subset U$, $f(z) \neq f(z_0)$ for all $z \in D(z_0, \delta)$, and

$$\left| \frac{z - z_0}{f(z) - f(z_0)} - \frac{1}{f'(z_0)} \right| \leq \varepsilon, \quad \text{whenever} \quad z \in D(z_0, \delta). \tag{2.4.4}$$

Now, by the continuity of h, there exists $\eta > 0$ such that $D(w_0, \eta) \subset V$ and $w \in D(z_0, \eta)$ implies $|h(w) - h(w_0)| < \delta$. For those $w \in D(w_0, \eta)$, we have that $h(w) \in U$, $f(h(w)) = w$, and $|h(w) - z_0| < \delta$. Hence, (2.4.4) yields

$$\left| \frac{h(w) - h(w_0)}{w - w_0} - \frac{1}{f'(h(w_0))} \right| = \left| \frac{h(w) - z_0}{f(h(w)) - f(z_0)} - \frac{1}{f'(h(w_0))} \right| \leq \varepsilon.$$

□

As a corollary, every branch of the nth root in a domain is holomorphic.

Theorem 2.58. *Let Ω be a domain with $0 \notin \Omega$, let $n \in \mathbb{N}$ with $n \geq 2$, and $h : \Omega \to \mathbb{R}$ a branch of the nth root in Ω. Then $h \in \mathcal{H}(\Omega)$ and*

$$h'(w) = \frac{1}{n(h(w))^{n-1}}, \quad w \in \Omega. \tag{2.4.5}$$

In particular, the Principal nth Root $\sqrt[n]{\cdot} : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$ and its derivative is given by

$$(\sqrt[n]{\cdot})'(w) = \frac{1}{n(\sqrt[n]{w})^{n-1}}, \quad w \in \mathbb{C} \setminus (-\infty, 0]. \tag{2.4.6}$$

Proof. It suffices to apply Lemma 2.57 with $U = \mathbb{C}$, $V = \Omega$ and $f : U \to \mathbb{C}$ the function $f(z) = z^n$. Since h is a branch of the nth root in Ω, one has $f(h(w)) = (h(w))^n = w$ and $f'(h(w)) = n\,(h(w))^{n-1} \neq 0$ for all $w \in \Omega$.

□

Let us examine a couple of branches of holomorphic square roots.

Example 2.59. By Theorem 2.58, the principal branch of the square root $\sqrt{\cdot} : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ is holomorphic in the domain $\mathbb{C} \setminus (-\infty, 0]$ with

$$(\sqrt{\cdot})'(z) = \frac{1}{2\sqrt{z}}, \quad z \in \mathbb{C} \setminus (-\infty, 0].$$

However, we may be interested in defining a square root function that is differentiable at some points of the half line $(-\infty, 0]$. Then we can for example consider $v = (1, 0) \in \mathbb{R}^2$, which gives $\ell_v = [0, +\infty)$ and the branch $h_v : \mathbb{C} \setminus [0, +\infty) \to \mathbb{C}$ of the square root; see Remark 2.56 for the existence of such branch. By Theorem 2.58, h_v is holomorphic in $\mathbb{C} \setminus [0, +\infty)$ and $h_v'(w) = (2h_v(w))^{-1}$ for all $w \in \mathbb{C} \setminus [0, +\infty)$. In particular, we have defined a holomorphic square root that is holomorphic at all w with $\mathrm{Re}(w) < 0$.

No branch of the square root is differentiable at 0, but we can define a branch that is differentiable at all points of $\{z \in \mathbb{C} \setminus \{0\} : \mathrm{Re}(z) \cdot \mathrm{Im}(z) = 0\}$. For example, taking $v = (1, 1)$, combining Remark 2.56 and Theorem 2.58, we get a holomorphic branch of the square root in $\mathbb{C} \setminus \ell_v$; where $\ell_v = \{z \in \mathbb{C} : \mathrm{Re}(z) = \mathrm{Im}(z) \geq 0\}$.

In the same spirit as in the definition of holomorphic nth roots, we define the complex logarithms based on branches. We want the logarithm to behave as an inverse of the exponential $z \mapsto e^z$.

Definition 2.60 (Logarithms)**.** *Let $z \in \mathbb{C} \setminus \{0\}$.* ***A logarithm*** *of z is any $w \in \mathbb{C}$ with $e^w = z$.* ***The logarithm*** *of z, denoted by $\langle \log z \rangle$, is the set of all logarithms of z. By Theorem 2.49(v),*

$$\langle \log z \rangle = \{\log |z| + i\theta \,:\, \theta \in \arg(z)\} = \{\log |z| + i\,(\mathrm{Arg}(z) + 2k\pi) \,:\, k \in \mathbb{Z}\}. \tag{2.4.7}$$

The ***principal logarithm*** *is the function $\mathbb{C} \setminus \{0\} \ni z \mapsto \mathrm{Log}\, z$ given by*

$$\mathrm{Log}\, z := \log |z| + i\,\mathrm{Arg}(z), \quad z \in \mathbb{C} \setminus \{0\}. \tag{2.4.8}$$

Clearly, $\langle \log z \rangle = \{\mathrm{Log}\, z + 2\pi i k \,:\, k \in \mathbb{Z}\}$ and $\mathrm{Log}\, x = \log |x|$ for all $x \in \mathbb{R} \setminus \{0\}$.

Finally, if $E \subset \mathbb{C} \setminus \{0\}$, ***a branch of the logarithm in*** *E is any continuous function $h : E \to \mathbb{C}$ with $e^{h(w)} = w$ for all $w \in E$. In other words, $h(w) \in \langle \log w \rangle$ for all $w \in E$.*

For example, for $z = i$, the logarithm of z is the set

$$\langle \log(i) \rangle = \{\log |i| + i\,(\mathrm{Arg}(i) + 2k\pi) \,:\, k \in \mathbb{Z}\} = \{i\left(\frac{\pi}{2} + 2k\pi\right) \,:\, k \in \mathbb{Z}\}.$$

The principal logarithm of i is the complex number $\mathrm{Log}(i) = i\frac{\pi}{2}$.

Unlike for real numbers, in general it is **not** true that $\mathrm{Log}(z_1 z_2) = \mathrm{Log}(z_1) + \mathrm{Log}(z_2)$. This can be seen, as in the comment subsequent to Corollary 1.12, with $z_1 = z_2 = -i$, where $\mathrm{Log}(z_1 z_2) = \mathrm{Log}(-1) = i\,\mathrm{Arg}(-1) = \pi i$ and $\mathrm{Log}(z_1) = \mathrm{Log}(z_2) = i\,\mathrm{Arg}(-i) = -\frac{\pi}{2} i$. This example also shows that, sometimes, $\mathrm{Log}(z^2) \neq 2\,\mathrm{Log}\, z$.

Nonetheless, this type of property holds to some extent for the set of all logarithms.

Proposition 2.61. *Let $z_1, z_2 \in \mathbb{C} \setminus \{0\}$. Then*

$$\langle \log(z_1 z_2) \rangle = \langle \log z_1 \rangle + \langle \log z_2 \rangle := \{w_1 + w_2 \,:\, w_1 \in \langle \log z_1 \rangle,\ w_2 \in \langle \log z_2 \rangle\}.$$

Proof. Bearing in mind that $\log(|z_1 z_2|) = \log |z_1| + \log |z_2|$, the assertion follows easily from the first identity in (2.4.7) and Corollary 1.12. □

Concerning the branches of a logarithm, we observe the following.

Remark 2.62. Obviously, the principal logarithm $\mathrm{Log} : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is a branch of the logarithm in the set $\mathbb{C} \setminus \{0\}$. Also, if $E \subset \mathbb{C} \setminus \{0\}$ is a set, and $h : E \to \mathbb{C}$ is a branch of the logarithm in E, then, by the expression (2.4.7), there exists a function $\alpha : E \to \mathbb{R}$ so that

$$\alpha(w) \in \arg(w) \quad \text{and} \quad h(w) = |w| + i\alpha(w), \quad \text{for all} \quad w \in E. \tag{2.4.9}$$

By the continuity of $w \mapsto h(w), |w|$, the function $\alpha : E \to \mathbb{R}$ is continuous in E too. Thus, $\alpha : E \to \mathbb{R}$ is a branch of the argument in E; see Definition 2.53. Consequenly, in the notation of Proposition 2.54, for each $v \in \mathbb{R}^2 \setminus \{(0,0)\}$, there is a branch of the logarithm $h : \mathbb{C} \setminus \ell_v \to \mathbb{C}$.

Moreover, if $\Omega \subset \mathbb{C} \setminus \{0\}$ is a domain and $h_1, h_2 : \Omega \to \mathbb{C}$ are two branches of the logarithm in Ω, there exists $n \in \mathbb{Z}$ such that

$$h_1(w) = h_2(w) + 2\pi k i, \quad \text{for all} \quad w \in \Omega. \tag{2.4.10}$$

Indeed, by (2.4.9), $h_1 - h_2 = \alpha_1 - \alpha_2$, for branches $\alpha_1, \alpha_2 : \Omega \to \mathbb{R}$ of the argument. By Proposition 2.54, there exists $k \in \mathbb{Z}$ with $\alpha_1(w) - \alpha_1(w) = 2k\pi$ for all $w \in \Omega$, and so we get (2.4.10).

We next show that all branches of the logarithm are holomorphic as a corollary of Lemma 2.57 applied to the exponential function.

Theorem 2.63 (Holomorphic Logarithm)**.** *Let $\Omega \subset \mathbb{C} \setminus \{0\}$ be open, and $h : \Omega \to \mathbb{C}$ a branch of the logarithm in Ω. Then $h \in \mathcal{H}(\Omega)$ and*

$$h'(w) = \frac{1}{w}, \quad w \in \Omega.$$

In particular, for $\Omega := \mathbb{C} \setminus (-\infty, 0]$, and $U := \{z \in \mathbb{C} : \mathrm{Im}(z) \in (-\pi, \pi)\}$, the Principal Logarithm function $w \mapsto \mathrm{Log}\, w$ is a holomorphic bijection from Ω to U, with

$$(\mathrm{Log}\, w)' = \frac{1}{w} \quad \textit{for all} \quad w \in \Omega. \tag{2.4.11}$$

Proof. We apply Lemma 2.57 with $U = \mathbb{C}$, $V = \Omega$ and $f : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ given by $f(z) = e^z$. Since h is a branch of the logarithm in Ω, we have $f(h(w)) = e^{h(w)} = w$ and $f'(h(w)) = e^{h(w)} \neq 0$ for all $w \in \Omega$.

□

Theorem 2.63 does not apply to $(-\infty, 0]$, since the function $w \mapsto \mathrm{Log}\, w$ is not even continuous there. To see this, note that for each $a < 0$, the sequences $z_n = a + \frac{i}{n}$ and $w_n = a - \frac{i}{n}$ converge both to a. However, $\mathrm{Log}\, z_n$ and $\mathrm{Log}\, w_n$ have different limits, as

$$\begin{aligned} \mathrm{Log}\, z_n &= \log|a| + i\,\mathrm{Arg}(a + \tfrac{i}{n}) \to \log|a| + i\pi, \\ \mathrm{Log}\, w_n &= \log|a| + i\,\mathrm{Arg}(a - \tfrac{i}{n}) \to \log|a| - i\pi. \end{aligned}$$

The definition of complex powers with arbitrary exponents are based on logarithmic functions.

Definition 2.64 (Complex powers)**.** *Let $z \in \mathbb{C} \setminus \{0\}$ and $w \in \mathbb{C}$. We define the set w-**power of** z, and denote it by z^w, as the set*

$$\langle z^w \rangle := e^{w \langle \log z \rangle} := \{e^{w\xi} : \xi \in \langle \log z \rangle\}. \tag{2.4.12}$$

*The elements of the set $\langle z^w \rangle$ are called w-powers of z. And **the principal** w-**power of** z is the number*

$$z^w := e^{w\, \mathrm{Log}\, z}. \tag{2.4.13}$$

For example, in the case $z = w = i$ we have, using the computations in the comments right after Definition 2.60,

$$\langle i^i \rangle = \{e^{i\xi} : \xi \in \langle \log i \rangle\} = \{e^{i \cdot i(\frac{\pi}{2}+2k\pi)} : k \in \mathbb{Z}\} = \{e^{-\frac{\pi}{2}+2k\pi} : k \in \mathbb{Z}\}, \quad \text{and}$$
$$i^i = e^{i \operatorname{Log} i} = e^{i \cdot i \frac{\pi}{2}} = e^{-\frac{\pi}{2}}.$$

A this point, it is worth mentioning the **falsity** of the equality $(e^z)^w = e^{zw}$; see Exercise 2.27.

In the next proposition, we show that the principal power is a holomorphic function. But we also need to verify that this definition of powers is consistent with previous definitions we gave in the case of integer exponents (Definition 1.5) or rational exponents (Definition 1.14).

Proposition 2.65 (Properties of complex powers)**.** *The power set and function satisfy the following properties.*

(i) If $w = n \in \mathbb{Z}$, $z \in \mathbb{C}$, then $\langle z^w \rangle = \{z^n\}$ and z^w (defined in (2.4.13)) coincides with z^n (as in Definition 1.5).

(ii) If $n \in \mathbb{N}$, $w = 1/n$, and $z \in \mathbb{C}$, then $\langle z^w \rangle = \langle \sqrt[n]{z} \rangle$, and the principal $1/n$-power $z^{1/n}$ coincides with the principal nth-root $\sqrt[n]{z}$ of z.

(iii) If $x, y \in (0, +\infty)$ then the principal y-power of x equals x^y. Here, x^y represents the usual power of real numbers.

(iv) If $w_1, w_2 \in \mathbb{C}$, $z \in \mathbb{C} \setminus \{0\}$, then $z^{w_1} z^{w_2} = z^{w_1+w_2}$.

(v) If $z_1, z_2 \in \mathbb{C} \setminus \{0\}$, $w \in \mathbb{C}$, then $\langle z_1^w \rangle \cdot \langle z_2^w \rangle = \langle (z_1 \cdot z_2)^w \rangle$.

(vi) If $w \in \mathbb{C}$, the principal w-power function $f_w : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, given by $f_w(z) = z^w$, is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$ and

$$(f_w)'(z) = wz^{w-1}, \quad z \in \mathbb{C} \setminus (-\infty, 0]. \tag{2.4.14}$$

Proof.

(i) By (2.4.13), $\langle z^w \rangle = \{e^{n\xi} : \xi \in \langle \log z \rangle\}$. Using Theorem 2.49(iv), $e^{n\xi} = (e^\xi)^n$, where this nth-power is in the sense of Definition 1.5. If $\xi \in \langle \log z \rangle$, then $e^\xi = z$. Therefore, $\langle z^w \rangle = \{z^n\}$. It is then obvious that z^n is the principal w-power of z.

(ii) If $w = 1/n$, $n \in \mathbb{N}$, for every $k \in \mathbb{Z}$, we have the equalities

$$e^{\frac{1}{n}(\log|z|+i(\operatorname{Arg}(z)+2k\pi))} = e^{\log \sqrt[n]{|z|}+i\frac{\operatorname{Arg}(z)+2k\pi}{n}} = e^{\log \sqrt[n]{|z|}} e^{i\frac{\operatorname{Arg}(z)+2k\pi}{n}} = \sqrt[n]{|z|} e^{i\frac{\operatorname{Arg}(z)+2k\pi}{n}}.$$

According to (2.4.7) and Theorem 1.15, this shows $\langle z^w \rangle = \langle \sqrt[n]{z} \rangle$. And for $k = 0$, the previous chain of equalities gives $z^w = \sqrt[n]{z}$.

(iii) The principal y-power of x is

$$e^{y \operatorname{Log} x} = e^{y(\log|x|+i\operatorname{Arg}(x))} = e^{y \log x} = x^y.$$

(iv) Use Theorem 2.49(ii) to write

$$z^{w_1} z^{w_2} = e^{w_1 \operatorname{Log} z} e^{w_2 \operatorname{Log} z} = e^{(w_1+w_2)\operatorname{Log} z} = z^{w_1+w_2}.$$

(v) By (2.4.12) and Theorem 2.49(ii),

$$\langle z_1^w \rangle \langle z_2^w \rangle = \{e^{w(\xi_1+\xi_2)} : \xi_1 \in \langle \log z_1 \rangle, \xi_2 \in \langle \log z_2 \rangle\}.$$

And $\langle (z_1 z_2)^w \rangle = \{e^{w\xi} : \xi \in \langle \log(z_1 z_2) \rangle\}$. But then Proposition 2.61 says that $\langle \log(z_1 z_2) \rangle = \langle \log z_1 \rangle + \langle \log z_2 \rangle$, and our claim $\langle z_1^w \rangle \langle z_2^w \rangle = \langle (z_1 z_2)^w \rangle$ then follows.

(vi) We combine Theorem 2.49(i) with Theorem 2.63 (and formula (2.4.11)) and the Chain Rule (Proposition 2.34) to obtain that

$$(f_w)'(z) = e^{w \operatorname{Log} z}\,\frac{w}{z} = z^w\,\frac{w}{z} = wz^{w-1};$$

where we also used (iv) of the current proposition in the last equality. □

Property (v) of Proposition 2.65 is false replacing the sets power with the principal powers, that is, $z_1^w \cdot z_2^w \neq (z_1 \cdot z_2)^w$ in general. For instance, if $w = 1/2$ and $z_1 = z_2 = -1$, then, z_1^w and z_2^w are both respectively the principal 2-roots of -1, and so $z_1^w = z_2^w = i$. However $(z_1 \cdot z_2)^w = 1$, and therefore $(z_1 \cdot z_2)^w \neq z_1^w \cdot z_2^w$.

2.5 Exercises

Exercise 2.1. *Using either the definitions or any of the corresponding characterizations, justify the following topological claims.*

(a) *The set* $A = \{z \in \mathbb{C} \,:\, |\operatorname{Re}(z^2) - 1| > 2\}$ *is open.*

(b) *If* $f : \mathbb{C} \to \mathbb{C}$ *is continuous in* $\mathbb{C}$*, then* $B = \{z \in \mathbb{C} \,:\, f(z) = 1\}$ *is closed.*

(c) *The set* $C = \{0\} \cup \bigcup_{n=1}^{\infty} S(0, 1/n)$ *is compact.*

(d) *The set* $D = \{z \in \mathbb{C} \,:\, \operatorname{Re}(z) \cdot \operatorname{Im}(z) \geq 0\}$ *is path-connected, but its interior* $\operatorname{int}(D)$ *is not even connected.*

(e) *If a set* $E \subset \mathbb{C}$ *is convex, then its closure* $\overline{E}$ *is convex as well.*

Exercise 2.2. *We say that a sequence* $\{z_n\}_n \subset \mathbb{C}$ *has the Cauchy property if for every* $\varepsilon > 0$ *there exists* $n_0 \in \mathbb{N}$ *so that* $|z_n - z_m| < \varepsilon$ *for all* $n, m \geq n_0$*. Prove that every sequence with the Cauchy property* $\mathbb{C}$ *is convergent and viceversa.*

Hint: Use that $\mathbb{R}$ *is complete.*

Exercise 2.3. *Find a countable set* S *that is dense in* $\mathbb{C}$*.*

Exercise 2.4. *Let* $A \subset \mathbb{C}$ *be a convex set. Prove the following statements.*

(a) *For every* $z \in \operatorname{int}(A)$ *and* $w \in \overline{A}$*, the half-open segment* $[z, w) := \{tw + (1-t)z \,:\, t \in [0, 1)\}$ *is contained in* $\operatorname{int}(A)$*. Deduce that* $\operatorname{int}(A)$ *is convex.*

Assume, additionally, that $\operatorname{int}(A) \neq \emptyset$*. Then*

(b) $\overline{\operatorname{int}(A)} = \overline{A}$ *and* $\operatorname{int}(\overline{A}) = \operatorname{int}(A)$*.*

(c) $\partial A = \partial(\overline{A}) = \partial(\operatorname{int}(A))$*.*

Exercise 2.5. *Define* $f : \mathbb{C} \to \mathbb{C}$ *by* $f = g + ih$*; where*

$$g(z) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}, \quad h(z) = x^2 y^3 \quad \text{for all } z = x + iy \in \mathbb{C}.$$

Determine at which points f *is continuous.*

Exercise 2.6. *Let* $A \subset \mathbb{C}$ *be a subset, let* $z \in A$*,* $w \in \mathbb{C} \setminus A$*, and* $\varphi : [0, 1] \to \mathbb{C}$ *a continuous function with* $\varphi(0) = z$ *and* $\varphi(1) = w$*. Show that there exists* $t \in [0, 1]$ *with* $\varphi(t) \in \partial A$*.*

Exercise 2.7. *Let* $\Omega \subset \mathbb{C}$ *be open and connected, and* $f : \Omega \to \mathbb{C}$ *continuous so that* $|f(z)^2 - 1| < 1$ *for all* $z \in \Omega$*. Show that either* $\operatorname{Re}(f(z)) > 0$ *for all* $z \in \Omega$ *or* $\operatorname{Re}(f(z)) < 0$ *for all* $z \in \Omega$*.*

Hint: Observe the behavior of $\mathrm{Re}(f)$, *and argue by contradiction. Also, take into account that* $\mathrm{Re}(f)(\Omega)$ *is a connected set.*

Exercise 2.8. *Show that:*

(a) The function $f(z) = |z|^2$ *is differentiable only at* $z = 0$, *with* $f'(0) = 0$.

(b) The function $g(z) = |z|$ *is not differentiable at any* $z \in \mathbb{C}$.

Hint: Use Theorem 2.35.

Exercise 2.9. *Let* $\Omega \subset \mathbb{C}$ *open and* $f : \Omega \to \mathbb{C}$ *differentiable at* $z_0 \in \Omega$ *with* $f'(z_0) \neq 0$. *Prove that there exists* $r > 0$ *so that* $f(z) \neq f(z_0)$ *for all* $z \in D(z_0, r)$.

Exercise 2.10. *Let* $f : \mathbb{C} \to \mathbb{C}$ *the function defined by*

$$f(x + iy) = x^2 + 2y + i(x^2 + y^2), \quad \textit{for all} \quad x + iy \in \mathbb{C}.$$

Determine at which $z \in \mathbb{C}$ *the function* f *is differentiable.*

Exercise 2.11. *Let* $f : \mathbb{C} \to \mathbb{C}$ *be holomorphic so that* $f(0) = i$ *and the real part* $u = \mathrm{Re}(f)$ *is*

$$u(x + iy) = 2x^3y - 2xy^3 + x^2 - y^2 \quad \textit{for all} \quad x + iy \in \mathbb{C}.$$

Find $v = \mathrm{Im}(f)$.

Hint: The Cauchy-Riemann Equations determine the partial derivatives of v. *Then somehow integrate those partial derivatives.*

Exercise 2.12. *Let* $f : \mathbb{C} \to \mathbb{C}$ *differentiable at* z_0 *with* $f'(z_0) \neq 0$. *Prove that* $\overline{f} := \mathrm{Re}(f) - i\,\mathrm{Im}(f)$ *is not differentiable at* z_0.

Suggestion: Suppose that $\overline{f}$ *is differentiable at* z_0. *Arrive at a contradiction.*

Exercise 2.13. *Let* $\Omega \subset \mathbb{C}$ *be open and connected, and* $f : \Omega \to \mathbb{C}$ *holomorphic. Show that* f *is constant in* Ω *in each of the following situations:*

(a) $f(\Omega) \subset \mathbb{R}$ *(*f *takes only real values) or* $f(\Omega) \subset i\mathbb{R}$ *(*f *takes only pure imaginary values).*

(b) $\mathrm{Re}(f) : \Omega \to \mathbb{R}$ *or* $\mathrm{Im}(f) : \Omega \to \mathbb{R}$ *is constant in* Ω.

(c) $\overline{f} := \mathrm{Re}(f) - i\,\mathrm{Im}(f)$ *is holomorphic in* Ω.

(d) The modulus of $|f|$ *is constant in* Ω.

(e) The principal argument of f, $\Omega \ni z \mapsto \mathrm{Arg}(f)(z) := \mathrm{Arg}(f(z))$ *is constant in* Ω. *Here, we additionally assume that* $f(z) \neq 0$ *for all* $z \in \Omega$.

Suggestion: Look at Corollary 2.37. For part (d) use part (c). In part (e), write $f(z)$ *in exponential form and look at (a).*

Exercise 2.14. *Show that if* $f : \Omega \to \mathbb{R}$ *is differentiable at* $z_0 = x_0 + iy_0 \in \Omega$, *then*

$$|f'(z_0)| = \|\nabla\,\mathrm{Re}(f)(x_0, y_0)\| = \|\nabla\,\mathrm{Im}(f)(x_0, y_0)\|;$$

where $\|\cdot\|$ *is the Euclidean norm in* $\mathbb{R}^2$; *that is,* $\|(a, b)\| = \sqrt{a^2 + b^2}$.

Exercise 2.15. *Let* $f : \mathbb{R}^2 \setminus \{(0,0) \to \mathbb{R}^2$ *be real-differentiable with* $f = (u, v)$, *and define* $\widetilde{u}(r, \theta) = u(r\cos\theta, r\sin\theta)$ *and* $\widetilde{v}(r, \theta) = v(r\cos\theta, r\sin\theta)$ *for* $r > 0$, $\theta \in \mathbb{R}$. *Show that the Cauchy-Riemann equations for* u *and* v *are equivalent to*

$$\frac{\partial \widetilde{u}}{\partial r} = \frac{1}{r}\frac{\partial \widetilde{v}}{\partial \theta}, \qquad \frac{\partial \widetilde{u}}{\partial \theta} = -r\frac{\partial \widetilde{v}}{\partial r}.$$

Exercise 2.16. *For every complex function $f = u + iv$, we define the differential operators*

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) f, \quad \frac{\partial f}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) f.$$

Verify that the Cauchy-Riemann equations for u, v are equivalent to

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

Use this to deduce that if $\Omega \subset \mathbb{C}$ is open and $f : \Omega \to \mathbb{C}$, then f is holomorphic if and only if f is real-differentiable and $\frac{\partial f}{\partial \overline{z}} = 0$ in Ω. Moreover, in such case, we have $f'(z) = \frac{\partial f}{\partial z}(z)$ for all $z \in \Omega$.

Exercise 2.17. *Let $f : D(0, 2) \to \mathbb{C}$ be holomorphic with f' continuous in $D(0, 2)$. Suppose that f is injective in $\overline{D}(0, 1)$ and that $f'(z) \neq 0$ for all $z \in \overline{D}(0, 1)$. Prove that there exists $\varepsilon > 0$ so that f is injective in $D(0, 1 + \varepsilon)$.*

Suggestion: Suppose that f is not injective in any disk $D(0, 1 + \frac{1}{n})$ with $n \in \mathbb{N}$, and use the Bolzano-Weierstrass Theorem 2.12 to derive a contradiction. The Inverse Function Theorem 2.39 plays a role here too.

Exercise 2.18. *For each $a > 0$, denote $\Omega = \{z \in \mathbb{C} : -a < \mathrm{Im}(z) < a\}$ and $\Omega' = \{w \in \mathbb{C} : \mathrm{Re}(w) > 0\}$. Find a conformal map $f : \mathbb{C} \to \mathbb{C}$ in $\mathbb{C}$ such that $f(\Omega) = \Omega'$.*

Suggestion: Start with the case $a = \pi/2$, and look at Theorem 2.43 to see how f' should be.

Exercise 2.19. *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be the function $u(x, y) = xy$. Prove that u is harmonic in $\mathbb{R}^2$ and find a harmonic conjugate v of u with $v(0, 0) = 0$.*

Exercise 2.20. *Let $\Omega \subset \mathbb{C}$ be open and connected and $u : \Omega \to \mathbb{R}$ be harmonic. Show that if v and $\widetilde{v}$ are two harmonic conjugates of u in Ω, then $v - \widetilde{v}$ is a constant function.*

Exercise 2.21. *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a harmonic function. Show that the function $f := \frac{\partial u}{\partial y} + i\frac{\partial u}{\partial x}$ is holomorphic in $\mathbb{C}$.*

Exercise 2.22. *Define $u : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ by $u(z) = \log |z|$. Prove that u is (real) harmonic in $\mathbb{C} \setminus \{0\}$, and that u has no harmonic conjugate in $\mathbb{C} \setminus \{0\}$.*

Suggestion: Write $u(x, y)$ in a simple form. Suppose that $v : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ is a harmonic conjugate of u. The Cauchy-Riemann equations for $u + iv$ will lead you to a contradiction.

Exercise 2.23. *Give an example of $f : \mathbb{C} \to \mathbb{C}$ holomorphic, and points $z, w \in \mathbb{C}$ with*

$$f(z) - f(w) \neq f'(\xi)(z - w) \quad \text{for all} \quad \xi \in [z, w].$$

Here $[z, w]$ denotes the segment line joining z and w. This shows that the real mean value theorem does not extend to complex differentiable functions.

Exercise 2.24. *Let $f : \mathbb{C} \to \mathbb{C}$ be defined by*

$$f(z) = \begin{cases} e^{-1/z^4} & \text{if} \quad z \in \mathbb{C} \setminus \{0\} \\ 0 & \text{if} \quad z = 0. \end{cases}$$

Prove that $f \in \mathcal{H}(\mathbb{C} \setminus \{0\})$, that the partial derivatives $\frac{\partial \,\mathrm{Re}(f)}{\partial x}, \frac{\partial \,\mathrm{Re}(f)}{\partial y}, \frac{\partial \,\mathrm{Im}(f)}{\partial x}, \frac{\partial \,\mathrm{Im}(f)}{\partial y}$ exist and satisfy the Cauchy-Riemann equations at $z_0 = 0$, and that $\lim_{z \to 0} f(z)$ does not even exist. In particular, f is not continuous at $z_0 = 0$.

Suggestion: It is not necessary to compute the partial derivatives of $\mathrm{Re}(f), \mathrm{Im}(f)$ at all points. Just remember the definition of partial derivatives (2.2.2)–(2.2.3) (at the point $(0, 0)$).

Exercise 2.25. *Verify that* $\overline{e^z} = e^{\overline{z}}$ *for all* $z \in \mathbb{C}$.

Exercise 2.26. *Find* $\max\{|e^{z^2}| \,:\, |z| \le 1\}$.

Exercise 2.27. *Give an example of two numbers* $z, w \in \mathbb{C} \setminus \{0\}$ *such that* $(e^z)^w \neq e^{zw}$. *By* $(e^z)^w$ *we understand the Principal* w*-power of* e^z.

Exercise 2.28. *Construct the following branches associated with the square root.*

(a) $f : \mathbb{C} \setminus (-\infty, -1] \to \mathbb{C}$ *holomorphic with* $(f(z))^2 = z + 1$ *for all* $z \in \mathbb{C} \setminus (-\infty, -1]$. *We can refer to* f *as a holomorphic square root of* $\sqrt{z+1}$ *in* $\mathbb{C} \setminus (-\infty, -1]$.

(b) $g : \mathbb{C} \setminus [1, +\infty) \to \mathbb{C}$ *holomorphic with* $(g(z))^2 = z - 1$ *for all* $z \in \mathbb{C} \setminus [1, +\infty)$. *We can refer to* g *as a holomorphic square root of* $\sqrt{z-1}$ *in* $\mathbb{C} \setminus [1, +\infty)$.

(c) For $\Omega := \mathbb{C} \setminus \{z \in \mathbb{C} \,:\, |\operatorname{Re}(z)| \ge 1\}$, $h : \Omega \to \mathbb{C}$ *holomorphic with* $(h(z))^2 = z^2 - 1$ *for all* $z \in \Omega$. *We can refer to* h *as a holomorphic square root of* $\sqrt{z^2-1}$ *in* Ω.

Exercise 2.29. *Let* $z, w \in \mathbb{C}$ *with* $z \neq 0$. *Prove the following.*

(a) If $\langle z^w \rangle$ *has exactly one element if and only if* $w \in \mathbb{Z}$.

(b) If $w \in \mathbb{Q}$ *and* $w = p/q$ *with* $p, q \in \mathbb{Z}$, $q > 0$, *and* $\gcd(p, q) = 1$, *then* $\langle z^w \rangle$ *has exactly* q *elements.*[3]

(c) If $w \in \mathbb{R} \setminus \mathbb{Q}$, *then* $\langle z^w \rangle$ *contains infinitely many different numbers.*

Exercise 2.30. *Show that* $\operatorname{Log}(1+i)^2 = 2\operatorname{Log}(1+i)$ *and* $\operatorname{Log}(-1+i)^2 \neq 2\operatorname{Log}(-1+i)$.

Exercise 2.31. *Given* $w \in \mathbb{C}$, *show that the set of all the solutions* $z \in \mathbb{C}$ *of the equation* $\sin z = w$ *is* $\{-i\xi \,:\, \xi \in \langle \log(iw + \varphi) \rangle,\ \varphi \in \langle \sqrt{1-w^2} \rangle\}$. *In particular, write down all the solutions* $z \in \mathbb{C}$ *of the equation* $\sin z = 2$.

Exercise 2.32. *Prove the following trigonometric-hyperbolic identities, for* $z = x + iy \in \mathbb{C}$:

(a) $(\cosh z)^2 - (\sinh z)^2 = 1$.

(b) $\cos z = \cos x \cosh y - i \sin x \sinh y$.

(c) $\sin z = \sin x \cosh y + i \cos x \sinh y$.

(d) $|\cos z|^2 = (\cos x)^2 + (\sinh y)^2$.

(e) $|\sin z|^2 = (\sin x)^2 + (\sinh y)^2$.

Exercise 2.33. *Compute the following sets and/or numbers.*

(a) The real and imaginary parts of e^{3-i}.

(b) The real and imaginary parts of $\cos(2 + 3i)$.

(c) $\langle \log(-1 + \sqrt{3}\, i) \rangle$ *and* $\operatorname{Log}(-1 + \sqrt{3}\, i)$.

(d) $\langle (-1)^i \rangle$ *and* $(-1)^i$.

(e) $\langle (1+i)^{(1+i)} \rangle$ *and* $(1+i)^{1+i}$.

[3] Here $\gcd(p, q)$ is the greatest common divisor of p and q, meaning the largest $d \in \mathbb{N}$ dividing both p and q.

Chapter 3

Series of complex functions

3.1 Series of complex numbers

Naturally, a series of complex numbers (or a complex series) is an expression of the form

$$z_1 + z_2 + \cdots + z_n + \cdots \quad \text{or} \quad \sum_{n=1}^{\infty} z_n.$$

The partial sums of $\sum_{n=1}^{\infty} z_n$ is the sequence $\{S_n\}_{n=1}^{\infty}$ given by $S_n = \sum_{n=1}^{\infty} z_n$, the sum of the first n terms of the series.

3.1.1 Convergence and absolute convergence

The convergence of series in $\mathbb{C}$ is defined exactly as in the real line.

Definition 3.1. *We say that a series $\sum_{n=1}^{\infty} z_n$ of complex numbers* ***converges to*** *$z_0 \in \mathbb{C}$ if the sequence of partial sums $\{\sum_{k=1}^{n} z_k\}_{n\in\mathbb{N}}$ converges to z_0, in which case we will denote $\sum_{n=1}^{\infty} z_n = z_0$. And if the limit of the partial sums does not exist, we say that the series $\sum_{n=1}^{\infty} z_n$* ***diverges.***

Moreover, we say that the series $\sum_{n=1}^{\infty} z_n$ is ***absolutely convergent*** *if the series of the modulus $\sum_{n=1}^{\infty} |z_n|$ is convergent.*

If a series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent, then it is convergent as well. Indeed, the Cauchy partial sums of $\sum_{n=1}^{\infty} z_n$ satisfy

$$\left|\sum_{n=1}^{M} z_n - \sum_{n=1}^{N} z_n\right| = \left|\sum_{n=N+1}^{M} z_n\right| \leq \sum_{n=N+1}^{M} |z_n| = \sum_{n=1}^{M} |z_n| - \sum_{n=1}^{N} |z_n|,$$

for $M > N \geq 1$, and the last term converges to 0 as $M, N \to \infty$. This is due to the fact that the sequence of partial sums of $\sum_{n=1}^{\infty} |z_n|$ converges, and thus have the Cauchy property. We have shown that the partial sums of $\sum_{n=1}^{\infty} z_n$ have the Cauchy property, and since $\mathbb{C}$ is complete (see Exercise 2.2), these partial sums converges in $\mathbb{C}$.

Needless to say, there are convergent series that are not absolutely convergent. For example, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \log 2$, whereas $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

Proposition 3.2. *Let $\{z_n\}_n \subset \mathbb{C}$ be a sequence with $\sum_{n=1}^{\infty} z_n$ convergent. The following hold.*

(i) $\lim\limits_{n\to\infty} z_n = 0$.

(ii) $\lim\limits_{N\to\infty} \sum_{n=N}^{\infty} z_n = 0$.

Proof.

(i) The series $\sum_{n=1}^{\infty} z_n$ converges to some $S \in \mathbb{C}$, and so the partial sums $S_N = \sum_{k=1}^{N} z_k$ satisfy $\lim_{N\to\infty} S_N = S$. Therefore

$$|z_n| = \left|\sum_{k=1}^{n} z_k - \sum_{k=1}^{n-1} z_k\right| = |S_n - S_{n-1}| \to |S - S| = 0 \text{ as } n \to \infty.$$

(ii) Let $S = \sum_{n=1}^{\infty} z_n$, and denote $S_m = \sum_{n=1}^{m} z_n$ for every m. For each $N \in \mathbb{N}$, $\sum_{n=N}^{\infty} z_n$ is (by Definition 3.1) the limit of the sequence $\sum_{n=N}^{M} z_n$ as $M \to \infty$. Thus, for each fixed N, we have

$$\sum_{n=N}^{\infty} z_n = \lim_{M\to\infty} \sum_{n=N}^{M} z_n = \lim_{M\to\infty} (S_M - S_{N-1}) = S - S_{N-1}.$$

And now, we let $N \to \infty$, obtaining

$$\lim_{N\to\infty} \sum_{n=N}^{\infty} z_n = \lim_{N\to\infty} (S - S_{N-1}) = S - S = 0.$$

□

An elementary example to study convergence or divergence is the *geometric series*.

Example 3.3 (Geometric Series)**.** For every $z \in D(0,1)$, that is $|z| < 1$, using the formula from Exercise 1.10, we see that the *geometric series* $\sum_{n=0}^{\infty} z^n$ is absolutely convergent and

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad \sum_{n=0}^{\infty} |z^n| = \frac{1}{1-|z|}. \tag{3.1.1}$$

Note that (3.1.1) also implies

$$\sum_{n=0}^{\infty} (-1)^n z^n = \frac{1}{1+z}, \quad |z| < 1.$$

However, when $|z| \geq 1$, the series $\sum_{n=0}^{\infty} z^n$ diverges by Proposition 3.2, as $\lim_{n\to\infty} |z^n| \neq 0$.

In the study of absolute convergence for complex series we can apply some of the convergence criteria that we already know for real numbers. Some of them are recorded in the following proposition, without proofs.

Proposition 3.4 (Convergence criteria)**.** *The following statements hold.*

(i) [Cauchy's Root Test] *Let* $\{a_n\}_n \subset \mathbb{R}$. *Then*

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1 \implies \sum_{n=1}^{\infty} |a_n| \text{ converges,}$$

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1 \implies \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

(ii) [D'Alembert's Ratio Test] *Let* $\{a_n\}_n \subset \mathbb{R} \setminus \{0\}$. *Then*

$$\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1 \implies \sum_{n=1}^{\infty} |a_n| \text{ converges,}$$

$$\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1 \implies \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

(iii) [Raabe's Test] *Let* $\{a_n\}_n \subset \mathbb{R} \setminus \{0\}$. *Then*

$$\liminf_{n\to\infty} n\left(1 - \frac{|a_{n+1}|}{|a_n|}\right) > 1 \implies \sum_{n=1}^{\infty} |a_n| \text{ converges,}$$
$$\limsup_{n\to\infty} n\left(1 - \frac{|a_{n+1}|}{|a_n|}\right) < 1 \implies \sum_{n=1}^{\infty} |a_n| \text{ diverges.}$$

(iv) [Cauchy's Condensation Test] *Let* $\{a_n\}_n \subset [0, +\infty)$ *be a non-increasing sequence. Then*

$$\sum_{n=1}^{\infty} a_n \ \text{ converges} \iff \sum_{n=1}^{\infty} 2^n a_{2^n} \ \text{ converges.}$$

(v) [Integral Test] *Let* $f : [0, \infty) \to [0, \infty)$ *be non-increasing. Then the series* $\sum_{n=1}^{\infty} f(n)$ *converges if and only if* $\int_1^{\infty} f(x)\,dx < \infty$. *Moreover, in such case, we have the bounds*

$$\int_1^{\infty} f(x)\,dx \le \sum_{n=1}^{\infty} f(n) \le f(1) + \int_1^{\infty} f(x)\,dx.$$

3.1.2 Operations with series. The Cauchy Product

Clearly we can multiply the terms $\{a_n\}_n \subset \mathbb{C}$ of a convergent series $\sum_{n=1}^{\infty} a_n$ by an scalar $\lambda \in \mathbb{C}$, and obtain a new convergent series $\sum_{n=1}^{\infty} \lambda a_n$. It is also easy to define the sum of two convergent series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ as the series obtaining by summing termwise: $\sum_{n=1}^{\infty}(a_n + b_n)$, which is a new convergent series so that $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$. However, the product of two series is a bit more complicated, and we define it through the *Cauchy product.*

We will use the notation $\mathbb{N}^* := \mathbb{N} \cup \{0\}$ in the sequel.

Definition 3.5 (Cauchy Product)**.** *Let* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *be two series of complex numbers. Define the complex numbers*

$$c_n := a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k} \quad \text{for all} \quad n \in \mathbb{N}^*. \tag{3.1.2}$$

The ***Cauchy Product*** *of* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *is the series* $\sum_{n=0}^{\infty} c_n$.

The Cauchy product of two convergent series is not necessarily convergent, even if the two series are the same; see Exercise 3.1. To guarantee the convergence of the Cauchy product, at least one of the series should be absolutely convergent.

Proposition 3.6. *Let* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *be two convergent series with at least one of them absolutely convergent. Then their Cauchy product* $\sum_{n=0}^{\infty} c_n$ *is convergent and*

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right).$$

Moreover, if both $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *are absolutely convergent, then* $\sum_{n=0}^{\infty} c_n$ *is absolutely convergent as well.*

Proof. Assume for instance that $\sum_{n=0}^{\infty} a_n$ is absolutely convergent. Define $s := \sum_{n=0}^{\infty} a_n \in \mathbb{C}$, $s_N := \sum_{n=0}^{N} a_n$, $r := \sum_{n=0}^{\infty} b_n \in \mathbb{C}$, $r_N := \sum_{n=0}^{N} b_n$, $t_N := \sum_{n=0}^{\infty} c_n$, for all $n \in \mathbb{N}^*$. Observe that we have, for all $n \in \mathbb{N}$,

$$t_N = \sum_{n=0}^{N}\sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{N} a_k \sum_{n=k}^{N} b_{n-k} = \sum_{k=0}^{N} a_k \sum_{n=0}^{N-k} b_n = \sum_{k=0}^{N} a_k\, r_{N-k} = s_N r + \sum_{k=0}^{N} a_k (r_{N-k} - r).$$

Because $s = \lim_{N\to\infty} s_N$, for the convergence of $\{t_N\}_N$ (to $s\cdot r$) it only remains to check that $\lim_{N\to\infty} \sum_{k=0}^{N} a_k(r_{N-k}-r) = 0$. Given $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that $|r_N - r| \le \varepsilon/(1+\sum_{n=0}^{\infty}|a_n|)$ for all $N \ge N_0$. So, for those $N > N_0$, we can write

$$\sum_{k=0}^{N} a_k(r_{N-k}-r) = \sum_{k=N-N_0+1}^{N} a_k(r_{N-k}-r) + \sum_{k=0}^{N-N_0} a_k(r_{N-k}-r) = \sum_{j=0}^{N_0-1} a_{N-j}(r_j - r) + \sum_{k=0}^{N-N_0} a_k(r_{N-k}-r).$$

The term $\sum_{j=0}^{N_0-1} a_{N-j}(r_j - r)$ is a sum of N_0-many sequences converging to 0 as $N\to\infty$, since $|a_N| \to 0$. Hence, taking limit superior in the above gives

$$\begin{aligned}\limsup_{N\to\infty}\left|\sum_{k=0}^{N} a_k(r_{N-k}-r)\right| &\le \limsup_{N\to\infty}\left|\sum_{k=0}^{N-N_0} a_k(r_{N-k}-r)\right| \le \limsup_{N\to\infty}\sum_{k=0}^{N-N_0}|a_k|\,|r_{N-k}-r| \\ &\le \frac{\varepsilon}{1+\sum_{n=0}^{\infty}|a_n|}\limsup_{N\to\infty}\sum_{k=0}^{N-N_0}|a_k| = \frac{\varepsilon}{1+\sum_{n=0}^{\infty}|a_n|}\sum_{n=0}^{\infty}|a_n| \le \varepsilon.\end{aligned}$$

Because $\varepsilon > 0$ is arbitrary, we get $\lim_{N\to\infty}\sum_{k=0}^{N} a_k(r_{N-k}-r) = 0$, as desired.

Assume now that both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent. Then, for each $N\in\mathbb{N}$ we have the bounds

$$\sum_{n=0}^{N}|c_n| = \sum_{n=0}^{N}\left|\sum_{k=0}^{n} a_k b_{n-k}\right| \le \sum_{k=0}^{N}|a_k|\sum_{n=k}^{N}|b_{n-k}| = \sum_{k=0}^{N}|a_k|\sum_{n=0}^{N-k}|b_n| \le \sum_{k=0}^{\infty}|a_k|\sum_{n=0}^{\infty}|b_n|;$$

where the last term is finite and independent of N. This shows that $\sum_{n=0}^{\infty}|c_n| < \infty$.

□

3.2 Sequences and series of functions

Definition 3.7 (Convergence of functions)**.** *Let $A\subset\mathbb{C}$ and $\{f_n : A\to\mathbb{C}\}_n$ a sequence of functions defined in A.*

- *We say that $\{f_n\}_n$ **converges pointwise** on A if for every $z\in A$, the sequence of complex numbers $\{f_n(z)\}_n$ converges in $\mathbb{C}$. This means that for every $z\in A$, there exists $f(z)\in\mathbb{C}$ such that for all $\varepsilon>0$ there exists $n_0 = n_0(z,\varepsilon)\in\mathbb{N}$ with $|f_n(z)-f(z)|<\varepsilon$ for all $n\ge n_0$.*
- *We say that $\{f_n\}_n$ **converges uniformly** on A to $f : A\to\mathbb{C}$ if for every $\varepsilon>0$ there exists $n_0 = n_0(\varepsilon)\in\mathbb{N}$ so that*

$$|f_n(z)-f(z)|<\varepsilon \quad \textit{for all} \quad n\ge n_0 \quad \textit{and all} \quad z\in A.$$

- *We say that $\{f_n\}_n$ is **Cauchy uniformly** on A if for every $\varepsilon>0$ there exists $n_0 = n_0(\varepsilon)\in\mathbb{N}$ so that*

$$|f_m(z)-f_n(z)|<\varepsilon \quad \textit{for all} \quad n,m\ge n_0 \quad \textit{and all} \quad z\in A.$$

Now we consider series of functions $\sum_{n=1}^{\infty} f_n$ defined in A.

- *We say that $\sum_{n=1}^{\infty} f_n$ converges **pointwise** on A if for each $z\in A$, the numerical series $\sum_{n=1}^{\infty} f_n(z)$ converges.*
- *We say that $\sum_{n=1}^{\infty} f_n$ converges **uniformly** on A if the sequence of functions given by the partial sums $\{S_n(z) = \sum_{k=1}^{n} f_k(z)\}_{n\in\mathbb{N}}$ is uniformly convergent on A.*
- *We say that $\sum_{n=1}^{\infty} f_n$ converges **absolutely** on A if the series of functions $\sum_{n=1}^{\infty}|f_n|$ is convergent.*

- *We say that $\sum_{n=1}^{\infty} f_n$ converges* ***absolutely–uniformly*** *on A if the series of functions $\sum_{n=1}^{\infty} |f_n|$ is uniformly convergent.*

It is important to be able to distinguish between these various notions of convergence.

Remark 3.8. Concerning Definition 3.7, we observe the following.

(1) *The difference between pointwise and uniform convergence is whether or not the index n_0 depends on the points $z \in A$.*

Notice that the uniform convergence $\{g_n\}_n \to g$ on A can be reformulated as

$$\lim_{n\to\infty} \left(\sup_{z\in A} |g_n(z) - g(z)| \right) = 0.$$

Uniform convergence is way stronger that pointwise convergence. For example, the uniform limit of continuous functions is continuous, as shown below by Proposition 3.10. However, this is not guaranteed with pointwise convergence, e.g., $f_n(x) = x^n$ on $x \in [0, 1]$.

(2) *A sequence of functions $\{g_n : A \to \mathbb{C}\}_n$ is Cauchy uniformly on A if and only if it is uniformly convergent on A.*

Proof. If $\{g_n\}_n$ is Cauchy uniformly on A, given $\varepsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that $|g_n(z) - g_m(z)| < \varepsilon/2$ whenever $n, m \geq n_0$ and $z \in A$. In particular, for each $z \in A$, the numerical sequence $\{g_n(z)\}_n$ is Cauchy and so there exists $g(z) \in \mathbb{C}$ with $g_n(z) \to g(z)$ as $n \to \infty$. Thus, for every $z \in A$ we can find $m_z \in \mathbb{N}$, $m_z \geq n_0$ such that $|g_{m_z}(z) - g(z)| \leq \varepsilon/2$. This implies

$$|g_n(z) - g(z)| \leq |g_n(z) - g_{m_z}(z)| + |g_{m_z}(z) - g(z)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and since n_0 is the same for all z, we conclude that g_n is uniformly convergent (to g) on A.

The converse is very easy to prove.

□

(3) *For series of functions $\sum_{n=1}^{\infty} f_n$, the convergence absolutely–uniformly implies (simultaneously) uniform, absolute, and pointwise convergence.*

To justify this, let $\sum_{n=1}^{\infty} f_n$ be absolutely–uniformly convergent on $A \subset \mathbb{C}$. Obviously this implies absolute convergence (simply by Definition 3.7). Now, $\sum_{n=1}^{\infty} f_n$ converging absolutely–uniformly means that $\sum_{n=1}^{\infty} |f_n|$ is uniformly convergent in A. Thanks to point (2) of the present remark, we can argue as in the comment after Definition 3.1 to deduce that also $\sum_{n=1}^{\infty} f_n$ converges uniformly in A. Indeed, for each $M, N \in \mathbb{N}$, $M > N$, one has

$$\sup_{z\in A} \left| \sum_{n=1}^{M} f_n(z) - \sum_{n=1}^{N} f_n(z) \right| \leq \sup_{z\in A} \sum_{n=N+1}^{m} |f_n(z)| = \sup_{z\in A} \left| \sum_{n=1}^{M} |f_n(z)| - \sum_{n=1}^{N} |f_n(z)| \right|.$$

But the sequence of partial sums of the series of functions $\sum_{n=1}^{\infty} |f_n|$ is Cauchy uniformly, so the last term goes to 0 as $M, N \to \infty$. This is then telling us that the sequence of partial sums of the series $\sum_{n=1}^{\infty} f_n$ is Cauchy uniformly on A, which, again by point (2), implies that $\sum_{n=1}^{\infty} |f_n|$ converges uniformly.

(4) *A series can converge absolutely and uniformly on a set A, and yet not absolutely–uniformly on A.* See Exercise 3.9.

3.2.1 The Weierstrass M-test. Continuity

One of the most useful results for convergence of functions is the Weierstrass M-test, which we will use systematically in this chapter.

Theorem 3.9 (Weierstrass M-test)**.** *Let $A \subset \mathbb{C}$, and a sequence of functions $f_n : A \to \mathbb{C}$ such that for every $n \in \mathbb{N}$ there exists $M_n > 0$ with $|f_n(z)| \le M_n$ for all $z \in A$, and so that $\sum_{n=1}^{\infty} M_n$ is finite. Then the series $\sum_{n=1}^{\infty} f_n$ converges absolutely–uniformly on A.*

Proof. Let $\varepsilon > 0$. Because $\sum_{n=1}^{\infty} M_n$ is convergent, by Proposition 3.2, there is $n_0 \in \mathbb{N}$ so that $\sum_{n=n_0}^{\infty} M_n < \varepsilon$. If $M > N \ge n_0$ we have

$$\sup_{z\in A}\left|\sum_{n=1}^{M}|f_n(z)| - \sum_{n=1}^{N}|f_n(z)|\right| = \sup_{z\in A}\sum_{n=N+1}^{M}|f_k(z)| \le \sup_{z\in A}\sum_{n=n_0}^{\infty}|f_n(z)| \le \sup_{z\in A}\sum_{n=n_0}^{\infty} M_n < \varepsilon.$$

Thus $\sum_{n=1}^{\infty}|f_n|$ is Cauchy uniformly on A, and thus Remark 3.8(2) says that $\sum_{n=1}^{\infty} f_n$ converges absolutely–uniformly on A. □

We finish this section showing that the uniform limit of continuous functions is continuous.

Proposition 3.10. *Let $A \subset \mathbb{C}$ and $\{f_n : A \to \mathbb{C}\}_n$ a sequence of continuous functions in A that converges uniformly on A to some $f : A \to \mathbb{C}$. Then f is continuous in A as well.*

Proof. Fix $z_0 \in A$ and let us check the continuity of f at z_0. Given $\varepsilon > 0$, by the uniform convergence of $\{f_n\}_n$ on A, we can find $N \in \mathbb{N}$ so that

$$\sup_{z\in A}|f_N(z) - f(z)| < \frac{\varepsilon}{3}. \tag{3.2.1}$$

This function f_N is continuous at z_0, so we can find $\delta > 0$ for which

$$|f_N(z) - f_N(z_0)| < \frac{\varepsilon}{3} \quad \text{for all} \quad z \in A \cap D(z_0,\delta). \tag{3.2.2}$$

Using (3.2.1) and (3.2.2) and the triangle inequality we obtain, for $z \in A \cap D(z_0,\delta)$,

$$\begin{aligned}|f(z) - f(z_0)| &\le |f(z) - f_N(z)| + |f_N(z) - f_N(z_0)| + |f_N(z_0) - f(z_0)| \\ &\le 2\sup_{w\in A}|f_N(w) - f(w)| + |f_N(z) - f_N(z_0)| < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

□

3.3 Power series

The main type of series of functions we will be studying is the power series. One of the main goals of the course is to show that every holomorphic function can be written as one of these series.

Recall that we are using the notation $\mathbb{N}^* := \mathbb{N} \cup \{0\}$.

Definition 3.11 (Power series)**.** *A **power series** is a series of functions of the form $\sum_{n=0}^{\infty} a_n(z - z_0)^n$, with $z_0, a_n \in \mathbb{C}$ for all $n \in \mathbb{N}^*$. We then say that z_0 is the center of series.*

3.3.1 The Radius and Disk of Convergence. Abel's Lemma

Definition 3.12 (Radius of Convergence). *Given a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, we define its **radius of convergence** $R \in [0,\infty]$ as*

$$R := \sup\left\{ r \geq 0 \,:\, \sum_{n=0}^{\infty} a_n r^n \text{ converges} \right\}. \tag{3.3.1}$$

*Then, the **disk of convergence** of the series is $D(z_0, R)$. In the case $R = \infty$, by $D(z_0, R)$ we mean the whole complex plane $\mathbb{C}$.*

Remark 3.13. A couple of preliminary observations are in order.

(1) If the numerical series $\sum_{n=0}^{\infty} a_n r^n$ converges for some $r > 0$, then $\sum_{n=0}^{\infty} |a_n| s^n$ converges for every $0 < s < r$.

Indeed, the convergence of the first series implies that $\lim_{n\to\infty} a_n r^n = 0$ by Proposition 3.2. In particular, there exists $C > 0$ with $|a_n| r^n \leq C$ for all $n \in \mathbb{N}^*$. Thus

$$\sum_{n=0}^{\infty} |a_n| s^n = \sum_{n=0}^{\infty} |a_n| r^n \left(\frac{s}{r}\right)^n \leq C \sum_{n=0}^{\infty} \left(\frac{s}{r}\right)^n = C\frac{r}{r-s} < \infty.$$

(2) The radius of convergence $R \in [0,\infty]$ of a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is also

$$R = \sup\left\{ r \geq 0 \,:\, \sum_{n=0}^{\infty} |a_n| r^n \text{ converges} \right\}. \tag{3.3.2}$$

To see this, denote by S the supremum in the right hand side of (3.3.2). Since absolute convergence implies convergence, we clearly have $S \leq R$. To show the reverse inequality, suppose that $S < R$ and let $\varepsilon > 0$ be so that $S < S + \varepsilon < R$ and $\sum_{n=0}^{\infty} a_n (S+\varepsilon)^n$ converges. The existence of such an ε is guaranteed by the definition of R (3.3.1). By (1) of the present remark, we get that $\sum_{n=0}^{\infty} |a_n| (S + \frac{\varepsilon}{2})^n$ converges, contradicting the definition of S.

Combining the idea of the proof of Remark 3.13(1) with Theorem 3.9, we obtain the following criterion for convergence of power series due to Abel.

Theorem 3.14 (Abel's Lemma). *Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series and assume there exists $z_1 \in \mathbb{C}$ so that* $\sup\{|a_n(z_1-z_0)^n| \,:\, n \in \mathbb{N}^*\} < \infty$. *Then the series of functions $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely–uniformly on each disk $\overline{D}(z_0, r)$ with $0 < r < |z_0 - z_1|$.*

Proof. Define $M := \sup\{|a_n(z_1-z_0)^n| \,:\, n \in \mathbb{N}^*\}$ and let $r > 0$ be so that $0 < r < |z_1 - z_0|$. The functions $\overline{D}(z_0, r) \ni z \mapsto a_n(z-z_0)^n$ satisfy

$$|a_n(z-z_0)^n| = |a_n(z_1-z_0)^n| \left|\frac{z-z_0}{z_1-z_0}\right|^n \leq |a_n(z_1-z_0)^n| \left(\frac{r}{|z_1-z_0|}\right)^n \leq M\left(\frac{r}{|z_1-z_0|}\right)^n.$$

Because $\sum_{n=0}^{\infty} M\left(\frac{r}{|z_1-z_0|}\right)^n < \infty$, Theorem 3.9 says that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely–uniformly on $\overline{D}(z_0, r)$. □

Observe that Theorem 3.14 provides a lower bound for the radius of convergence R of a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. Namely, if z_1 is as in Theorem 3.14, then in particular $\sum_{n=0}^{\infty} a_n r^n$ converges for all $0 \leq r < |z_0 - z_1|$, and so $R \geq |z_0 - z_1|$.

3.3.2 Convergence of Power Series: The Cauchy-Hadamard Theorem

In proper subdisks of the disk of convergence, the convergence of the power series is absolutely–uniform, and the series always diverges outside the disk of convergence. Moreover, there is a formula for the radius of convergence in terms of $\{a_n\}_n$. This is the content of the next theorem.

Theorem 3.15 (Cauchy-Hadamard Theorem)**.** *Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series with radius of convergence $R \in [0, \infty]$. The following is satisfied.*

(i) If $0 < r < R$, the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely-uniformly on $\overline{D}(z_0, r)$.

(ii) If $z \in D(z_0, R)$, the numerical series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely. We will express this by saying that the series converges (absolutely) pointwise in $D(z_0, R)$.

(iii) For all $z \in \mathbb{C}$ so that $|z-z_0| > R$, the numerical series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ diverges.[1]

(iv) The radius of convergence R is given by the formula

$$R = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}}. \tag{3.3.3}$$

In the case $\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|} = 0$, we have $R = \infty$; and if $\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, we have $R = 0$.

(v) If $a_n \neq 0$ for all n and $\lim\limits_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} \in [0, +\infty]$, then

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}. \tag{3.3.4}$$

Proof.

(i) If $0 < r < R$, then we can find $\varepsilon > 0$ so that $r < r+\varepsilon < R$ and $\sum_{n=0}^{\infty} a_n(r+\varepsilon)^n$ is convergent. Taking any $z_1 \in S(z_0, r+\varepsilon)$ so that $|z_1 - z_0| = r+\varepsilon$, then $\sup\{|a_n(z_1-z_0)^n| : n \in \mathbb{N}^*\} < \infty$, and so the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely–uniformly on $\overline{D}(z_0, s)$ for all $s < |z_1 - z_0| = r+\varepsilon$, which of course includes the disk $\overline{D}(z_0, r)$.

(ii) Assume $R > 0$ (otherwise there is nothing to prove), and $z \in D(z_0, R)$. Clearly we can find $0 < r < R$ with $z \in \overline{D}(z_0, r)$, e.g., taking $|z-z_0| < r < R$. By (i) there is absolute–uniform convergence of the power series in $\overline{D}(z_0, r)$, and in particular absolute convergence at $z = z_0$.

(iii) And if $|z_1 - z_0| > R$, suppose, for the sake of contradiction, that $\sum_{n=0}^{\infty} a_n(z_1-z_0)^n$ is convergent. Then $\sup\{|a_n(z_1-z_0)^n| : n \in \mathbb{N}^*\} < \infty$ and Theorem 3.14 says that the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges (absolutely–uniformly) on each disk $\overline{D}(z_0, r)$ with $0 < r < |z_1 - z_0|$. If $\varepsilon > 0$ is so that $|z_1 - z_0| > R+\varepsilon$, and we put $z = z_0 + (R+\varepsilon) \in \overline{D}(z_0, R+\varepsilon)$, the above gives the convergence of the numerical series $\sum_{n=0}^{\infty} a_n(R+\varepsilon)^n$, contradicting the definition of R; see (3.3.1).

(iv)[2] Consider first the case where $\limsup\limits_{n\to\infty} |a_n|^{1/n} \in (0, +\infty)$ and denote $r = \frac{1}{\limsup\limits_{n\to\infty} |a_n|^{1/n}}$. Recall that $\limsup\limits_{n\to\infty} |a_n|^{1/n} = \lim\limits_{n\to\infty} \sup\limits_{m\geq n} |a_m|^{1/m}$, and hence $r = \lim\limits_{n\to\infty} 1/\big(\sup\limits_{m\geq n} |a_m|^{1/m}\big)$. So, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ so that

$$r - \frac{\varepsilon}{2} \leq \frac{1}{\sup\limits_{m\geq n} |a_m|^{1/m}} \leq r + \frac{\varepsilon}{2} \quad \text{for all} \quad n \geq n_0. \tag{3.3.5}$$

[1]Of course, this statement is vacuous when $R = \infty$.

[2]This proof can be very much simplified applying directly the Root Test (Proposition 3.4) to series of the form $\sum_{n=0}^{\infty} |a_n| s^r$, in combination with formula (3.3.2). But we offer here the full argument to remind the reader the idea of proof of the Root Test Criterion.

From the left inequality of (3.3.5), we get, in particular, that $|a_n| \leq \left(r - \frac{\varepsilon}{2}\right)^{-1/n}$ for all $n \geq n_0$. Thus we have the estimates

$$\sum_{n=0}^{\infty} |a_n|(r-\varepsilon)^n = \sum_{n=0}^{n_0-1} |a_n|(r-\varepsilon)^n + \sum_{n=n_0}^{\infty} |a_n|(r-\varepsilon)^n \leq \sum_{n=0}^{n_0-1} |a_n|(r-\varepsilon)^n + \sum_{n=n_0}^{\infty} \left(\frac{r-\varepsilon}{r-\frac{\varepsilon}{2}}\right)^n < \infty.$$

Therefore, the series $\sum_{n=0}^{\infty} |a_n|(r-\varepsilon)^n$ converges and so, by (3.3.1), $r - \varepsilon \leq R$. And the second inequality of (3.3.5) leads us to $|a_{n_k}| \geq \left(r + \frac{\varepsilon}{2}\right)^{-1/n_k}$ for a subsequence $n_k \to \infty$, as $k \to \infty$. Hence,

$$\sum_{n=0}^{\infty} |a_n|(r+\varepsilon)^n \geq \sum_{k=0}^{\infty} |a_{n_k}|(r+\varepsilon)^{n_k} \geq \sum_{k=0}^{\infty} \left(\frac{r+\varepsilon}{r+\frac{\varepsilon}{2}}\right)^{n_k} = \infty.$$

By relation (3.3.2) in Remark 3.13, the above yields $R \leq r+\varepsilon$. We have shown that $r-\varepsilon \leq R \leq r+\varepsilon$ for arbitrary $\varepsilon > 0$, and so $R = r$, as desired.

Now, if $\limsup_{n\to\infty} |a_n|^{1/n} = \infty$, then for every $r > 0$ we can find a subsequence $(n_k)_k \to \infty$ with $|a_{n_k}| \geq \left(\frac{r+1}{r}\right)^{n_k}$. This gives

$$\sum_{n=0}^{\infty} |a_n| r^n \geq \sum_{k=0}^{\infty} |a_{n_k}| r^{n_k} \geq \sum_{k=0}^{\infty} \left(\frac{r+1}{r}\right)^{n_k} r^{n_k} = \sum_{k=0}^{\infty} (r+1)^{n_k} = \infty.$$

Since $r > 0$ is arbitrary, this means, by e.g. formula (3.3.2), that $R = 0$.

Finally, in the case $\limsup_{n\to\infty} |a_n|^{1/n} = 0$, for every $r > 0$ we can find $n_0 \in \mathbb{N}$ such that $|a_n| \leq \left(\frac{1}{r+1}\right)^n$ for all $n \geq n_0$. Thus,

$$\sum_{n=0}^{\infty} |a_n| r^n = \sum_{n=0}^{n_0-1} |a_n| r^n + \sum_{n=n_0}^{\infty} |a_n| r^n \leq \sum_{n=0}^{n_0-1} |a_n| r^n + \sum_{n=n_0}^{\infty} \left(\frac{r}{r+1}\right)^n < \infty.$$

Because $r > 0$ is arbitrary, we have shown that $R = \infty$.

(v) Define $r := \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} \in [0, +\infty]$. Let us begin with the case $r < \infty$. We have for every $\varepsilon > 0$, that

$$\lim_{n\to\infty} \frac{|a_{n+1}|(r+\varepsilon)^{n+1}}{|a_n|(r+\varepsilon)^n} = (r+\varepsilon) \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \begin{cases} \frac{r+\varepsilon}{r} & \text{if } r > 0 \\ \infty & \text{if } r = 0. \end{cases}$$

The limit is greater than 1 in any case, and by the Ratio Test (see Proposition 3.4), this implies that $\sum_{n=0}^{\infty} |a_n|(r+\varepsilon)^n$ does not converges. Identity (3.3.2) leads us to $R \leq r + \varepsilon$, and because $\varepsilon > 0$ is arbitrary, we get $R \leq r$. This in particular proves the result in the case $r = 0$.

Consider now the case $0 < r < \infty$, and $0 < \varepsilon < r$. Again we use the Ratio Test for numerical series:

$$\lim_{n\to\infty} \frac{|a_{n+1}|(r-\varepsilon)^{n+1}}{|a_n|(r-\varepsilon)^n} = (r-\varepsilon) \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{r-\varepsilon}{r} < 1.$$

Thus the series $\sum_{n=0}^{\infty} |a_n|(r-\varepsilon)^n$ converges and by the same reasoning we get that $r - \varepsilon \leq R$ for all $\varepsilon > 0$, implying that $r \leq R$. We conclude $R = r$.

And when $r = \infty$, we use an identical argument to show that, for each $M > 0$, the series $\sum_{n=0}^{\infty} |a_n| M^n$ converges, which implies $R = \infty = r$. □

We now apply Theorem 3.15 to concrete examples of power series.

Example 3.16. Let us determine the radius and disk of convergence in the following cases.

(i) $\sum_{n=0}^{\infty} \frac{(1+i)^n}{n^n}(z-i)^n$. The series is centered at i and the coefficients are $a_n = \frac{(1+i)^n}{n^n}$. To find the radius of convergence R, we use formula (3.3.3). We have $|a_n| = |1+i|^n/n^n = \left(\sqrt{2}/n\right)^n$. So,

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[n]{\left(\sqrt{2}/n\right)^n} = \limsup_{n\to\infty} \frac{\sqrt{2}}{n} = 0.$$

Therefore $R = \infty$. So the series converges pointwise in all of $\mathbb{C}$, and absolutely–uniformly on each bounded subset of $\mathbb{C}$ (as these are all contained in disks of the form $D(z_0, N)$ for $N \in \mathbb{N}$).

(ii) $\sum_{n=0}^{\infty}(1+ni)\left(\frac{z+i}{2}\right)^n$. The center is $z_0 = -i$ and the coefficients are $a_n = \frac{1+ni}{2^n}$ for all $n \in \mathbb{N}^*$. So the moduli are $|a_n| = \frac{\sqrt{1+n^2}}{2^n}$, and

$$\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} \left(\frac{\sqrt{1+n^2}}{2^n}\right)^{1/n} = \frac{1}{2}\limsup_{n\to\infty} \left(\sqrt{1+n^2}\right)^{1/n} = \frac{1}{2}.$$

Therefore, the radius and disk of convergence of the series are $R = 2$ and $D(-i, 2)$. Within each closed subsdisk of $D(-i, 2)$, the series converges absolutely–uniformly. On the the open disk $D(-i, 2)$, we have pointwise convergence, and outside the disk $\overline{D}(-i, 2)$ the series diverges at every point. These are all conclusions from Theorem 3.15.

But, *what is the situation when $z \in \partial D(-i, 2)$?* Unfortunately, Theorem 3.15 is useless here and we need to study the convergence by other methods. If z is such that $|z+i| = 2$, then the numerical series $\sum_{n=0}^{\infty}(1+ni)\left(\frac{z+i}{2}\right)^n$ has general term equal to $b_n = \frac{1+ni}{2^n}(z+i)^n$. But then $|b_n| = |1+ni| = \sqrt{1+n^2}$, which of course does not converge to 0. According to Proposition 3.2, the series $\sum_{n=0}^{\infty}(1+ni)\left(\frac{z+i}{2}\right)^n$ diverges.

(iii) $\sum_{n=1}^{\infty} \frac{1+ni}{n^3} z^n$. The center is $z_0 = 0$ and the coefficients are $a_n = \frac{1+ni}{n^3}$ for all $n \in \mathbb{N}$. Hence

$$\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} \left(\frac{\sqrt{1+n^2}}{n^3}\right)^{1/n} = 1.$$

The radius and disk of convergence are $R = 1$ and $D(0, 1)$. By Theorem 3.15 we have absolute–uniform convergence in closed subdisks of $D(0, 1)$, pointwise convergence in $D(0, 1)$ and diverge in all of $\mathbb{C}\backslash\overline{D}(0, 1)$. In the boundary $\partial D(0, 1)$, again Theorem 3.15 is inconclusive. But if $|z| = 1$, the numerical series $\sum_{n=0}^{\infty} \frac{1+ni}{n^3} z^n$ has general term equal to $b_n(z) = \frac{1+ni}{n^3} z^n$, with

$$|b_n(z)| = \frac{\sqrt{1+n^2}}{n^3}|z|^n = \frac{\sqrt{1+n^2}}{n^3} \leq \frac{\sqrt{2}\,n}{n^3} = \frac{\sqrt{2}}{n^2}.$$

Because $\sum_{n=1}^{\infty} \frac{\sqrt{2}}{n^2} < \infty$, we have that $\sum_{n=0}^{\infty} \frac{1+ni}{n^3} z^n$ is convergent for each $z \in \partial D(0, 1)$. Moreover, since the bound above is independent of $z \in \partial D(0, 1)$, Theorem 3.9 tells us that $\partial D(0, 1) \ni z \mapsto \sum_{n=1}^{\infty} \frac{1+ni}{n^3} z^n$ converges absolutely–uniformly.

(iv) $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n$. The coefficients are $a_n = \frac{(n!)^2}{(2n)!}$, and the center is $z_0 = 0$. Perhaps in this case formula (3.3.3) is not the easiest way to calculate the radius of convergence, especially if all we know is that $\lim_{n\to\infty} \sqrt[n]{(n!)^2} = \lim_{n\to\infty} \sqrt[n]{(2n)!} = \infty$. We can try out Theorem 3.15(v) instead:

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{\frac{(n!)^2}{(2n)!}}{\frac{((n+1)!)^2}{(2(n+1))!}} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2} = 4.$$

Therefore, the radius of convergence is $R = 4$ and $D(0, 4)$ is the disk of convergence. We know very well what the convergence is in $D(0, 4)$ and in $\mathbb{C} \setminus \overline{D}(0, 4)$ from Theorem 3.15. However, if $z \in \partial D(0, 4)$, since $\lim_{n\to\infty} \frac{(n!)^2}{(2n)!} 4^n = \infty$,[3] the series $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n$ diverges.

[3]This is a consequence of Wallis' product formula, which actually shows that $\frac{(n!)^2}{(2n)!} 4^n \sim \sqrt{n}$.

(v) Let us treat some series with infinitely many terms equal to 0. Let $\varphi : \mathbb{N}^* \to \mathbb{N}^*$ be an increasing function, and consider the series $\sum_{n=0}^{\infty} a_n z^{\varphi(n)}$. In oder to apply the Cauchy-Hadamard formula (3.3.3), we can define new coefficients

$$b_k = \begin{cases} a_n & \text{if} \quad k = \varphi(n), \text{ for some } n \in \mathbb{N}^*, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\sum_{n=0}^{\infty} a_n z^{\varphi(n)} = \sum_{n=0}^{\infty} b_k z^k$ and their radius of covergence R satisfies, by (3.3.3),

$$\begin{aligned} R^{-1} = \limsup_{k\to\infty} |b_k|^{1/k} &= \lim_{k\to\infty} \sup\{|b_j|^{1/j} \,:\, j \geq k\} = \lim_{k\to\infty} \sup\{|a_n|^{1/\varphi(n)} \,:\, j \geq k,\, j = \varphi(n), n \in \mathbb{N}^*\} \\ &= \lim_{k\to\infty} \sup\{|a_n|^{1/\varphi(n)} \,:\, \varphi(n) \geq \varphi(k),\, n \in \mathbb{N}^*\} \\ &= \lim_{k\to\infty} \sup\{|a_n|^{1/\varphi(n)} \,:\, n \geq k,\, n \in \mathbb{N}^*\} = \limsup_{n\to\infty} |a_n|^{1/\varphi(n)}. \end{aligned}$$

In the fourth equality we used that if $\{c_k\}_n \subset \mathbb{R}$ is non-decreasing, then $\lim_{k\to\infty} c_k = \lim_{k\to\infty} c_{\varphi(k)}$. So we can again derive a formula for R in terms of $\{a_n\}_n$.

For example, let us examine $\sum_{n=0}^{\infty} a_n z^{2n}$. By the above, the radius of convergence R satisfies

$$R^{-1} = \limsup_{n\to\infty} |a_n|^{1/2n} = \left(\limsup_{n\to\infty} \sqrt[n]{|a_n|}\right)^{1/2}.$$

If it is difficult to figure out $\limsup_{n\to\infty} \sqrt[n]{|a_n|}$, we can try a variation of the Ratio Formula (3.3.4), always provided that $a_n \neq 0$ from some N on. Note that (3.3.4) cannot be applied as it is for the series $\sum_{n=0}^{\infty} a_n z^{2n}$, because the coefficients of the terms of the form z^{2n+1} are all zero. Thus we go back to the Ratio Test for numerical series (Proposition 3.4), and use that

$$\lim_{n\to\infty} \frac{|a_{n+1}| r^{2(n+1)}}{|a_n| r^{2n}} = r^2 \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|},$$

for $r > 0$, to obtain

$$\sum_{n=0}^{\infty} |a_n| r^n = \begin{cases} \text{converges,} & \text{if} \quad \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1/r^2, \\ \text{diverges,} & \text{if} \quad \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1/r^2. \end{cases}$$

By (3.3.2), this clearly shows that $R = \left(\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}\right)^{1/2}$.

3.3.3 Convergence on the Boundary

We have learnt from Example 3.16 that it is not so easy to determine the convergence of a power series on the boundary of its disk of convergence. Theorem 3.15 is inconclusive in this respect: on the boundary the series may converge or diverge at all points, or converge only at some points. We next prove a criterion for convergence on the boundary, covering a reasonably big amount of cases. We first need the useful Abel's Summation by Parts formula.

Lemma 3.17 (Abel's Summation by Parts)**.** *Let $\{a_n\}_n$, $\{b_n\}_n$ be sequences of complex numbers and denote $B_n = \sum_{k=0}^{n} b_n$ for every $n \in \mathbb{N}^*$. Then for all $M, N \in \mathbb{N}^*$ with $M > N$ we have*

$$\sum_{n=N}^{M} a_n b_n = a_M B_M - a_N B_{N-1} - \sum_{n=N}^{M-1} (a_{n+1} - a_n) B_n.$$

Proof. It is enough to spell out the last term of the right hand side:

$$\begin{aligned}
-\sum_{n=N}^{M-1}(a_{n+1}-a_n)B_n &= -\sum_{n=N}^{M-1} a_{n+1}B_n + \sum_{n=N}^{M-1} a_n B_n = -\sum_{n=N+1}^{M} a_n B_{n-1} + \sum_{n=N}^{M-1} a_n B_n \\
&= -a_M B_{M-1} + \sum_{n=N+1}^{M-1} a_n(B_n - B_{n-1}) + a_N B_N \\
&= -a_M B_{M-1} + \sum_{n=N+1}^{M-1} a_n b_n + a_N B_N \\
&= -a_M B_{M-1} - a_M b_M + \sum_{n=N}^{M} a_n b_n - a_N b_N + a_N B_N \\
&= -a_M B_M + \sum_{n=N}^{M} a_n b_n + a_N B_{N-1}.
\end{aligned}$$

□

Theorem 3.18 (Picard's Criterion)**.** *The following statements hold.*

(i) If $\{a_n\}_n \subset \mathbb{C}$ are such that $\sum_{n=0}^{\infty}|a_{n+1}-a_n| < \infty$ and $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=0}^{\infty} a_n z^n$ converges for all z with $|z| = 1$ and $z \neq 1$.

(ii) In particular, if $\{a_n\}_n \subset \mathbb{R}$ converges monotonically to 0, then the series $\sum_{n=0}^{\infty} a_n z^n$ converges for all z with $|z| = 1$ and $z \neq 1$.

Proof. We first prove part (i). Let $z \in \mathbb{C}$ with $|z| = 1$ and $z \neq 1$. Let us show that the partial sums of $\sum_{n=0}^{\infty} a_n z^n$ satisfy the Cauchy property. Indeed, if $M > N$ are naturals, Lemma 3.17 permits to write

$$\sum_{n=0}^{M} a_n z^n - \sum_{n=0}^{N} a_n z^n = \sum_{n=N+1}^{M} a_n z_n = a_M \sum_{n=0}^{M} z^n - a_{N+1}\sum_{n=0}^{N} z^n - \sum_{n=N+1}^{M-1}(a_{n+1}-a_n)\sum_{k=0}^{n} z^k.$$

Taking moduli, using the triangle inequality, and computing the geometric sum, we get

$$\begin{aligned}
\left|\sum_{n=0}^{M} a_n z^n - \sum_{n=0}^{N} a_n z^n\right| &\le |a_M|\left|\sum_{n=0}^{M} z^n\right| + |a_{N+1}|\left|\sum_{n=0}^{N} z^n\right| + \sum_{n=N+1}^{M-1}|a_{n+1}-a_n|\left|\sum_{k=0}^{n} z^k\right| \\
&= |a_M|\left|\frac{1-z^{M+1}}{1-z}\right| + |a_{N+1}|\left|\frac{1-z^{N+1}}{1-z}\right| + \sum_{n=N+1}^{M-1}|a_{n+1}-a_n|\left|\frac{1-z^{n+1}}{1-z}\right| \\
&\le \frac{2}{|1-z|}\left(|a_M| + |a_{N+1}| + \sum_{n=N+1}^{M-1}|a_{n+1}-a_n|\right);
\end{aligned}$$

where we used the crude estimate $|1-z^m| \le 1 + |z|^m = 2$ in the last inequality. Because $a_n \to 0$, we have that $|a_M|, |a_{N+1}| \to 0$ as $N, M \to \infty$. And the term $\sum_{n=N+1}^{M-1}|a_{n+1}-a_n|$ also tends to 0 as $N, M \to \infty$ because it coincides with the difference of partial sums

$$\sum_{n=0}^{M-1}|a_{n+1}-a_n| - \sum_{n=0}^{N}|a_{n+1}-a_n|,$$

of the convergent series $\sum_{n=0}^{\infty}|a_{n+1}-a_n|$. We conclude that the partial sums of $\sum_{n=0}^{\infty} a_n z^n$ have the Cauchy property.

Now, to prove (ii), observe that $\sum_{n=0}^{N}|a_{n+1}-a_n| = \left|\sum_{n=0}^{N}(a_{n+1}-a_n)\right|$ as the sequence $\{a_n\}_n$ is monotone. But the last term is equal to $|a_{N+1}-a_0|$, whose limit is $|a_0|$. Therefore the series $\sum_{n=0}^{N}|a_{n+1}-a_n|$ converges, and we can apply (i). □

Here there is an obvious generalization of Theorem 3.18 for arbitrary radius and center.

Corollary 3.19. *Let $\{a_n\}_n \subset \mathbb{C}$, $z_0 \in \mathbb{C}$, and $r > 0$. Assume that $\sum_{n=0}^{\infty} |r^{n+1}a_{n+1} - r^n a_n| < \infty$ and $\lim_{n\to\infty} r^n a_n = 0$. Then $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for all $z \in \partial D(z_0, r) \setminus \{z_0 + r\}$.*

In particular, if a sequence $\{a_n r^n\}_n \subset \mathbb{R}$ converges to 0 monotonically and $\lim_{n\to\infty} r^n a_n = 0$, then $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for all $z \in \partial D(z_0, r) \setminus \{z_0 + r\}$.

Proof. Writing $\sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} a_n r^n \left(\frac{z-z_0}{r}\right)^n$, it suffices to apply Theorem 3.18 for $a_n r^n$ in place of a_n and $w = (z-z_0)/r$ in place of z. □

For example, the series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ has radius of convergence equal to 1, and so there is pointwise converge in $D(0,1)$ and absolute–uniform convergence in disks $\overline{D}(0,r)$ for all $r < 1$. The coefficients $a_n = \frac{1}{n} \downarrow 0$ and if $z \in \partial D(0,1) \setminus \{1\}$, we can apply Theorem 3.18(ii) to conclude that $\sum_{n=1}^{\infty} \frac{z^n}{n}$ is convergent. In the case $z = 1$, the series clearly diverges.

3.3.4 Differentiability of Power Series

If $R > 0$ is the radius of convergence of a power series $f(z) = \sum_{n=0} a_n(z-z_0)^n$, then Theorem 3.15 implies that, on every disk $\overline{D}(z_0, r)$ with $r < R$, f is the uniform limit (as $N \to \infty$) of the partial sums $\sum_{n=0}^{N} a_n(z-z_0)^n$ on $z \in D(z_0, r)$. By Proposition 3.10, f is continuous at every $z \in D(z_0, R)$. So, power series are continuous on their disk of convergence. Our next objective is to show that they are actually infinitely differentiable on the disk. The main technical difficulty is, as usual, exchange the order of derivatives with limits of partial sums.

Theorem 3.20 (Differentiability of Power Series)**.** *Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series with radius of convergence $R > 0$. Then the function $f : D(z_0, R) \to \mathbb{C}$ defined as*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, \quad z \in D(z_0, R),$$

is holomorphic in $D(z_0, R)$ and

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}, \quad z \in D(z_0, R).$$

Moreover the power series of f' above has radius of convergence equal to R.

Proof. Define the function $D(z_0, R) \ni z \mapsto g(z) := \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$. By Theorem 3.15, (3.3.3), we have $\limsup_{n\to\infty} |a_n|^{1/n} = 1/R$ and so

$$\limsup_{n\to\infty} (n|a_n|)^{1/n} = \limsup_{n\to\infty} n^{1/n} |a_n|^{1/n} = \frac{1}{R}.$$

Then the radius of convergence of $\sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$ is also R (again by (3.3.3)). This gives plenty of information. First, we have confirmed that g is well defined in $D(z_0, R)$. Also, for every $0 < r < R$, the series $\sum_{n=1}^{\infty} n|a_n| r^{n-1}$ is convergent (also thanks to (3.3.2)). Therefore, using Proposition 3.2(ii), we get that

$$\lim_{N\to\infty} \sum_{n=N}^{\infty} n|a_n| r^{n-1} = 0. \tag{3.3.6}$$

Moreover, from Theorem 3.15 we learnt that $\sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$ converges (even uniformly) on each disk $\overline{D}(z_0, r)$ with $0 < r < R$, so

$$g(z) = \lim_{N\to\infty} \sum_{n=1}^{N} n a_n (z-z_0)^{n-1}, \quad z \in \overline{D}(z_0, r). \tag{3.3.7}$$

Now, to check that f is holomorphic with $f' = g$ on $D(z_0, R)$, let us fix $z \in D(z_0, R)$, take r with $|z - z_0| < r < R$. In $D(z_0, r)$, the series defining f and g converge, and we can write, for all $w \in D(z_0, r)$ and $N \in \mathbb{N}$:

$$f(w) = f_N(w) + f^N(w), \text{ with } f_N(w) := \sum_{n=0}^{N} a_n(w - z_0)^n, \quad f^N(w) := \sum_{n=N+1}^{\infty} a_n(w - z_0)^n,$$

$$g(w) = g_N(w) + g^N(w), \text{ with } g_N(w) := \sum_{n=1}^{N} na_n(w - z_0)^{n-1}, \quad g^N(w) := \sum_{n=N+1}^{\infty} na_n(w - z_0)^{n-1}.$$

But notice that f_N is just a polynomial function, with $f_N' = g_N$ on $D(z_0, R)$. Thus, the idea is that we only need to verify that *f^N is differentiable at z, with $(f^N)'(z) = g^N(z)$ for sufficiently large N*. Let us make this rigorous. Given $\varepsilon > 0$, these observations along with (3.3.6) and (3.3.7) yield the existence of $N \in \mathbb{N}$ and $\delta > 0$ with $0 < \delta < r - |z - z_0|$ such that

$$|g(z) - g_N(z)| < \frac{\varepsilon}{3}, \quad \sum_{n=N}^{\infty} n|a_n|r^{n-1} < \frac{\varepsilon}{3}, \quad \text{and} \quad \left|\frac{f_N(w) - f_N(z)}{w - z} - g_N(z)\right| < \frac{\varepsilon}{3}, \tag{3.3.8}$$

whenever $w \in D(z, \delta) \setminus \{z\}$. Using the estimates of (3.3.8), we can write, for all $w \in D(z, \delta) \setminus \{z\} \subset D(z_0, r)$:

$$\begin{aligned}
\left|\frac{f(w) - f(z)}{w - z} - g(z)\right| &= \left|\frac{f^N(w) - f^N(z)}{w - z}\right| + \left|\frac{f_N(w) - f_N(z)}{w - z} - g_N(z)\right| + |g(z) - g_N(z)| \\
&< \left|\sum_{n=N+1}^{\infty} a_n \frac{(w - z_0)^n - (z - z_0)^n}{w - z}\right| + \frac{2\varepsilon}{3} = \left|\sum_{n=N+1}^{\infty} \frac{a_n(w - z)}{w - z} \sum_{k=0}^{n-1} (w - z_0)^{n-1-k}(z - z_0)^k\right| + \frac{2\varepsilon}{3} \\
&= \left|\sum_{n=N+1}^{\infty} a_n \sum_{k=0}^{n-1} (w - z_0)^{n-k}(z - z_0)^k\right| + \frac{2\varepsilon}{3} \le \sum_{n=N+1}^{\infty} |a_n| \sum_{k=0}^{n-1} |w - z_0|^{n-1-k}|z - z_0|^k + \frac{2\varepsilon}{3} \\
&\le \sum_{n=N+1}^{\infty} |a_n| \sum_{k=0}^{n-1} r^{n-1-k}r^k + \frac{2\varepsilon}{3} = \sum_{n=N+1}^{\infty} n|a_n|r^{n-1} + \frac{2\varepsilon}{3} < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.
\end{aligned}$$

We may conclude that f is differentiable at z, with $f'(z) = g(z)$. $\square$

According to Theorem 3.20, any power series is holomorphic on the disk of convergence and its derivative is new power series (obtaining by differentiating termwise in the original series) with the same radius of convergence. We can apply the theorem repeatedly to derive the following.

Corollary 3.21 (C^∞ regularity of power series). *Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series with radius of convergence $R > 0$, and let $D(z_0, R) \ni z \mapsto f(z) := \sum_{n=0}^{\infty} a_n(z - z_0)^n$. Then,*

(i) $f \in C^\infty(D(z_0, R))$.

(ii) $f^{(k)}(z) = \sum_{n=k}^{\infty} n(n - 1) \cdots (n - k + 1)a_k(z - z_0)^{n-k}$ *for all* $z \in D(z_0, R)$.

(iii) The coefficients $\{a_n\}_{n \in \mathbb{N}^*}$ are unique and satisfy $a_n = \frac{f^{(n)}(z_0)}{n!}$ for all $n \in \mathbb{N}^*$. In particular, on $D(z_0, R)$, the series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ is the Taylor series of f centered at z_0.

Proof. By iterating Theorem 3.20, we get that f is infinitely many times differentiable in Ω, with each $f^{(k)}$ given by the power series $\sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1)(z-z_0)^{n-k}$ on $z \in D(z_0, R)$, whose radius of convergence is equal to R. And evaluating at $z = z_0$, we get $f^{(k)}(z) = k!\, a_k$. $\square$

Similarly, we can obtain primitives (anti-derivatives) of power series by *integrating* termwise in the series.

Corollary 3.22 (Antiderivatives of Power Series)**.** *Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series with radius of convergence $R > 0$, and define $D(z_0, R) \ni z \mapsto f(z) := \sum_{n=0}^{\infty} a_n(z-z_0)^n$. Then the power series*

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1}$$

has radius of convergence equal to R, and the function $g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1}$ is holomorphic in $D(z_0, R)$ with $g'(z) = f(z)$ for all $z \in D(z_0, R)$.

Proof. Because

$$\limsup_{n\to\infty} \left(\frac{|a_n|}{n+1}\right)^{1/(n+1)} = \limsup_{n\to\infty} \left(\frac{|a_n|}{n+1}\right)^{1/(n+1)} = \limsup_{n\to\infty} \left(|a_n|^{1/n}\right)^{n/(n+1)} = \frac{1}{R},$$

the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1}$ is equal R by Theorem 3.15. Applying Theorem 3.20 to this power series, we get that $g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1}$ is holomorphic in $D(z_0, R)$ with

$$g'(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n = f(z), \quad z \in D(z_0, R).$$

□

3.4 Analytic Functions

Definition 3.23 (Analytic function)**.** *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ a function. If $z_0 \in \Omega$, we say that f is **analytic at** z_0 if there exists $r > 0$ with $D(z_0, r) \subset \Omega$ and a sequence $\{a_n\}_{n\in\mathbb{N}^*} \subset \mathbb{C}$ so that*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \quad \textit{for all} \quad z \in D(z_0, r).$$

*If f is analytic at every $z_0 \in \Omega$, we say that f is **analytic in** Ω. We denote the family of all analytic functions in Ω by $\mathcal{A}(\Omega)$.*

Remark 3.24. If $f : \Omega \to \mathbb{C}$ is analytic at $z_0 \in \Omega$ and $f(z) = \sum_{n=0} a_n(z-z_0)^n$ for all $z \in D(z_0, r) \subset \Omega$, then in particular $\sum_{n=0} a_n s^n$ is convergent (to $f(z_0 + s) \in \mathbb{C}$) for all $0 < s < r$, and therefore $r \leq R$, the radius of convergence of the power series $\sum_{n=0} a_n(z-z_0)^n$; recall Definition 3.12. Consequenly, f coincides with a power series on a disk $D(z_0, r)$ contained in its disk of convergence $D(z_0, R)$. By Corollary 3.21, $f \in C^\infty(D(z_0, r))$ and

$$f(z) = \sum_{n=0} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \quad \text{for all} \quad z \in D(z_0, r).$$

And observe that in the case where $f^{(n)}(z_0) = 0$ for all $n \in \mathbb{N}^*$ (understanding that $f^{(0)} = f$), f is identically zero on $D(z_0, r)$.

Note that Remark 3.24 implies that we only need to assume analyticity of f at a point z_0 to guarantee C^∞ regularity on a whole disk $D(z_0, r)$ around z_0.

3.4.1 Analyticity of Power Series

The next step is showing that actually analyticity at a point z_0 implies analyticity on a disk around z_0. To prove this, we will show that all power series are analytic functions on their disk of convergence. We first need a Fubini-type property for summation of *iterated series* to make our proof entirely rigorous.

Proposition 3.25. *Let $C : \mathbb{N}^* \times \mathbb{N}^* \to \mathbb{C}$ be a function such that*

$$\text{either} \quad \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} |C(n,k)| < \infty \quad \text{or} \quad \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} |C(n,k)| < \infty.$$

Then we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C(n,k) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} C(n,k).$$

Proof. Assume, without loss of generality, that $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} |C(n,k)| < \infty$. Consequently

$$\sum_{k=0}^{\infty} C(n,k) \in \mathbb{C}, \sum_{k=0}^{\infty} |C(n,k)| \in \mathbb{C} \text{ for all } n \quad \text{and} \quad \sum_{n=0}^{\infty} C(n,k) \in \mathbb{C}, \sum_{n=0}^{\infty} |C(n,k)| \in \mathbb{C} \text{ for all } k,$$

which we will be using systematically (and implicitly) in the proof. Now, let $\varepsilon > 0$. For each $n \in \mathbb{N}^*$, we have that $\varphi(n) := \sum_{k=0}^{\infty} |C(n,k)| \in \mathbb{C}$, as a consequence of the assumption. Moreover, the assumption says that $\sum_{n=0}^{\infty} \varphi(n)$ converges, and so the partial sums have the Cauchy property. Thus there exists $N_0 \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{M}\sum_{k=0}^{\infty} |C(n,k)| = \sum_{n=N+1}^{M} \varphi(n) \leq \frac{\varepsilon}{3}, \quad \text{for all} \quad M, N \geq N_0. \tag{3.4.1}$$

The convergence of $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} |C(n,k)|$ also implies that $L := \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C(n,k) \in \mathbb{C}$. Thus we can find $N_1 \geq N_0$ such that

$$\left|\sum_{n=0}^{N_1}\sum_{k=0}^{\infty} C(n,k) - L\right| \leq \frac{\varepsilon}{3}. \tag{3.4.2}$$

But also $\lim_{K\to\infty} \sum_{k=0}^{K} C(n,k) \in \mathbb{C}$ for all n, and so we can find K_1 (depending on N_1 and ε) such that, for every $K \geq K_1$:

$$\left|\sum_{n=0}^{N_1}\sum_{k=0}^{\infty} C(n,k) - \sum_{n=0}^{N_1}\sum_{k=0}^{K_1} C(n,k)\right| \leq \frac{\varepsilon}{3}.$$

This estimate, in combination with (3.4.2), gives

$$\left|\sum_{k=0}^{K}\sum_{n=0}^{N_1} C(n,k) - L\right| = \left|\sum_{n=0}^{N_1}\sum_{k=0}^{K} C(n,k) - L\right| \leq \frac{2\varepsilon}{3} \quad \text{for all} \quad K \geq K_1. \tag{3.4.3}$$

It is now tempting to freeze K and let $N_1 \to \infty$ in (3.4.3), but we are not allowed to do so because here K is at least K_1, which depends on N_1. We need to show an estimate like (3.4.3) replacing N_1 with every $N \geq N_1$. But we can use (3.4.1) with $N > N_1 \geq N_0$ (and (3.4.3) itself) to get that, for all $K \geq K_1$ and $N \geq N_1$,

$$\begin{aligned}\left|\sum_{k=0}^{K}\sum_{n=0}^{N} C(n,k) - L\right| &\leq \left|\sum_{k=0}^{K}\sum_{n=0}^{N} C(n,k) - \sum_{k=0}^{K}\sum_{n=0}^{N_1} C(n,k)\right| + \left|\sum_{k=0}^{K}\sum_{n=0}^{N_1} C(n,k) - L\right| \\ &\leq \sum_{k=0}^{K}\sum_{n=N_1+1}^{N} |C(n,k)| + \frac{2\varepsilon}{3} \leq \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.\end{aligned}$$

Now we can first freeze K and let $N \to \infty$ to obtain $\left|\sum_{k=0}^{K}\sum_{n=0}^{\infty} C(n,k) - L\right| \leq \varepsilon$, and then let $K \to \infty$ to conclude $|\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} C(n,k) - L| \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we get $\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} C(n,k) = L$, as desired.

□

Theorem 3.26 (Analyticity of Power Series)**.** *Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series with radius of convergence $R>0$. Then the function $D(z_0,R) \ni z \mapsto f(z) := \sum_{n=0}^{\infty} a_n(z-z_0)^n$ is analytic in $D(z_0,R)$.*

Proof. Obviously we already have that f is analytic at z_0. So let $z_1 \in D(z_0,R)\setminus\{z_0\}$ and let $r>0$ be so that $r+|z_1-z_0|<R$. Note than then $D(z_1,r) \subset D(z_0,R)$. For every $z \in D(z_1,r)$ and $n \in \mathbb{N}^*$, we write $(z-z_0)^n = ((z-z_1)+(z_1-z_0))^n$ and apply the Binomial Formula (1.1.3) to obtain

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k}(z-z_1)^k(z_1-z_0)^{n-k} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C(n,k); \tag{3.4.4}$$

$$\text{where}\quad C(n,k) := \begin{cases} a_n\binom{n}{k}(z-z_1)^k(z_1-z_0)^{n-k} & \text{if } 0 \le k \le n \\ 0 & \text{if } k>n. \end{cases}$$

It is now convenient to be able to apply Proposition 3.25 to change the order of summation $\sum_n \sum_k \to \sum_n \sum_k$, for which we will check first that $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}|C(n,k)|<\infty$. Indeed,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}|C(n,k)| \le \sum_{n=0}^{\infty}\sum_{k=0}^{n}|a_n|\binom{n}{k}|z-z_1|^k|z_1-z_0|^{n-k} = \sum_{n=0}^{\infty}|a_n|\left(|z-z_1|+|z-z_0|\right)^n.$$

But $s:=|z-z_1|+|z_1-z_0|<r+|z_1-z_0|<R$, and R is the radius of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, so e.g. (3.3.2) says that series $\sum_{n=0}^{\infty}|a_n|s^n$. Therefore $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}|C(n,k)|<\infty$.

Continuing with the equalities of (3.4.4), we use Proposition 3.25 to arrive at

$$f(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C(n,k) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} C(n,k) = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} a_n\binom{n}{k}(z-z_1)^k(z_1-z_0)^{n-k} = \sum_{k=0}^{\infty} b_k(z-z_1)^k; \tag{3.4.5}$$

$$\text{where}\quad b_k := \sum_{n=k}^{\infty} a_n\binom{n}{k}(z_1-z_0)^{n-k}, \quad k \in \mathbb{N}^*.$$

Let us justify why $b_k \in \mathbb{C}$ for all k. Because $\binom{n}{k} \le \frac{n^k}{k!}$, we have the estimate

$$\sum_{n=k}^{\infty}|a_n|\binom{n}{k}|z_1-z_0|^{n-k} \le \frac{1}{k!|z_1-z_0|^k}\sum_{n=k}^{\infty}|a_n|n^k|z_1-z_0|^n,$$

and we check whether this series converges using e.g. the Root Test; see Proposition 3.4. We have

$$\limsup_{n\to\infty}\left(|a_n|n^k|z_1-z_0|^n\right)^{1/n} = |z_1-z_0|\limsup_{n\to\infty}|a_n|^{1/n} = \frac{|z_1-z_0|}{R},$$

by virtue of formula (3.3.3), as R is the radius of convergence of the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. But $|z_1-z_0|<R$, so the previous limit superior is smaller than 1, and so the numerical series $\sum_{n=k}^{\infty}|a_n|n^k|z_1-z_0|^n$ is convergent, as so is the series defining the number b_k. From (3.4.5), we see that f is analytic at z_1. ☐

Corollary 3.27. *Let $\Omega \subset \mathbb{C}$ be open and $z_0 \in \Omega$. If f is analytic at z_0, then f is analytic on a disk $D(z_0,r) \subset \Omega$.*

Proof. By Remark 3.24, f is a power series on a disk $D(z_0,r)$ contained in the disk of convergence of this series. By Theorem 3.26, f is then analytic in $D(z_0,r)$. ☐

3.4.2 Examples

Let us find the power series expansions of some of the elementary functions we constructed in Section 2.4.

Example 3.28. The complex exponential $\mathbb{C} \ni z \mapsto e^z \in \mathbb{C}$ from Definition 2.48 is analytic in $\mathbb{C}$, and

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \text{for all} \quad z \in \mathbb{C}. \tag{3.4.6}$$

To see this, let us recall known results from real analysis:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \cos y = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} y^{2n}, \quad \sin y = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} y^{2n+1}, \quad x, y \in \mathbb{R}.$$

This is shown via computing the Taylor series at the origin of the functions $\mathbb{R} \ni x \mapsto e^x$, $\mathbb{R} \ni y \mapsto \cos y, \sin y$. The series of $\cos y$ and $\sin y$ above converge absolutely, and so we can sum termwise and take into account (1.1.2) to obtain

$$\begin{aligned}\cos y + i \sin y &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} y^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} y^{2n+1} = \sum_{n=0}^{\infty} \frac{i^{2n}}{(2n)!} y^{2n} + \sum_{n=0}^{\infty} \frac{i \cdot i^{2n}}{(2n+1)!} y^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(iy)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(iy)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \left(\frac{(iy)^{2n}}{(2n)!} + \frac{(iy)^{2n+1}}{(2n+1)!} \right) = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!}.\end{aligned}$$

For all $x, y \in \mathbb{R}$, the two numerical series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $\sum_{n=0}^{\infty} \frac{(iy)^n}{n!}$ converge absolutely (e.g. by the Root Test from Proposition 3.4), and so their Cauchy product is absolutely convergent, and converges to the product of the series by Proposition 3.6. From (3.1.2), the general term of the Cauchy product is

$$c_n(x, y) = \sum_{k=0}^{n} \frac{x^k}{k!} \frac{(iy)^{n-k}}{(n-k)!} = \sum_{k=0}^{n} \frac{1}{n!} \binom{n}{k} x^k (iy)^{n-k} = \frac{(x+iy)^n}{n!},$$

after applying Newton's binomial formula. This shows the identity

$$e^{x+iy} = e^x (\cos y + i \sin y) = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{(iy)^n}{n!} \right) = \sum_{n=0}^{\infty} c_n(x, y) = \sum_{n=0}^{\infty} \frac{(x+iy)^n}{n!},$$

thus showing (3.4.6). Using the definitions (2.4.2) in combination with (3.4.6), we get that $z \mapsto \sin z, \cos z$ are analytic in $\mathbb{C}$ with

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}, \quad \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \quad z \in \mathbb{C}. \tag{3.4.7}$$

Example 3.29. Let us prove that the principal branch of the logarithm (Definition 2.60) satisfies

$$\operatorname{Log}(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}, \quad |z| \leq 1,\ z \neq 1. \tag{3.4.8}$$

Indeed, it is immediate that the radius of convergence of the series above is $R = 1$, by (3.3.3). Defining $f(z) = -\sum_{n=1}^{\infty} \frac{1}{n} z^n$, by Corollary 3.21, the series obtained by differentiating termwise

$$g(z) := -\sum_{n=1}^{\infty} z^{n-1} = -\sum_{n=0}^{\infty} z^n, \quad |z| < 1,$$

is holomorphic on $D(0,1)$ with $g' = f$. But g is a geometric series, whose value is $g(z) = \frac{1}{1-z}$. Now, the points of the form $w = 1-z$, with $|z| < 1$, clearly satisfies $\mathrm{Re}(w) > 0$, contained in the domain of holomorphicity of Log, see Theorem 2.63. Thus $(\mathrm{Log})'(1-z) = \frac{-1}{1-z} = f'(z)$ for all $|z| < 1$. By Corollary 2.37, $D(0,1) \ni z \mapsto \mathrm{Log}(1-z)$ and f differ by a constant, but evaluating at $z = 0$, we see that $f(0) = 0 = \mathrm{Log}(1)$, from which we obtain (3.4.8) for all $|z| < 1$. Now, for $z \in \partial D(0,1)$ with $z \neq 1$, we have from Picard's Criterion 3.18 that $-\sum_{n=1}^{\infty} \frac{1}{n} z^n$ is convergent. If $r \in (0,1)$ then $rz \in D(0,1)$, and so

$$\mathrm{Log}(1-rz) = -\sum_{n=1}^{\infty} \frac{1}{n}(rz)^n = -\sum_{n=1}^{\infty} \frac{1}{n} z^n r^n.$$

By the continuity of Log in $\mathbb{C} \setminus (-\infty, 0]$ and Exercise 3.15, we deduce

$$\mathrm{Log}(1-z) = \lim_{r \to 1^-} \mathrm{Log}(1-rz) = -\lim_{r \to 1^-} \sum_{n=1}^{\infty} \frac{z^n}{n} r^n = -\sum_{n=1}^{\infty} \frac{1}{n} z^n,$$

and this proves completely (3.4.8). Note that (3.4.8) also implies

$$\mathrm{Log}(1+z) = -\sum_{n=1}^{\infty} \frac{(-z)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n, \quad |z| \leq 1,\ z \neq -1.$$

3.4.3 Operations with Power Series and Analytic Functions

As expected, linear combinations of power series is another power series in the appropriate disks of convergence. The same holds for the product, using the Cauchy product to obtain the coefficients of the new series.

Proposition 3.30. *Let $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ and $\sum_{n=0}^{\infty} b_n (z-z_0)^n$ be two power series centered at $z_0 \in \mathbb{C}$ with radius of convergence $R_1 > 0$ and $R_2 > 0$ respectively. Then,*

(i) If $\lambda \in \mathbb{C} \setminus \{0\}$, the power series $\sum_{n=0}^{\infty} \lambda a_n (z-z_0)^n$ has radius of convergence R_1, and

$$\sum_{n=0}^{\infty} \lambda a_n (z-z_0)^n = \lambda \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad z \in D(z_0, R_1).$$

(ii) The power series $\sum_{n=0}^{\infty} (a_n + b_n)(z-z_0)^n$ has radius of convergence $R \geq \min\{R_1, R_2\}$, and

$$\sum_{n=0}^{\infty} (a_n + b_n)(z-z_0)^n = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=0}^{\infty} b_n (z-z_0)^n, \quad z \in D\left(z_0, \min\{R_1, R_2\}\right). \tag{3.4.9}$$

(iii) The power series $\sum_{n=0}^{\infty} c_n (z-z_0)^n$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, $n \in \mathbb{N}^*$, has radius of convergence $R \geq \min\{R_1, R_2\}$ and

$$\sum_{n=0}^{\infty} c_n (z-z_0)^n = \left(\sum_{n=0}^{\infty} a_n (z-z_0)^n\right)\left(\sum_{n=0}^{\infty} b_n (z-z_0)^n\right), \quad z \in D\left(z_0, \min\{R_1, R_2\}\right). \tag{3.4.10}$$

Proof.

(i) That $\sum_{n=0}^{\infty} \lambda a_n (z-z_0)^n$ has radius of convergence R_1 is immediate from the definition (3.3.1), and the equality because the sums for all $z \in D(z_0, R_1)$ follows from the $\mathbb{C}$-linearity of limits (of partial sums, in this case).

(ii) For every $r < \min\{R_1, R_2\}$, the two series $\sum_{n=0}^{\infty} a_n r^n$ and $\sum_{n=0}^{\infty} b_n r^n$ are convergent, and so is $\sum_{n=0}^{\infty} (a_n + b_n) r^n$. Therefore $R \geq \min\{R_1, R_2\}$, and the equality (3.4.9) clearly holds.

(iii) For every $r < \min\{R_1, R_2\}$, and every $z \in D(z_0, r)$, the numerical series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $\sum_{n=0}^{\infty} b_n(z-z_0)^n$ converge absolutely; see Theorem 3.15. By Propostion 3.6, the series $\sum_{n=0}^{\infty} c_{n,z}$ also converges (even absolutely), where

$$c_{n,z} := \sum_{k=0}^{n} a_k(z-z_0)^k b_{n-k}(z-z_0)^{n-k} = (z-z_0)^n \sum_{k=0}^{n} a_k b_{n-k} = (z-z_0)^n c_n,$$

and (also thanks to Proposition 3.6), we have

$$\left(\sum_{n=0}^{\infty} a_n(z-z_0)^n\right)\left(\sum_{n=0}^{\infty} b_n(z-z_0)^n\right) = \sum_{n=0}^{\infty} c_{n,z} = \sum_{n=0}^{\infty} c_n(z-z_0)^n.$$

This holds for all $z \in D(z_0, r)$, $0 < r < \min\{R_1, R_2\}$, from which we get the bound $R \geq \{R_1, R_2\}$ and the equality (3.4.10) for all $z \in D(z_0, \min\{R_1, R_2\})$. □

Consequently, analyticity is closed under multiplication with scalars, sums, and multiplications of functions.

Corollary 3.31. *Let $\Omega \subset \mathbb{C}$ be open, and $f, g \in \mathcal{A}(\Omega)$, $\lambda \in \mathbb{C}$. Then also*

$$\lambda f \in \mathcal{A}(\Omega), \quad f + g \in \mathcal{A}(\Omega), \quad f \cdot g \in \mathcal{A}(\Omega).$$

Proof. For every $z_0 \in \Omega$, we have expansions $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $g(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$ on $z \in D(z_0, r)$. By Proposition (3.30), we have expansions for λf, $f+g$, and $f \cdot g$ into power series centered z_0 on the disk $D(z_0, r)$ as well, and so those functions are analytic at z_0. □

The division of power series and/or analytic functions is a slightly more delicate issue and we are not yet able to prove that dividing analytic functions gives another analytic function. If $f, g \in \mathcal{A}(\Omega)$ and $g \neq 0$ on Ω, then $f, g \in \mathcal{H}(\Omega)$ (Theorem 3.20), and $h = f/g \in \mathcal{H}(\Omega)$ as well; see Proposition 2.34. But we do not (yet) know that then $h \in \mathcal{A}(\Omega)$. This will be proven in Chapter 4. Nonetheless, assuming a priori that h is analytic, we can deduce an expression for the coefficients of the power series of h in terms of those of f and g.

Namely, let $f, g \in \mathcal{A}(\Omega)$, $z_0 \in \Omega$ with $g(z_0) \neq 0$ and $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$, $g(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$, $h(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$ on $z \in D(z_0, r)$. Let us express the coefficients c_n in terms of a_n, b_n. We can assume that the convergence of the three series is absolutely–uniform on $D(z_0, r)$, and because $f = h \cdot g$, (3.4.10) says that

$$f(z) = \left(\sum_{n=0}^{\infty} c_n(z-z_0)^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n(z-z_0)^n\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} c_k b_{n-k}(z-z_0)^n, \quad z \in D(z_0, r);$$

whence $a_n = \sum_{k=0}^{n} c_k b_{n-k}$ for all $n \in \mathbb{N}^*$. Therefore

$$c_0 = \frac{a_0}{b_0}, \quad c_1 = \frac{1}{b_0}(a_1 - c_0 b_1), \quad \ldots, \quad c_n = \frac{1}{b_0}\left(a_n - \sum_{k=0}^{k-1} c_k b_{n-k}\right), \quad n \in \mathbb{N}^*.$$

3.4.4 Identity Principles for Analytic Functions

Analytic functions have a rather rigid structure. In particular, if at some point of a domain all its derivatives are zero, the function is automatically everywhere zero on that domain. This is the content of the next theorem.

Theorem 3.32 (1st Identity Theorem)**.** *Let Ω be open and connected, and $f : \Omega \to \mathbb{C}$ analytic in Ω. The following statements are equivalent*

(i) $f(z) = 0$ for all $z \in \Omega$.

(ii) There exists $z_0 \in \Omega$ such that $f(z_0) = f^{(n)}(z_0) = 0$ for all $n \in \mathbb{N}$.

Proof. The impplication $(i) \implies (ii)$ is obvious. Conversely, assume (ii) and define

$$E := \{z \in \Omega \,:\, f^{(n)}(z) = 0 \text{ for all } n \in \mathbb{N}^*\}.$$

The set E is nonempty because $z_0 \in E$ and obviously $E \subset \Omega$.

Let us show that E is open. Indeed, if $z \in E$, then $f^{(n)}(z) = 0$ for all $n \in \mathbb{N}^*$, and since f is analytic at z, there exists $\delta > 0$ so that $f(w) = 0$ for all $w \in D(z, \delta)$; as we pointed out in Remark 3.24. Of course this implies $f^{(n)}(z) = 0$ for all $n \in \mathbb{N}^*$ and all $w \in D(z, \delta)$ as well, which shows that $D(z, \delta) \subset E$. Consequently, E is open.

Now, let us prove that $E = F \cap \Omega$, for some closed set F. Each function $f^{(n)} : \Omega \to \mathbb{C}$ is continuous on Ω (because $f \in C^\infty(\Omega)$), and since $\{0\}$ is closed, Proposition 2.20 says that $\left(f^{(n)}\right)^{-1}(\{0\}) = F_n \cap \Omega$ for some $F_n \subset \mathbb{C}$ closed. But then $F = \bigcap_{n=0}^\infty F_n$ is closed as well (by virtue of Proposition 2.4), and clearly $E = F \cap \Omega$.

Since E is nonempty, $E = F \cap \Omega$, with F closed, and Ω is open and connected, by Proposition 2.27, we conclude $E = \Omega$, which means $f \equiv 0$ in Ω. □

A particular consequence of Theorem 3.32 is that if f is analytic on a domain Ω and $f \equiv 0$ on some open subset $U \subset \Omega$, then $f \equiv 0$ on Ω as well. Actually the distribution of zeros of (non-null) analytic functions is even more rigid: their zeros are *isolated.* This is a fundamental principle for analytic functions, which is stated and proved in the next theorem.

For a function $f : \Omega \to \mathbb{C}$, we denote the set of *zeros of f* in Ω by

$$\mathcal{Z}_\Omega(f) := \{z \in \Omega \,:\, f(z) = 0\}.$$

For every A, we denote by A' the set of accumulation points of A; recall Definition 2.5.

Theorem 3.33 (2^{nd} Identity Theorem)**.** *Let Ω be open and connected, and $f : \Omega \to \mathbb{C}$ analytic in Ω. The following statements are equivalent*

(i) $f(z) = 0$ for all $z \in \Omega$.

(ii) There exist $z_0 \in \Omega$ and a sequence $\{z_k\}_k \subset \Omega \setminus \{z_0\}$ such that $\lim_{k\to\infty} z_k = z_0$ and $f(z_k) = 0$ for all $k \in \mathbb{N}$. By Proposition 2.9, this is the same as saying that $(\mathcal{Z}_\Omega(f))' \cap \Omega \neq \emptyset$.

Proof. The implication $(i) \implies (ii)$ is obvious. Conversely, assume that (ii) holds and let $z_0 \in \Omega$ and $\{z_k\}_k$ be as in (ii). Since f is analytic at z_0, Remark 3.24 says that we can write

$$f(z) = \sum_{n=0}^\infty \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \quad z \in D(z_0, r) \subset \Omega.$$

Suppose, seeking a contradiction, that $f \not\equiv 0$ on Ω. By Theorem 3.32, there is some $m_0 \in \mathbb{N}^*$ such that $f^{(m_0)}(z_0) \neq 0$. Let $m \in \mathbb{N}^*$ be the smallest nonnegative integer with the property $f^{(m)}(z_0) \neq 0$. The series above then becomes

$$f(z) = \sum_{n=m}^\infty \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n = (z - z_0)^m g(z); \quad \text{where} \quad g(z) = \sum_{n=0}^\infty \frac{f^{(n+m)}(z_0)}{(n+m)!}(z - z_0)^n$$

for all $z \in D(z_0, r)$. Because $z_k \to z_0$, we may assume that $z_k \in D(z_0, r)$ for all k. The assumptions says that $0 = f(z_k) = (z - z_0)^m g(z_k)$, whence $g(z_k) = 0$ because $z_k \neq z_0$. But $g : D(z_0, r) \to \mathbb{C}$ is a continuous function, and $z_k \to z_0$ implies $g(z_k) \to g(z_0) = \frac{f^{(m)}(z_0)}{m!} \neq 0$. This is a contradiction. □

Theorems 3.32 and 3.33 are not true outside the class of analytic functions. For instance, the real function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0, \end{cases}$$

is of class $C^\infty(\mathbb{R}, \mathbb{R})$ with $f^{(n)}(0) = 0$ for all $n \geq 0$, and obviously f is not identically null in $\mathbb{R}$.

3.5 Exercises

Exercise 3.1. *Prove that the Cauchy product of the series* $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ *with itself is not a convergent series.*

Exercise 3.2. *If* $\{z_n\}_n$, $\{w_n\}_n$ *are sequences complex numbers, prove, justifying carefully all the steps, that:*

(i) $\sum_{n=1}^{\infty} z_n$ *converges if and only if* $\sum_{n=1}^{\infty} \mathrm{Re}(z_n)$ *and* $\sum_{n=1}^{\infty} \mathrm{Im}(z_n)$ *converge.*

(ii) If $|z_n| \leq |w_n|$ *for all* $n \in \mathbb{N}$, *and* $\sum_{n=1}^{\infty} w_n$ *converges absolutely, then* $\sum_{n=1}^{\infty} z_n$ *converges absolutely too.*

(iii) $\sum_{n=1}^{\infty} z_n$ *converges absolutely if and only if* $\sum_{n=1}^{\infty} \mathrm{Re}(z_n)$ *and* $\sum_{n=1}^{\infty} \mathrm{Im}(z_n)$ *converge absolutely.*

Exercise 3.3. *Use Exercise 3.2 to show that if* $\{z_n\}_n \subset \mathbb{C}$ *is such that there exists* $0 < \theta < \pi/2$ *so that* $|\mathrm{Arg}(z_n)| \leq \theta$ *for all* $n \in \mathbb{N}$, *then*

$$\sum_{n=1}^{\infty} z_n \text{ converges} \iff \sum_{n=1}^{\infty} z_n \text{ converges absolutely.}$$

Suggestion: Use the assumption on $\mathrm{Arg}(z_n)$ *to study the proportion between* $\mathrm{Im}(z_n)$ *and* $\mathrm{Re}(z_n)$.

Exercise 3.4. *Consider the sequence of functions* $\{f_n : \mathbb{C} \to \mathbb{C}\}_n$ *given by*

$$f_n(z) = \frac{n + e^z}{1 + n|z|^2}, \quad n \in \mathbb{N},\ z \in \mathbb{C}.$$

Find $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ *so that* $\{f_n\}_n$ *converges pointwise to* f *on* $\mathbb{C} \setminus \{0\}$. *Then show that this convergence is uniform on each set* $A_R := \{z \in \mathbb{C} : 1/R \leq |z| \leq R\}$, *with* $R > 0$.

Suggestion: We remind that $|e^z| = e^{\mathrm{Re}(z)}$, *see* (2.4.1), *which helps when estimating.*

Exercise 3.5. *For* $\Omega := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, *consider the sequence of functions* $\{f_n : \Omega \to \mathbb{C}\}_n$ *given by* $f_n(z) = \tan(nz)$, $z \in \Omega$, $n \in \mathbb{N}$. *Prove that* f_n *converges pointwise to the constant function* $f(z) = i$ *for all* $z \in \Omega$, *and that the convergence is uniform on each set of the form* $\{z \in \mathbb{C} : \mathrm{Im}(z) \geq \varepsilon\}$ *with* $\varepsilon > 0$.

Exercise 3.6. *Let* $K \subset \mathbb{C}$ *be compact, and* $\{f_n : K \to \mathbb{R}\}_n$ *a sequence of real-valued and continuous functions on* K *such that* $\{f_n\}_n$ *converges pointwise to a continuous* $f : K \to \mathbb{R}$, *and that* $f_n(z) \leq f_{n+1}(z)$ *for all* $z \in K$, $n \in \mathbb{N}$. *Prove that* $\{f_n\}_n$ *converges to* f *uniformly.*

Exercise 3.7. *Let* $A \subset \mathbb{C}$ *be a set, and* $\{f_n\}_n$ *sequence of continuous functions* $f_n : \overline{A} \to \mathbb{C}$ *converging uniformly on* A *(to some function* $f : A \to \mathbb{C}$*). Show that also* $\{f_n\}_n$ *converges uniformly on* $\overline{A}$.

As a corollary, show that if a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$, $\{a_n\} \subset \mathbb{C}$, $z_0 \in \mathbb{C}$, *converges uniformly in some set* A, *then it converges uniformly on* $\overline{A}$.

Exercise 3.8. *Prove that if a power series* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, $\{a_n\}_n \subset \mathbb{C}$, $z_0 \in \mathbb{C}$, *converges uniformly in all of* $\mathbb{C}$, *then there exists* $n_0 \in \mathbb{N}$ *so that* $a_n = 0$ *for all* $n \geq n_0$.

Suggestion: Show first that the uniform convergence of $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ *on* $z \in \mathbb{C}$, *implies that* $\{|a_n(z - z_0)^n|\}_n$ *converges to* 0 *uniformly on* $z \in \mathbb{C}$ *as well.*

Exercise 3.9. *Consider the series of functions* $\sum_{n=1}^{\infty} f_n$, *with* $f_n(x) = \frac{(-1)^n}{n} x^n$ *for all* $x \in A := [0, 1)$, $n \in \mathbb{N}$. *Prove that* $\sum_{n=1}^{\infty} f_n$ *converges absolutely and uniformly on* A *but not absolutely-uniformly on* A. *This amounts to show that:*

(a) *for each* $x \in A$, *the numerical series* $\sum_{n=1}^{\infty} |f_n(x)|$ *converges;*

(b) *the sequence of functions given by the partial sums* $\{\sum_{n=1}^{N} f_n : A \to \mathbb{R}\}_N$ *converges uniformly on* A;

(c) *the sequence of functions given by the partial sums* $\{\sum_{n=1}^{N} |f_n| : A \to \mathbb{R}\}_N$ *does **not** converge uniformly on* A.

Hint: For (b) and (c), it is easier to study the truth/falsity of the corresponding Cauchy property.

Exercise 3.10. *If* $\Omega := \{z \in \mathbb{C} : \mathrm{Re}(z) > 1\}$, *prove the following about the series of functions* $\sum_{n=1}^{\infty} \frac{1}{n^z}$, $z \in \Omega$:

(a) $\sum_{n=1}^{\infty} \frac{1}{n^z}$ *converges absolutely for each* $z \in \Omega$.

(b) $\sum_{n=1}^{\infty} \frac{1}{n^z}$ *converges absolutely–uniformly on each set* $\{z \in \mathbb{C} : \mathrm{Re}(z) \geq 1 + \varepsilon\}$, *with* $\varepsilon > 0$.

(c) $\sum_{n=1}^{\infty} \frac{1}{n^z}$ *does not converge uniformly on* Ω.

Clarification: Here n^z *is the principal* z*-power of* n, *that is,* $n^z = e^{z \,\mathrm{Log}\, n} = e^{z \log n}$. *For parts (b) and (c), it's perhaps easier to prove/disprove the corresponding Cauchy property.*

Exercise 3.11. *Let* $f : \mathbb{R} \to [0, \infty)$ *be a decreasing function with* $\int_0^{\infty} f(x)\,dx = \infty$. *Prove that the series* $\sum_{n=1}^{\infty} f(n) z^n$ *has radius of convergence* $R \leq 1$.

Exercise 3.12. *Determine the disk and the radius of convergence of the following power series.*

a) $\sum_{n=1}^{\infty} \frac{1}{n^2} z^n$ b) $\sum_{n=1}^{\infty} \dfrac{(3+4i)^n}{(1+\frac{i}{n})^{n^3}} (z-i)^n$ c) $\sum_{n=1}^{\infty} \left(\sqrt[n]{n} - 1\right)^n (z-1)^n$ d) $\sum_{n=1}^{\infty} n! z^n$

e) $\sum_{n=1}^{\infty} \left(2 + (-1)^n\right)^n z^n$ f) $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$ g) $\sum_{n=1}^{\infty} (\log n)^2 z^n$ h) $\sum_{n=1}^{\infty} \frac{n^n}{1+2^n n^n} z^n$

i) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}} z^n$ j) $\sum_{n=1}^{\infty} \frac{n^n}{n!} z^n$ k) $\sum_{n=1}^{\infty} z^{n^2}$ l) $\sum_{n=1}^{\infty} 2^n z^{n!}$

m) $\sum_{n=1}^{\infty} \frac{1}{n^2} z^{n!}$ n) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\log n} z^{3n-1}$ o) $\sum_{n=1}^{\infty} 3^{n^2} z^{1+2+\cdots+n}$ p) $\sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^3} z^{3n}$.

If needed, feel free to use Stirling's Theorem: $\lim\limits_{n\to\infty} \frac{n! e^n}{n^n \sqrt{2\pi n}} = 1$.

Exercise 3.13. *For the series a)–p) in Exercise 3.12, study the convergence in the boundary of their disk of convergence.*

Exercise 3.14. *For every* $m \in \mathbb{N}$, *find a power series with disk of convergence* $D(0,1)$ *and so that it diverges precisely at* m *points of the boundary* $\partial D(0,1)$ *of* $D(0,1)$.

Suggestion: Look at Theorem 3.18 and recall that there are precisely m m^{th}*-roots of unity.*

Exercise 3.15. *Let* $\xi \in \mathbb{C}\backslash\{0\}$ *be so that* $\sum_{n=0}^{\infty} a_n \xi^n$ *is convergent. Prove that the series of functions* $\{\sum_{n=0}^{\infty} a_n r^n \xi^n\}_n$, *defined on* $r \in [0,1]$, *converges uniformly on* $[0,1]$. *Then deduce that*

$$\lim_{r \to 1^-} \sum_{n=1}^{\infty} a_n \xi^n r^n = \sum_{n=1}^{\infty} a_n \xi^n.$$

Suggestion: Use Abel's Summation by Parts formula; Lemma 3.17.

Exercise 3.16. *Prove the following power series expansions.*

(a) $\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$ *for all* $z \in \mathbb{C}$.

(b) $\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$ *for all* $z \in \mathbb{C}$.

(c) $\frac{e^z}{1-z} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!}\right) z^n$ *for all* $|z| < 1$.

(d) $\frac{1}{(1-z)^m} = \sum_{n=0}^{\infty} \binom{n+m-1}{m-1} z^n$ *for all* $|z| < 1$*, and a fixed* $m \in \mathbb{N}$.

Suggestion: In (c), use the Cauchy Product, Definition 3.5, Proposition 3.6, of two known series.

Exercise 3.17. *Show that the following series converge in the given sets and calculate their sum.*

(a) $\sum_{n=0}^{\infty} nz^n$, *for* $|z| < 1$.

(b) $\sum_{n=0}^{\infty} n^2 z^n$, *for* $|z| < 1$.

(c) $\sum_{n=0}^{\infty} (2^n - 1) z^n$, *for* $|z| < 1/2$.

(d) $\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n}$, *for* $\theta \in \mathbb{R}$, $0 < |\theta| \leq \pi$.

(e) $\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n}$, *for* $\theta \in \mathbb{R}$, $0 < |\theta| \leq \pi$.

Suggestion: In (d), look at the logarithmic expansion (3.4.8)*, in Example 3.29.*

Exercise 3.18. *Express the following functions as power series centered at* $z = 0$ *and* $z = i$.

$$(a)\ \frac{1}{z-2}. \qquad (b)\ \frac{1}{(z-2)^2}. \qquad (c)\ \frac{1}{z^2 - z - 2}.$$

Suggestion: In (c), use partial fraction decomposition.

Exercise 3.19. *For the function* $f(z) = \sin z$, $z \in \mathbb{C}$, *show that* $\mathcal{Z}(f) = \{k\pi \,:\, k \in \mathbb{Z}\}$ *(the zeros of* f*). This function is analytic in* $\mathbb{C}$*, not identically null, and vanishes in a sequence. Explain why this does not contradict the 2nd Identity Theorem 3.33.*

Exercise 3.20. *Let* $f \in \mathcal{A}(D(0,1))$ *such that* $f(\frac{1}{n^2}) = \frac{n^4}{(n^2-1)^2}$ *for all* $n \in \mathbb{N}$, $n \geq 2$*. Find the explicit formula for* $f(z)$ *for all* $z \in D(0,1)$.

Exercise 3.21. *Let* $f \in \mathcal{A}(D(0,1))$ *such that* $f(\frac{1}{n}) = \frac{n^2}{n^2+1}$ *for all* $n \in \mathbb{N}$, $n \geq 2$*. Find the explicit formula for* $f(z)$ *for all* $z \in D(0,1)$*, and calculate* $f^{(n)}(0)$ *for all* $n \in \mathbb{N}$.

Chapter 4

Complex Integration. The Fundamental Theorems

In this chapter we cover some of the most important theorems of the course, for the class of holomorphic functions, as well as some of their consequences. Here is a brief summary:

- *The Local Cauchy-Goursat Integral Theorems*: holomorphic maps have null integrals over triangles, and over closed paths in convex domains; see Theorem 4.20 and Corollary 4.23.
- *The Cauchy Integral formula*: an expression for a holomorphic map via path-integrals against a rational function (Theorem 4.27 and Corollary 4.29).
- C^∞*-regularity for holomorphic functions and the Cauchy Formulae for the derivatives*: holomorphic maps are infinitely differentiable and the derivatives have integral expressions against rational functions; see Theorem 4.32.
- *Analyticity of holomorphic maps*: holomorphic functions are analytic; see Theorem 4.39.
- *The Morera Theorem*: a characterization of holomorphicity via null-integral condition over triangles; see Theorem 4.36.
- *The Weierstrass Convergence Theorem*: the locally uniform limit of holomorphic functions is holomorphic; see Theorem 4.37.
- *The Maximum Modulus Principles*: the modulus of holomorphic maps attain their maximum in the boundary; see Theorems 4.48, 4.50.
- *The Liouville Theorem*: holomorphic maps in $\mathbb{C}$ are either unbounded or constant; see Theorem 4.45.
- *The Fundamental Theorem of Algebra*: every complex polynomial of degree n has precisely n roots counted with multiplicty; see Theorem 4.47.

4.1 Contour Integration

The contour integral is a type of integral defined for complex-valued functions over a sufficiently suitable class of *curves* or *paths*. In these notes, we will use the terminology *complex path-integration.*

4.1.1 Continuous and Piecewise C^1-paths

In Section 2.3 we briefly discussed the differentiability of curves $\gamma : (a, b) \to \mathbb{C}$ in order to deal with angle-preserving and conformal maps; see Definition 2.40. Here we extend this concept to *piecewise continuous* or *piecewise* C^1 curves.

Definition 4.1 (Path)**.** *A* ***path*** *is any continuous function* $\gamma : [a, b] \to \mathbb{C}$ *with* $a, b \in \mathbb{R}$*, and* $a \leq b$*. Then the* ***trace*** *of the* γ *is the set*

$$\gamma^* := \gamma([a, b]) = \{\gamma(t) \,:\, t \in [a, b]\}.$$

We also say that the path $\gamma : [a, b] \to \mathbb{C}$ *is* ***closed*** *if* $\gamma(a) = \gamma(b)$*.*

Observe that the trace $\gamma^* = \gamma([a, b])$ of a path $\gamma : [a, b] \to \mathbb{C}$ is always a compact set, as the image of the compact set $[a, b]$ by a continuous function; see Proposition 2.25.

There are two basic operations with paths that we will use systematically.

Definition 4.2 (Reverse path and Composition of paths)**.** *If* $\gamma : [a, b] \to \mathbb{C}$ *is a path, the* ***reverse path*** γ^- ***of*** γ *is the path* $\gamma^- : [a, b] \to \mathbb{C}$ *given by*

$$\gamma^-(t) := \gamma(a + b - t), \quad t \in [a, b].$$

In particular, $\gamma^-(a) = \gamma(b)$*,* $\gamma^-(b) = \gamma(a)$ *and* $\gamma^* = (\gamma^-)^*$*.*

Also, if $\gamma_1 : [a, b] \to \mathbb{C}$ *and* $\gamma_2 : [c, d] \to \mathbb{C}$ *are two paths with* $\gamma_1(b) = \gamma_2(c)$*, the* ***concatenation or composition*** *of* γ_1 *and* γ_2 *is the path* $\gamma_1 \star \gamma_2 : [0, 1] \to \mathbb{C}$ *given by*

$$(\gamma_1 \star \gamma_2)(t) = \begin{cases} \gamma_1(a + (b - a)2t) & \text{if } t \in [0, 1/2] \\ \gamma_2(c + (d - c)(2t - 1)) & \text{if } t \in [1/2, 1]. \end{cases} \tag{4.1.1}$$

The continuity of γ_1 *and* γ_2 *and* $\gamma_1(b) = \gamma_2(c)$ *imply the continuity of* $\gamma_1 \star \gamma_2$ *in* $[0, 1]$*.*

Example 4.3. Some instances of paths are the following:

- Given $z, w \in \mathbb{C}$, the segment line $[z, w]$ joining z and w can be described via the path $\gamma : [0, 1] \to \mathbb{C}$, $\gamma(t) = tw + (1 - t)z$ for all $t \in [0, 1]$. Note that this path γ has certain *orientation*, meaning that the initial and terminal points are z and w respectively. The reverse path $\gamma^- : [0, 1] \to \mathbb{C}$ given by $\gamma(t) = tz + (1 - t)w$ for all $t \in [0, 1]$, has initial and terminal points equal to w and z respectively. The traces of these paths are $\gamma^* = (\gamma^-)^* = [z, w]$.

- Given $z_0 \in \mathbb{C}$, $r > 0$, $n \in \mathbb{N}$, the trace γ^* of the path $\gamma : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) = z_0 + re^{int}$, $t \in [0, 2\pi]$, is the circle $S(z_0, r)$. However, γ travels on the circle n times and counterclockwise. The reverse path γ^- of γ is $\gamma^-(t) = z_0 + re^{-int}$, $t \in [0, 2\pi]$, which takes precisely n *loops* on the circle $S(z_0, r)$ but in the clockwise direction.

- The set $\partial Q = \{z \in \mathbb{C} \,:\, \max\{|\operatorname{Re}(z)|, |\operatorname{Im}(z)|\} = 1\}$ is the boundary of the *unit square of* $\mathbb{R}^2$, which can be written as the trace of the concatenation $\gamma_1 \star \gamma_2 \star \gamma_3 \star \gamma_4$ of the paths

$$\gamma_1(t) = 1 + ti, \quad \gamma_2(t) = -t + i, \quad \gamma_3(t) = -1 - ti, \quad \gamma_4(t) = t - i, \quad t \in [-1, 1].$$

The closed path $\gamma := \gamma_1 \star \gamma_2 \star \gamma_3 \star \gamma_4$ travels ∂Q counterclockwise with $1 - i$ as initial (and terminal) point.

According to Definition 2.40, a path $\gamma : [a, b] \to \mathbb{C}$ is differentiable at a point $t \in (a, b)$ when the real functions $\operatorname{Re}(\gamma), \operatorname{Im}(\gamma) : [a, b] \to \mathbb{R}$ are differentiable at t_0. The **one-sided derivatives** of γ and the points a, b are defined in the natural way:

$$\gamma'_+(a) := \lim_{t \to a^+} \frac{\gamma(t) - \gamma(a)}{t - a}, \quad \gamma'_-(b) := \lim_{t \to b^-} \frac{\gamma(t) - \gamma(b)}{t - b}. \tag{4.1.2}$$

This enables us to define paths that are C^1 in $[a, b]$ except at finitely many points.

Definition 4.4 (Piecewise C^1-paths)**.** *We say that* $\gamma : [a, b] \to \mathbb{C}$ *is a* ***C^1-path*** *if* γ *is differentiable at all point of* (a, b)*, the one-sided derivatives* (4.1.2) *exist (meaning that they are complex numbers), and the derivative* $\gamma' : [a, b] \to \mathbb{C}$ *is continuous in* $[a, b]$*.*

More generally, we say that $\gamma : [a, b] \to \mathbb{C}$ *is a* ***piecewise C^1-path*** *if there exist finitely many points* $a = t_1 < t_2 < \cdots < t_{N-1} < t_N = b$ *so that each restricted curve* $\gamma_{|_{[t_n,t_{n+1}]}} : [t_n, t_{n+1}] \to \mathbb{C}$ *is a* C^1*-path for all* $n \in \{1, \ldots, N-1\}$*.*

Remark 4.5. If $\gamma : [a, b] \to \mathbb{C}$ is a C^1-path (resp. piecewise C^1-path), the reverse path $\gamma^- : [a, b] \to \mathbb{C}$ is C^1 (resp. piecewise C^1) as well.

Also, it is clear that if $\gamma_1 : [a, b] \to \mathbb{C}$, $\gamma_2 : [c, d] \to \mathbb{C}$ are piecewise C^1-paths with $\gamma_1(b) = \gamma_2(c)$, the composition $\gamma_1 \star \gamma_2 : [0, 1] \to \mathbb{C}$ is also piecewise C^1.

Furthermore, if $\gamma : [a, b] \to \mathbb{C}$ is piecewise C^1, then there are C^1-paths $\gamma_1, \ldots, \gamma_N$ with $\gamma_n : [a_n, b_n] \to \gamma^*$ for all $n = 1, \ldots, N$, $\gamma_{n-1}(b_{n-1}) = \gamma_n(a_n)$ for all $n = 2, \ldots, N$ and $\gamma \circ \phi = \gamma_1 \star \cdots \star \gamma_N$, where $\phi : [0, 1] \to [a, b]$ is given by $\phi(t) = a + t(b - a)$ for all $t \in [0, 1]$. This tells us that every piecewise C^1-path is, up to a *reparametrisation*, the concatenation of finitely many C^1-paths. Definition 4.7 below will clarify this concept.

Definition 4.6. *If* $\gamma : [a, b] \to \mathbb{C}$ *is a piecewise* C^1*-path, the* ***length*** *of* γ *is*

$$\text{length}(\gamma) := \int_a^b \left|\gamma'(t)\right| \, \mathrm{d}t. \tag{4.1.3}$$

We will sometimes use the notation $\text{length}(\gamma) = \ell(\gamma)$ *to shorten.*

We observe that $[a, b] \ni t \mapsto |\gamma'(t)|$ is continuous (possibly) except at finitely many points, as γ is a piecewise C^1-path, and so the integral (4.1.3) is well-defined.

For example, if $\gamma : [0, 2\pi] \to \mathbb{C}$ is given by $\gamma(t) = e^{int}$ for $t \in [0, 2\pi]$, then $\gamma'(t) = ine^{it}$ and

$$\text{length}(\gamma) = \int_0^{2\pi} \left|ine^{it}\right| \, \mathrm{d}t = \int_0^{2\pi} n \, \mathrm{d}t = 2\pi n.$$

Also, if $\varphi : [0, 1] \to \mathbb{C}$ is given by $\varphi(t) = tw + (1 - t)z$, $t \in [0, 1]$ and $z, w \in \mathbb{C}$, then $\varphi'(t) = w - z$ and $\text{length}(\varphi) = |w - z|$.

Paths with the same trace and *orientation* can be represented by means of many different mappings $\gamma : [a, b] \to \mathbb{C}$. To express this rigorously we need to define the following concept.

Definition 4.7 (Reparametrisation of paths)**.** *Let* $\gamma : [a, b] \to \mathbb{C}$ *be a piecewise* C^1*-path. We say that a piecewise* C^1*-path* $\eta : [c, d] \to \mathbb{C}$ *is a* ***reparametrisation of*** γ *if there exists a* C^1 *and increasing bijection* $\phi : [c, d] \to [a, b]$ *with* $\eta(s) = \gamma(\phi(s))$ *for all* $s \in [c, d]$*.*

In such case, we say that γ *and* η *are* ***equivalent paths.***

A bijection $\phi : [c, d] \to [a, b]$ of class C^1 with $\phi'(t) > 0$ for all $t \in [a, b]$ (as the one appearing in Definition 4.7) is often called an *orientation-preserving change of variables* between $[c, d]$ and $[a, b]$.

An observation that follows from the Change of Variables in the Riemann integral is that two equivalent paths $\gamma : [a, b] \to \mathbb{C}$, $\eta : [c, d] \to \mathbb{C}$ have the same length. Indeed, if ϕ is as in Definition 4.7, then

$$\text{length}(\eta) = \int_c^d \left|\eta'(s)\right| \, \mathrm{d}s = \int_c^d \left|(\gamma \circ \phi)'(s)\right| \, \mathrm{d}s = \int_c^d |\gamma'(\phi(s))| |\phi'(s)| \, \mathrm{d}s = \int_a^b |\gamma'(t)| \, \mathrm{d}t = \text{length}(\gamma).$$

4.1.2 Complex Path-Integral and Arc-Length Integral

The integral of complex functions $h : [a, b] \to \mathbb{C}$ whith both real and imaginary part Riemann-integrable is defined in the obvious way. The integration *along a piecewise* C^1 *path* γ involves the derivative γ'.

Definition 4.8 (Integral and Complex Path-Integral)**.** *Let $h : [a,b] \to \mathbb{C}$ be a function with* $\mathrm{Re}(h), \mathrm{Im}(h) : [a,b] \to \mathbb{R}$ *Riemann integrable in* $[a,b]$*. The* ***integral of*** h ***in*** $[a,b]$ *is defined by*

$$\int_a^b h(t)\,\mathrm{d}t := \int_a^b \mathrm{Re}(h(t))\,\mathrm{d}t + i\int_a^b \mathrm{Im}(h(t))\,\mathrm{d}t. \tag{4.1.4}$$

Now, if $\gamma : [a,b] \to \mathbb{C}$ is a piecewise C^1-path and $f : \gamma^* \to \mathbb{C}$ is continuous, we define the ***path-integral of*** f ***along*** γ *by*

$$\int_\gamma f(z)\,\mathrm{d}z := \int_a^b f(\gamma(t))\cdot\gamma'(t)\,\mathrm{d}t = \int_a^b \mathrm{Re}\left(f(\gamma(t))\gamma'(t)\right)\,\mathrm{d}t + i\int_a^b \mathrm{Im}\left(f(\gamma(t))\gamma'(t)\right)\,\mathrm{d}t. \tag{4.1.5}$$

We remark that the product $f(\gamma(t))\cdot\gamma'(t)$ appearing in (4.1.5) is the complex product. Also, note that the functions $[a,b] \ni t \to \mathrm{Re}\left((f\circ\gamma)(t)\cdot\gamma'(t)\right), \mathrm{Im}\left((f\circ\gamma)(t)\cdot\gamma'(t)\right)$, being continuous on $[a,b]$ except (possibly) at finitely many points, are Riemann-integrable in $[a,b]$ and (4.1.5) is well-defined.

Example 4.9. Let $z_0 \in \mathbb{C}$, $r > 0$, $k \in \mathbb{Z}$, and consider the curve $\gamma : [0,2\pi] \to \mathbb{C}$ given by $\gamma(t) = z_0 + re^{ikt}$, $t \in [0,2\pi]$. If $f(z) = \frac{1}{z-z_0}$ for $z \in S(z_0,r)$, then clearly f is continuous and the path-integral of f on γ is

$$\int_\gamma f(z)\,\mathrm{d}z = \int_0^{2\pi} \frac{1}{\gamma(t)-z_0}\gamma'(t)\,\mathrm{d}t = \int_0^{2\pi} \frac{1}{re^{ikt}}\cdot ikre^{ikt}\,\mathrm{d}t = \int_0^{2\pi} ik\,\mathrm{d}t = 2\pi ki.$$

We will discuss more about this type of integral in Corollary 4.19 below.

The notion of *arc-length integral* over a path is *inspired* by the Definition 4.6 of length.

Definition 4.10 (Arc-Length Integral)**.** *Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise C^1-path and $f : \gamma^* \to \mathbb{C}$ be continuous. The* ***arc-length integral of*** f ***on*** γ *is defined by*

$$\int_\gamma f(z)|\mathrm{d}z| := \int_a^b f(\gamma(t))|\gamma'(t)|\,\mathrm{d}t = \int_a^b \mathrm{Re}(f(\gamma(t)))|\gamma'(t)|\,\mathrm{d}t + i\int_a^b \mathrm{Im}(f(\gamma(t)))|\gamma'(t)|\,\mathrm{d}t. \tag{4.1.6}$$

We now establish some basic properties concerning the previous integrations.

Proposition 4.11. *Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise C^1-path, $h : [a,b] \to \mathbb{R}$ Riemann integrable, and $f, g : \gamma^* \to \mathbb{C}$ continuous. The following properties hold.*

(i) *If $\xi \in \mathbb{C}$, then* $\int_\gamma(\xi f + g) = \xi\int_\gamma f + \int_\gamma g$.

(ii) $\int_{\gamma^-} = -\int_\gamma f$.

(iii) *If $\eta : [c,d] \to \mathbb{C}$ is another piecewise C^1-path equivalent to γ, then*

$$\int_\eta f = \int_\gamma f.$$

(iv) *If $\sigma : [c,d] \to \mathbb{C}$ is another piecewise C^1-path with $\gamma(b) = \sigma(c)$, then*

$$\int_{\gamma\star\sigma} f = \int_\gamma f + \int_\sigma f.$$

(v) $\left|\int_a^b h(t)\,\mathrm{d}t\right| \le \int_a^b |h(t)|\,\mathrm{d}t$.

(vi) $\left|\int_\gamma f(z)\,\mathrm{d}z\right| \le \int_\gamma |f(z)||\mathrm{d}z| \le (\sup\{|f(w)| : w \in \gamma^*\})\,\mathrm{length}(\gamma)$.

Proof.

(i) Clearly the path-integral of a sum of functions is the sum of the two path-integrals, so we may assume that $g = 0$. Now, if $\xi \in \mathbb{R}$, the property holds immediately from the $\mathbb{R}$-linearity of the mappings $\mathrm{Re}, \mathrm{Im} : \mathbb{C} \to \mathbb{R}$ and Definition 4.8. Now, consider the case $\xi = i$. Then,

$$\begin{aligned}\int_\gamma \xi f &= \int_a^b i f(\gamma(t))\gamma'(t)\,\mathrm{d}t = \int_a^b \left[-\operatorname{Im}\left(f(\gamma(t))\gamma'(t)\right) + i \operatorname{Re}\left(f(\gamma(t))\gamma'(t)\right)\right]\,\mathrm{d}t \\ &= \int_a^b -\operatorname{Im}\left(f(\gamma(t))\gamma'(t)\right)\,\mathrm{d}t + i\int_a^b \operatorname{Re}\left(f(\gamma(t))\gamma'(t)\right)\,\mathrm{d}t \\ &= -\int_a^b \operatorname{Im}\left(f(\gamma(t))\gamma'(t)\right)\,\mathrm{d}t + i\int_a^b \operatorname{Re}\left(f(\gamma(t))\gamma'(t)\right)\,\mathrm{d}t \\ &= i\left[\int_a^b \operatorname{Re}\left(f(\gamma(t))\gamma'(t)\right)\,\mathrm{d}t + i\int_a^b \operatorname{Im}\left(f(\gamma(t))\gamma'(t)\right)\,\mathrm{d}t\right] = i\int_\gamma f = \xi\int_\gamma f.\end{aligned}$$

The third equality is from the definition of path-integral (4.1.5), the fourth is a consequence of the (already proven) $\mathbb{R}$-linearity of the path integral, and the fifth is again by definition (4.1.5).

Finally, the arbitrary case $\xi \in \mathbb{C}$ follows from combining the previous cases.

(ii) Since $\gamma^- : [a,b] \to \mathbb{C}$ is defined by $\gamma^-(t) = \gamma(b+a-t)$, one has $(\gamma^-)'(t) = -\gamma'(b+a-t)$ for all $t \in [a,b]$. Therefore,

$$\begin{aligned}\int_{\gamma^-} f &= \int_a^b f(\gamma^-(t)) \cdot (\gamma^-)'(t)\,\mathrm{d}t = -\int_a^b f(\gamma(b+a-t)) \cdot \gamma'(b+a-t)\,\mathrm{d}t \\ &= \int_b^a f(\gamma(s)) \cdot \gamma'(s)\,\mathrm{d}s = -\int_a^b f(\gamma(t)) \cdot \gamma'(t)\,\mathrm{d}t = -\int_\gamma f,\end{aligned}$$

after applying the change of variables $s = b+a-t$ in the integral in the third equality.

(iii) There exists a C^1 bijection $\phi : [c,d] \to [a,b]$ with $\varphi'(s) > 0$ for all $s \in (a,b]$ and $\eta(s) = \gamma(\phi(s))$ for all $s \in [a,b]$. By the Chain Rule we have $\eta'(s) = \gamma'(\phi(s))\phi'(s)$ for all s in $[c,d]$ except for finitely many points. Applying this and the Change of Variables to the Riemann Integrals from path-integral Definition 4.8, we obtain

$$\int_\eta f = \int_c^d f(\eta(s))\eta'(s)\,\mathrm{d}s = \int_c^d f(\gamma(\phi(s)))\gamma'(\phi(s))\phi'(s)\,\mathrm{d}s = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t = \int_\gamma f;$$

where in the middle Riemann integrals, the derivatives are defined except at finitely many points.

(iv) Clearly there exist reparametrisations $\widetilde{\gamma} : [0,1/2] \to \mathbb{C}$, $\widetilde{\eta} : [1/2,1] \to \mathbb{C}$ of γ and η respectively, so that the composite path $\gamma \star \eta : [0,1] \to \mathbb{C}$ defined via formula (4.1.1) satisfies

$$(\gamma \star \eta)(t) = \begin{cases} \widetilde{\gamma}(t) & \text{if } t \in [0,1/2] \\ \widetilde{\eta}(t) & \text{if } t \in [1/2,1]. \end{cases}$$

Using (iii), we get

$$\int_{\gamma\star\eta} f = \int_{\widetilde{\gamma}} f + \int_{\widetilde{\eta}} f = \int_\gamma f + \int_\eta f.$$

(v) The case where $\int_a^b h = 0$ is trivial, so we assume $\int_a^b h \neq 0$. Observe that in such case, the definition (4.1.4) gives

$$\begin{aligned}\left|\int_a^b h(t)\,\mathrm{d}t\right|^2 &= \left|\int_a^b \mathrm{Re}(h(t))\,\mathrm{d}t + i\int_a^b \mathrm{Im}(h(t))\,\mathrm{d}t\right|^2 = \left(\int_a^b \mathrm{Re}(h(t))\,\mathrm{d}t\right)^2 + \left(\int_a^b \mathrm{Im}(h(t))\,\mathrm{d}t\right)^2 \\ &= \int_a^b \left(\int_a^b \mathrm{Re}(h(t))\,\mathrm{d}t\right)\mathrm{Re}(h(s))\,\mathrm{d}s + \int_a^b \left(\int_a^b \mathrm{Im}(h(t))\,\mathrm{d}t\right)\mathrm{Im}(h(s))\,\mathrm{d}s \\ &= \int_a^b \left[\left(\int_a^b \mathrm{Re}(h(t))\,\mathrm{d}t\right)\mathrm{Re}(h(s)) + \left(\int_a^b \mathrm{Im}(h(t))\,\mathrm{d}t\right)\mathrm{Im}(h(s))\right]\,\mathrm{d}s \\ &\leq \int_a^b \left\|\left(\int_a^b \mathrm{Re}(h(t))\,\mathrm{d}t, \int_a^b \mathrm{Im}(h(t))\,\mathrm{d}t\right)\right\| \|(\mathrm{Re}(h(s)), \mathrm{Im}(h(s)))\|\,\mathrm{d}s \\ &= \left\|\left(\int_a^b \mathrm{Re}(h(t))\,\mathrm{d}t, \int_a^b \mathrm{Im}(h(t))\,\mathrm{d}t\right)\right\| \int_a^b \|(\mathrm{Re}(h(s)), \mathrm{Im}(h(s)))\|\,\mathrm{d}s \\ &= \left|\int_a^b h(t)\,\mathrm{d}t\right| \int_a^b |h(t)|\,\mathrm{d}t.\end{aligned}$$

The inequality is due to Cauchy-Schwarz inequality: $u, v \in \mathbb{R}^2$ implies $\langle u, v\rangle \leq \|u\|\|v\|$. The fifth equality is the $\mathbb{R}$-linearity of the integral, and the sixth one is by the definition of complex integral (4.1.4). The above clearly implies the desired estimate.

(vi) To obtain the first inequality we apply property (v) and formula (4.1.6):

$$\left|\int_\gamma f\right| = \left|\int_a^b f(\gamma(t)) \cdot \gamma'(t)\,\mathrm{d}t\right| \leq \int_a^b \left|f(\gamma(t)) \cdot \gamma'(t)\right|\,\mathrm{d}t = \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,\mathrm{d}t = \int_\gamma |f(z)||\mathrm{d}z|.$$

For the second inequality, note that $|f|$ is bounded in γ^*, as $|f| : \gamma^* \to \mathbb{R}$ is continuous and $\gamma^* = \gamma([a, b])$ is compact, as a continuous image of a compact set; see Proposition 2.25. We then apply the definition of arc-length integral and use the linearity and monotonicity of the Riemann-integral (for real-valued functions):

$$\int_\gamma |f(z)||\mathrm{d}z| = \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,\mathrm{d}t \leq \sup_{w\in\gamma^*} |f(w)| \int_a^b |\gamma'(t)|\,\mathrm{d}t = \left(\sup_{w\in\gamma^*} |f(w)|\right) \mathrm{length}(\gamma);$$

where the last equality is the just the definition of length; (4.1.3). □

A consequence of Proposition 4.11 is that one can interchange limit and integral when the convergence is uniform.

Corollary 4.12. *Let $\gamma : [a, b] \to \mathbb{C}$ a piecewise C^1-path, and $\{f_n : \gamma^* \to \mathbb{C}\}_n$ a sequence of continuous functions in γ^* converging uniformly to $f : \gamma^* \to \mathbb{C}$. Then,*

$$\lim_n \int_\gamma f_n(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z.$$

Proof. By Proposition 3.10, one has that $f : \gamma^* \to \mathbb{C}$ is a continuous functions, and so $\int_\gamma f$ is well-defined. Also, since $f_n \to f$ uniformly on γ^*, we actually have that

$$\lim_{n\to\infty} \sup_{w\in\gamma^*} |f_n(w) - f(w)| = 0.$$

Thus, by Proposition 4.11(vi), the above implies

$$\begin{aligned}\lim_{n\to\infty} \left|\int_\gamma f_n(z)\,\mathrm{d}z - \int_\gamma f(z)\,\mathrm{d}z\right| &= \lim_{n\to\infty} \left|\int_\gamma (f_n(z) - f(z))\,\mathrm{d}z\right| \leq \lim_{n\to\infty} \int_\gamma |f_n(z) - f(z)|\,|\mathrm{d}z| \\ &\leq \left(\lim_{n\to\infty} \sup_{w\in\gamma^*} |f_n(w) - f(w)|\right) \mathrm{length}(\gamma) = 0.\end{aligned}$$

□

4.1.3 Primitives and a Fundamental Theorem of Calculus

Definition 4.13. *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be a function We say that $F : \Omega \to \mathbb{C}$ is* ***a primitive of*** f ***in*** Ω *if $F \in \mathcal{H}(\Omega)$ and $F'(z) = f(z)$ for all $z \in \Omega$.*

Remark 4.14. Primitives are unique up to an additive constant when Ω is a domain. Indeed, if $F, G : \Omega \to \mathbb{C}$ are two primitives of $f : \Omega \to \mathbb{C}$, then $(F - G)' = F' - G' = f - f = 0$ on Ω and Corollary 2.37 implies that $F - G$ is constant in Ω.

We now show a version of the Fundamental Theorem of Calculus for the complex path-integral.

Theorem 4.15. *Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \to \mathbb{C}$ be continuous, and $F : \Omega \to \mathbb{C}$ be primitive of f in Ω. If $\gamma : [a, b] \to \Omega$ is a piecewise C^1-path, then*

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)). \tag{4.1.7}$$

In particular, if $\gamma : [a, b] \to \mathbb{C}$ is additionally a closed path, one has $\int_\gamma f(z)\,\mathrm{d}z = 0$.

Proof. Let us first prove (4.1.7) in the case where γ is a C^1-path (not only piecewise C^1-path). As shown in Lemma 2.41 (there, the paths were defined in $(-\varepsilon, \varepsilon)$ but the result is identical for C^1 paths in (a, b)), we have that $F \circ \gamma : (a, b) \to \mathbb{C}$ is differentiable at all point $t \in (a, b)$ with $(F \circ \gamma)'(t) = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t)$. Moreover, since $F' = f$ and f is continuous, we have that $F \circ \gamma \in C^1(a, b)$. Also note that $F \circ \gamma : [a, b] \to \mathbb{C}$ is continuous in $[a, b]$ because so are $\gamma : [a, b] \to \Omega$ and $f : \Omega \to \mathbb{C}$. Thus $\mathrm{Re}(F \circ \gamma), \mathrm{Im}(F \circ \gamma) : [a, b] \to \mathbb{C}$ are continuous in $[a, b]$ and $C^1(a, b)$ (in the real sense), and the Fundamental Theorem of Calculus applies for both functions:

$$\begin{aligned}
\int_\gamma f &= \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t = \int_a^b (F \circ \gamma)'(t)\,\mathrm{d}t = \int_a^b \mathrm{Re}\left((F \circ \gamma)'(t)\right)\mathrm{d}t + i\int_a^b \mathrm{Im}\left((F \circ \gamma)'(t)\right)\mathrm{d}t \\
&= \int_a^b \left(\mathrm{Re}(F \circ \gamma)\right)'(t)\ \mathrm{d}t + i\int_a^b \left(\mathrm{Im}(F \circ \gamma)\right)'(t)\,\mathrm{d}t \\
&= \mathrm{Re}(F \circ \gamma)(b) - \mathrm{Re}(F \circ \gamma)(a) + i\left(\mathrm{Im}(F \circ \gamma)(b) - \mathrm{Im}(F \circ \gamma)(a)\right) = F(\gamma(b)) - F(\gamma(a)).
\end{aligned}$$

This proves the assertion when γ is a C^1-path. If γ is piecewise C^1-path, by Remark 4.5, there are C^1-paths $\gamma_1, \ldots, \gamma_N$ with $\gamma_n : [a_n, b_n] \to \gamma^*$ and $\gamma_{n-1}(b_{n-1}) = \gamma_n(a_n)$ for all $n = 2, \ldots, N$ and $\eta = \gamma_1 \star \cdots \star \gamma_N$, where $\eta : [0, 1] \to \mathbb{C}$ is a reparametrisation of γ. By Proposition 4.11(iv) and the proven C^1 case, we deduce

$$\int_\gamma f = \sum_{n=1}^N \int_{\gamma_n} f = \sum_{n=1}^N \left(F(\gamma_n(b_n)) - F(\gamma_n(a_n))\right) = F(\gamma_N(b_N)) - F(\gamma_1(a_1)) = F(\gamma(b)) - F(\gamma(a)).$$

□

Theorem 4.15 in the particular case of γ equal to a line segment, we deduce the following.

Corollary 4.16. *Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \to \mathbb{C}$ be holomorphic with f' continuous in Ω.*[1] *If $w, \xi \in \Omega$ so that $[w, \xi] \subset \Omega$, then*

$$f(\xi) - f(w) = \int_{[w,\xi]} f'(z)\,\mathrm{d}z = \int_0^1 f'\left(w + t(\xi - w)\right) \cdot \gamma'(t)\,\mathrm{d}t. \tag{4.1.8}$$

Proof. We apply Theorem 4.15 for $\gamma(t) = w + t(\xi - w)$, $t \in [0, 1]$. □

[1]We will see in Theorem 4.32 that assuming that f' is continuous is unnecessary, as holomorphic functions are of class C^∞.

Example 4.17. Let us apply Theorem 4.15 to some concrete examples.

(1) If $f(z) = z^2$, $z \in \mathbb{C}$, then clearly f has the primitive $F(z) = \frac{z^3}{3}$ in $\mathbb{C}$. Therefore, using Theorem 4.15, for any piecewise C^1-path $\gamma : [a,b] \to \mathbb{C}$, one has

$$\int_\gamma z^2 \, \mathrm{d}z = F(\gamma(b)) - F(\gamma(a)) = \frac{\gamma(b)^3}{3} - \frac{\gamma(a)^3}{3}.$$

(2) If $f : \mathbb{C}\setminus\{0\} \to \mathbb{C}$ is given by $f(z) = 1/z$, then f has no primitive in any disk $D(0,r)$. Indeed, if $\gamma : [0, 2\pi] \to \partial D(0,r)$ is given by $\gamma(t) = re^{it}$, then γ is a piecewise C^1-path, and

$$\int_\gamma \frac{1}{z} \, \mathrm{d}z = \int_0^{2\pi} \frac{\gamma'(t)}{\gamma(t)} \, \mathrm{d}t = \int_0^{2\pi} \frac{rie^{it}}{re^{it}} \, \mathrm{d}t = 2\pi i \neq 0.$$

So Theorem 4.15 says that there is no $F \in \mathcal{H}(D(0,r))$ with $F' = f$ on $D(0,r)$. Although the principal logarithm $\mathrm{Log} : \mathbb{C}\setminus(-\infty, 0] \to \mathbb{C}$ satisfies $\mathrm{Log}'(z) = 1/z$, it is not valid as a primitive of $1/z$ in any disk $D(0,r)$, due to the discontinuity of Log at every point $z \in (-\infty, 0]$.

(3) If $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is given by $f(z) = 1/z^2$, then $F(z) = -1/z$ is a primitive of f in $\mathbb{C} \setminus \{0\}$. Therefore, for any piecewise C^1-path $\gamma : [a,b] \setminus \mathbb{C} \setminus \{0\}$, we have, by Theorem 4.15, that

$$\int_\gamma \frac{1}{z^2} \, \mathrm{d}z = \frac{-1}{\gamma(b)} - \frac{-1}{\gamma(a)} = \frac{1}{\gamma(a)} - \frac{1}{\gamma(b)}.$$

4.1.4 Differentiation under the Integral sign

Using Theorem 4.15, we prove the following result on differentiation under the integral sign will permit us to handle several technicalities in the coming sections.

Theorem 4.18. *Let $\Omega \subset \mathbb{C}$ be open, $\gamma : [a,b] \to \mathbb{C}$ a piecewise C^1-path, and $\varphi : \gamma^* \times \Omega \to \mathbb{C}$ a continuous functions such that for every $\xi \in \gamma^*$ the function $\Omega \ni z \mapsto \varphi(\xi, z)$ is holomorphic in Ω, and $\gamma^* \times \Omega \ni (\xi, z) \mapsto \frac{\partial \varphi}{\partial z}(\xi, z)$ is continuous in $\gamma^* \times \Omega$. Then, the function $F : \Omega \to \mathbb{C}$ given by*

$$F(z) = \int_\gamma \varphi(\xi, z) \, \mathrm{d}\xi, \quad z \in \Omega,$$

is holomorphic in Ω and

$$F'(z) = \int_\gamma \frac{\partial \varphi}{\partial z}(\xi, z) \, \mathrm{d}\xi, \quad z \in \Omega.$$

Proof. Fix $z_0 \in \Omega$ and $r > 0$ so that $\overline{D}(z_0, r) \subset \Omega$. For any $w \in D(z_0, r)$, consider the C^1-path $\ell_w : [0,1] \to D(z_0, r)$ defined by $\ell_w(t) = z_0 + t(w - z_0)$ for all $t \in [0,1]$, and apply Theorem 4.15 for the function $D(z_0, r) \ni z \mapsto \frac{\partial \varphi}{\partial z}(\xi, z)$ along this path to obtain

$$\varphi(\xi, w) - \varphi(\xi, z_0) = \int_{\ell_w} \frac{\partial \varphi}{\partial z}(\xi, z) \, \mathrm{d}z = \int_0^1 \frac{\partial \varphi}{\partial z}(\xi, \ell_w(t))(w - z_0) \, \mathrm{d}t,$$

for all $\xi \in \gamma^*$. Taking into account this identity, we see that the differentiability of F at z_0 amounts to study the existence of the following limits, as $w \in D(z_0, r) \setminus \{z_0\}$:

$$\lim_{w \to z_0} \frac{F(w) - F(z_0)}{w - z_0} = \lim_{w \to z_0} \frac{\int_\gamma (\varphi(\xi, w) - \varphi(\xi, z_0)) \, \mathrm{d}\xi}{w - z_0} = \lim_{w \to z_0} \int_\gamma \left(\int_0^1 \frac{\partial \varphi}{\partial z}(\xi, \ell_w(t)) \, \mathrm{d}t \right) \mathrm{d}\xi. \tag{4.1.9}$$

So, let $\{w_n\}_n \subset D(z_0, r) \setminus \{z_0\}$ be a sequence converging to z_0. We now claim that the sequence of functions $\{h_n\}_n$ converges uniformly to h on $\gamma^* \times [0,1]$, where h_n and h are define by:

$$\gamma^* \times [0,1] \ni (\xi, t) \mapsto h_n(\xi, t) := \frac{\partial \varphi}{\partial z}(\xi, \ell_{w_n}(t)), \quad n \in \mathbb{N}$$

$$\gamma^* \times [0,1] \ni (\xi, t) \mapsto h(\xi, t) := \frac{\partial \varphi}{\partial z}(\xi, z_0).$$

Indeed, suppose, seeking a contradiction, that h_n does not converge uniformly to h on the set $\gamma^* \times [0,1]$. Then there exist $\varepsilon > 0$, a subsequence $\{n_k\}_k$ and sequences $\{\xi_k\}_k \subset \gamma^*$, $\{t_k\}_k \subset [0,1]$ for which

$$\left|\frac{\partial \varphi}{\partial z}(\xi_k, z_0 + t_k(w_{n_k} - z_0)) - \frac{\partial \varphi}{\partial z}(\xi_k, z_0)\right| = |h_{n_k}(\xi_k, t_k) - h(\xi_k, t_k)| \geq \varepsilon, \quad \text{for all} \quad k \in \mathbb{N}. \tag{4.1.10}$$

By the compactness of the sets γ^*, $[0,1]$ and $\overline{D}(z_0, r)$, we can assume, passing to subsequences that $\xi_k \to \xi \in \gamma^*$, $t_k \to t \in [0,1]$ and $w_{n_k} \to z_0$; see Bolzano-Weierstrass Theorem 2.12. By the assumption, $\gamma^* \times \Omega \ni (\xi, w) \mapsto \frac{\partial \varphi}{\partial z}(\xi, w)$ is continuous, and so letting $k \to \infty$ in (4.1.10) leads to a contradiction. We have proven that

$$\lim_{n\to\infty} \sup_{\xi\in\gamma^*} \sup_{t\in[0,1]} \left|\frac{\partial \varphi}{\partial z}(\xi, \ell_{w_n}(t)) - \frac{\partial \varphi}{\partial z}(\xi, z_0)\right| = 0.$$

By Proposition 4.11(v), this clearly shows

$$\lim_{n\to\infty} \sup_{\xi\in\gamma^*} \left|\int_0^1 \frac{\partial \varphi}{\partial z}(\xi, \ell_{w_n}(t))\, \mathrm{d}t - \frac{\partial \varphi}{\partial z}(\xi, z_0)\right| = 0. \tag{4.1.11}$$

For every n, consider the mapping $\gamma^* \ni \xi \mapsto g_n(\xi) := \int_0^1 \frac{\partial \varphi}{\partial z}(\xi, \ell_{w_n}(t))\, \mathrm{d}t$ and apply Proposition 4.11(v) to get, for every $\xi, \xi' \in \gamma^*$:

$$|g(\xi) - g(\xi')| \leq \int_0^1 \left|\frac{\partial \varphi}{\partial z}(\xi, \ell_{w_n}(t)) - \frac{\partial \varphi}{\partial z}(\xi', \ell_{w_n}(t))\right| \mathrm{d}t \leq \sup_{u\in\overline{D}(z_0,r)} \left|\frac{\partial \varphi}{\partial z}(\xi, u) - \frac{\partial \varphi}{\partial z}(\xi', u)\right|.$$

For every $\xi \in \gamma^*$, the last term tends to 0 as $\gamma^* \ni \xi' \to \xi$, as otherwise Theorem 2.12 would give $\varepsilon > 0$ and sequences $\gamma^* \supset \{\xi_n\} \to \xi$, $\overline{D}(z_0, r) \supset \{u_n\}_n \to u \in \overline{D}(z_0, r)$, and the contradiction:

$$0 = \left|\frac{\partial \varphi}{\partial z}(\xi, u) - \frac{\partial \varphi}{\partial z}(\xi, u)\right| = \liminf_{n\to\infty} \left|\frac{\partial \varphi}{\partial z}(\xi, u_n) - \frac{\partial \varphi}{\partial z}(\xi_n, u_n)\right| \geq \inf_{n\in\mathbb{N}} \left|\frac{\partial \varphi}{\partial z}(\xi, u_n) - \frac{\partial \varphi}{\partial z}(\xi_n, u_n)\right| \geq \varepsilon;$$

the second equality due to the continuity of $\frac{\partial \varphi}{\partial z}$ on $\gamma^* \times \Omega$. Thus $\{g_n : \gamma^* \to \mathbb{C}\}_n$ are continuous functions which, by (4.1.12), converge to $\frac{\partial \varphi}{\partial z}(\xi, z_0)$ uniformly on $\xi \in \gamma^*$. Proposition 4.12 then yields

$$\lim_{n\to\infty} \int_\gamma \left(\int_0^1 \frac{\partial \varphi}{\partial z}(\xi, \ell_{w_n}(t))\right) \mathrm{d}\xi = \int_\gamma \frac{\partial \varphi}{\partial z}(\xi, z_0)\, \mathrm{d}\xi. \tag{4.1.12}$$

Since $\{w_n\}_n$ is any sequence in $D(z_0, r) \setminus \{z_0\}$ converging to z_0, we can conclude from the combination of (4.1.10) and (4.1.12) that

$$\lim_{w\to z_0} \frac{F(w) - F(z_0)}{w - z_0} = \int_\gamma \frac{\partial \varphi}{\partial z}(\xi, z_0)\, \mathrm{d}\xi.$$

□

Let us apply Theorem 4.18 to very important particular situation, which we will use to prove the *Cauchy Integral Formula.*

Corollary 4.19. *Let $z_0 \in \mathbb{C}$, $r > 0$, $k \in \mathbb{Z}$, and let $\gamma : [0, 2\pi] \to \mathbb{C}$ the path $\gamma(t) = z_0 + re^{ikt}$, $t \in [0, 2\pi]$. Then*

$$\int_\gamma \frac{1}{\xi - z}\, \mathrm{d}\xi = 2\pi k i, \quad \textit{for all} \quad z \in D(z_0, r).$$

Proof. Clearly $\gamma^* = S(z_0, r)$, and we define $\Omega := \mathbb{C} \setminus S(z_0, r)$ and the function $\varphi : \gamma^* \times \Omega \to \mathbb{C}$ by

$$\varphi(\xi, z) = \frac{1}{\xi - z}, \quad (\xi, z) \in \gamma^* \times \Omega.$$

For each $\xi \in \gamma^*$, the function $\Omega \ni z \mapsto \varphi(\xi, z)$ is holomorphic in Ω, and

$$\frac{\partial \varphi}{\partial z}(\xi, z) = \frac{1}{(\xi - z)^2}, \quad (\xi, z) \in \gamma^* \times \Omega,$$

is continuous in $\gamma^* \times \Omega$. By Theorem 4.18, the function

$$F(z) = \int_\gamma \varphi(\xi, z)\, \mathrm{d}\xi = \int_\gamma \frac{1}{\xi - z}\, \mathrm{d}\xi, \quad z \in \Omega,$$

is holomorphic in Ω with

$$F'(z) = \int_\gamma \frac{\partial \varphi}{\partial z}(\xi, z)\, \mathrm{d}\xi = \int_\gamma \frac{1}{(\xi - z)^2}\, \mathrm{d}\xi, \quad z \in \Omega. \tag{4.1.13}$$

We claim that $F'(z) = 0$ for all $z \in D(z_0, r)$. Indeed, given $z \in D(z_0, r)$, let $\varepsilon > 0$ be so that $z \in D(z_0, r - \varepsilon)$, and define $U = D(z_0, r + \varepsilon) \setminus \overline{D}(z_0, r - \varepsilon)$. Clearly $g(\xi) := \frac{1}{(\xi - z)^2}$, $\xi \in U$, defines a continuous function which has a holomorphic primitive $G(\xi) = \frac{1}{z - \xi}$, $\xi \in U$. Since the path $\gamma : [0, 2\pi] \to S(z_0, r)$ is closed and takes values in U, we can apply Theorem 4.15 to deduce that

$$\int_\gamma \frac{1}{(\xi - z)^2}\, \mathrm{d}\xi = 0,$$

and then (4.1.13) implies that $F'(z) = 0$. Since $D(z_0, r)$ is open and connected, by Corollary 2.37, F is constant in $D(z_0, r)$. But then Example 4.9 shows that

$$F(z) = F(z_0) = \int_\gamma \frac{1}{\xi - z_0}\, \mathrm{d}\xi = 2\pi k i, \quad z \in D(z_0, r).$$

□

4.2 The Cauchy-Integral Theorem

4.2.1 The Cauchy-Goursat Theorem in a triangle

Naturally, by a triangle $T \subset \mathbb{C}$ we understand the union of three segment lines $[a, b], [b, c], [c, a]$; where $a, b, c \in \mathbb{C}$ are not align in the plane. Note that here T is only the boundary of the solid triangle Δ generated by a, b, c. Since Δ is clearly the *convex envelope of* T, given any triangle T, we will denote by $\mathrm{co}(T)$ the corresponding solid triangle. In other words, $T = \partial(\mathrm{co}(T))$; recall the Definition 2.5 of boundary. Also, we will always assume (without loss of generality in the next theorems) that the segments $[u, v]$ forming the edges of T are parametrized by the C^1-path $[0, 1] \ni t \mapsto u + t(v - u)$.

Theorem 4.20 (Cauchy-Goursat). *Let $\Omega \subset \mathbb{C}$ be open, T be a triangle such that $\mathrm{co}(T) \subset \Omega$, $z_0 \in \Omega$, and $f : \Omega \to \mathbb{C}$ continuous in Ω and holomorphic in $\Omega \setminus \{z_0\}$. Then*

$$\int_T f(z)\, \mathrm{d}z = 0.$$

Proof. Let $[a, b], [b, c], [c, a]$ be the edges of T. We need to consider several cases depending on the location of the distinguished point z_0.

<u>**Case 1:**</u> $z_0 \notin \mathrm{co}(T)$. We will compare $\int_T f$ with the integral $\int_{T_n} f$ over smaller and smaller subtriangles T_n of $\mathrm{co}(T)$. Then, in those T_n, we will compare f with its first-degree Taylor polynomial, which admits a holomorphic primitive, and so its integral over any triangle is null.

Define $T_0 := T$ and join the midpoints of the edges $[a,b],[b,c],[c,a]$ of T by three segment lines, which are naturally contained in $\mathrm{co}(T)$. These three lines form a triangle T_0^4, and moreover split $\mathrm{co}(T)$ into the convex envelopes of four triangles $T_0^1, T_0^2, T_0^3, T_0^4$. Parametrizing all these triangles with segment lines following the same orientation (clockwise/counterclockwise) as T_0, we claim that

$$\int_T f = \int_{T_0} f = \sum_{j=1}^{4} \int_{T_0^j} f. \tag{4.2.1}$$

Indeed, denote by ℓ_k^j, $k = 1,2,3$, $j = 1,2,3,4$ the kth edge of T_0^j, with orientations determined by the orientation of T_0^j. Denote $\mathcal{L} = \{(k,j) \in \{1,2,3\} \times \{1,2,3,4\} : \ell_k^j \subset T\}$. Then Proposition 4.11(iv) gives

$$\sum_{j=1}^{4} \int_{T_0^j} f = \sum_{j=1}^{4}\sum_{k=1}^{3} \int_{\ell_k^j} f = \sum_{(k,j)\in\mathcal{L}} \int_{\ell_k^j} f + \sum_{(k,j)\notin\mathcal{L}} \int_{\ell_k^j} f = \int_T f + \sum_{(k,j)\notin\mathcal{L}} \int_{\ell_k^j} f.$$

Now, the set $\{1,2,3\}\times\{1,2,3,4\}\setminus\mathcal{L}$ corresponds to the segments ℓ_k^j that are in the interior region of the triangle T_0. These are precisely 6 segments, and more precisely the segments $\ell_{1,4}, \ell_{2,4}, \ell_{3,4}$ (edges of T_0^4) along with their reverse paths $\ell_{1,4}^-, \ell_{2,4}^-, \ell_{3,4}^-$. So, by Proposition 4.11(ii), we see that

$$\sum_{(k,j)\notin\mathcal{L}} \int_{\ell_k^j} f = \sum_{k=1}^{3} \left(\int_{\ell_{k,4}} f + \int_{\ell_{k,4}^-} f \right) = \sum_{k=1}^{3} \left(\int_{\ell_{k,4}} f - \int_{\ell_{k,4}} f \right) = 0,$$

and then (4.2.1) follows. By the triangle inequality, there must exist at least some $j \in \{1,2,3,4\}$, giving raise to a triangle T_0^j which we denote by T_1 from now on, so that

$$\mathrm{co}(T_1) \subset \mathrm{co}(T_0), \quad \mathrm{length}(T_1) = \frac{1}{2}\mathrm{length}(T_0), \quad \text{and} \quad \left|\int_{T_1} f\right| \geq \frac{1}{4}\left|\int_{T_0} f\right|.$$

Repeating the same tiling procedure for T_1 in place of T_0, we obtain a new triangle T_2 with the properties

$$\mathrm{co}(T_2) \subset \mathrm{co}(T_1), \quad \mathrm{length}(T_2) = \frac{1}{4}\mathrm{length}(T_0), \quad \text{and} \quad \left|\int_{T_2} f\right| \geq \frac{1}{4^2}\left|\int_{T_0} f\right|.$$

By induction we obtain a sequence of triangles $\{T_n\}_{n=0}^{\infty}$ with the properties

$$\mathrm{co}(T_n) \subset \mathrm{co}(T_{n-1}), \quad \mathrm{length}(T_n) = \frac{1}{2^n}\mathrm{length}(T_0), \quad \text{and} \quad \left|\int_{T_n} f\right| \geq \frac{1}{4^n}\left|\int_{T_0} f\right|, \quad n \in \mathbb{N}. \tag{4.2.2}$$

By Lemma 2.15, there exists a unique $w_0 \in \bigcap_{n=0}^{\infty} \mathrm{co}(T_n) \subset \Omega \setminus \{z_0\}$. And f is differentiable at w_0, so, given $\varepsilon > 0$ we can find $r > 0$ with $D(w_0, r) \subset \Omega$ and

$$|f(w) - f(w_0) - f'(w_0)(w - w_0)| \leq \varepsilon |w - w_0|, \quad w \in D(w_0, r). \tag{4.2.3}$$

Since $\lim_{n\to\infty} \mathrm{diam}(\mathrm{co}(T_n)) = 0$ by (4.2.2) and $w_0 \in \bigcap_{n=0}^{\infty} \mathrm{co}(T_n)$, we can find $n_0 \in \mathbb{N}$ so that $|w_0 - w| < r$ for all $w \in \mathrm{co}(T_n)$, $n \geq n_0$. Now, the polynomial $w \mapsto f(w_0) + f'(w_0)(w - w_0)$ is continuous and clearly has a primitive (in all of $\mathbb{C}$). Since triangles are closed piecewise C^1-paths, Theorem 4.15 tells us that

$$\int_{T_n} \big(f(w_0) + f'(w_0)(w - w_0)\big)\, \mathrm{d}w = 0, \quad \text{for all} \quad n \in \mathbb{N} \cup \{0\}. \tag{4.2.4}$$

Applying first (4.2.4), then Proposition 4.11(vi), then (4.2.3), then Proposition 4.11(vi) again, and finally the second property of (4.2.2), we obtain, for $n \geq n_0$,

$$\begin{aligned}\left|\int_{T_n} f\right| &= \left|\int_{T_n} \big(f(w) - f(w_0) - f'(w_0)(w-w_0)\big)\ \mathrm{d}w\right| \\ &\leq \int_{T_n} |f(w) - f(w_0) - f'(w_0)(w-w_0)||\mathrm{d}w| \\ &\leq \int_{T_n} \varepsilon|w-w_0||\mathrm{d}w| \leq \varepsilon\,(\mathrm{length}(T_n))^2 \leq \frac{(\mathrm{length}(T_0))^2}{4^n}\varepsilon.\end{aligned}$$

By the third property of (4.2.2), we may conclude

$$\left|\int_T f\right| = \left|\int_{T_0} f\right| \leq 4^n \left|\int_{T_n} f\right| \leq 4^n \frac{(\mathrm{length}(T_0))^2}{4^n}\varepsilon = (\mathrm{length}(T_0))^2\,\varepsilon,$$

implying that $\int_T f = 0$ because $\varepsilon > 0$ was arbitrary.

Case 2: $z_0 \in \{a,b,c\}$. Without loss of generality, we can assume $z_0 = a$. By the continuity of f in the compact set $\mathrm{co}(T)$, there exists $M > 0$ so that $|f(z)| \leq M$ for all $z \in \mathrm{co}(T)$; see Proposition 2.25. Given $\varepsilon > 0$ we can find points $\xi_1 \in [a,b]$ and $\xi_2 \in [c,a]$ such that if T_1 denotes the triangle with edges $[a,\xi_1],[\xi_1,\xi_2],[\xi_2,a]$, then $\mathrm{length}(T_1) \leq \varepsilon/M$. We also define the triangles T_2, with edges $[\xi_1,b],[b,\xi_2],[\xi_2,\xi_1]$, and T_3, with edges $[b,c],[c,\xi_2],[\xi_2,b]$. We again consider the orientations in T_1,T_2,T_3 determined from the one in T. As in **Case 1**, we use Proposition 4.11(ii) to write

$$\begin{aligned}\sum_{j=1}^{3}\int_{T_j} f &= \int_{[a,\xi_1]} f + \int_{[\xi_1,\xi_2]} f + \int_{[\xi_2,a]} f + \int_{[\xi_1,b]} f + \int_{[b,\xi_2]} f + \int_{[\xi_2,\xi_1]} f + \int_{[b,c]} f + \int_{[c,\xi_2]} f + \int_{[\xi_2,b]} f \\ &= \int_{[a,\xi_1]} f + \int_{[\xi_1,\xi_2]} f + \int_{[\xi_2,a]} f + \int_{[\xi_1,b]} f + \int_{[b,\xi_2]} f - \int_{[\xi_1,\xi_2]} f + \int_{[b,c]} f + \int_{[c,\xi_2]} f - \int_{[b,\xi_2]} f \\ &= \int_{[a,\xi_1]} f + \int_{[\xi_1,b]} f + \int_{[b,c]} f + \int_{[c,\xi_2]} f + \int_{[\xi_2,a]} f = \int_T f. \end{aligned} \tag{4.2.5}$$

Since $z_0 \notin T_2 \cup T_3$, by **Case 1**, we have

$$\int_{T_2} f = \int_{T_3} f = 0.$$

Also, by Proposition 4.11(vi), we can estimate

$$\left|\int_{T_1} f\right| \leq \left(\sup_{w\in T_1} |f(w)|\right) \mathrm{length}(T_1) \leq \left(\sup_{w\in \mathrm{co}(T)} |f(w)|\right) \mathrm{length}(T_1) \leq M\frac{\varepsilon}{M} = \varepsilon.$$

Using these two observations in (4.2.5), we can conclude

$$\left|\int_T f\right| \leq \left|\int_{T_1} f\right| \leq \varepsilon,$$

and since $\varepsilon > 0$ we get that $\int_T f = 0$.

Case 3: $z_0 \in \mathrm{co}(T) \setminus \{a,b,c\}$. In this case we can form triangles T_j, $j = 1,2,3$ with $\mathrm{co}(T_j) \subset \mathrm{co}(T)$ and so that z_0 is a vertex of each of them. Applying **Case 2**, we have $\int_{T_j} f = 0$ for $j = 1,2,3$, and providing the triangles with the suitable orientation, we have

$$\int_T f = \sum_{j=1}^{3}\int_{T_j} f = 0.$$

□

4.2.2 Cauchy Theorem in Convex Sets. Existence of Primitives

A first application of Theorem 4.20 is the existence of primitives of holomorphic mappings in convex domains. We first prove, without utilizing Theorem 4.20, a version for merely continuous functions that will be useful later on.

Lemma 4.21. *$\Omega \subset \mathbb{C}$ be open and convex, and let $f : \Omega \to \mathbb{C}$ be continuous in Ω and with the property that, for every triangle T with $\mathrm{co}(T) \subset \Omega$, one has*

$$\int_T f = 0. \tag{4.2.6}$$

Then there exists $F : \Omega \to \mathbb{C}$ holomorphic with $F' = f$ in Ω.

Proof. If we fix a point $w_0 \in \Omega$, then the segment lines $[w_0, w]$ are entirely contained in Ω by the convexity of Ω. This enables to define our primitive F by the formula

$$F(w) = \int_{[w_0,w]} f, \quad w \in \Omega. \tag{4.2.7}$$

Here, we understand that the integral is along the path $\gamma(t) = w_0 + t(w - w_0)$, $t \in [0, 1]$. Because f is continuous on Ω, the function F is well-defined. Let us now fix $w \in \Omega$ and prove that $F'(w) = f(w)$. Given $\varepsilon > 0$, we can find $r > 0$ so that $D(w, r) \subset \Omega$ and

$$|f(z) - f(w)| < \varepsilon, \quad z \in D(w, r). \tag{4.2.8}$$

Then, if $\xi \in D(w, r)$, we define T_ξ as the triangle with edges $[w_0, w], [w, \xi], [\xi, w_0]$. By the convexity of Ω, these segments lines are contained in Ω, as well as $\mathrm{co}(T_\xi) \subset \Omega$. By the definition of F in (4.2.7) we get

$$F(\xi) - F(w) = \int_{[w_0,\xi]} f - \int_{[w_0,w]} f = \int_{[w,\xi]} f - \left(\int_{[w_0,w]} f + \int_{[w,\xi]} f + \int_{[\xi,w_0]} f \right) = \int_{[w,\xi]} f - \int_{T_\xi} f;$$

and by the assumption (4.2.6) applied for T_ξ, this means that $F(\xi) - F(w) = \int_{[w,\xi]} f$. Then we can write

$$\begin{aligned} |F(\xi) - F(w) - f(w)(\xi - w)| &= \left| \int_{[w,\xi]} (f(z) - f(w)) \, \mathrm{d}z \right| \le \left(\sup_{z\in[w,\xi]} |f(z) - f(w)| \right) \mathrm{length}([w, \xi]) \\ &\le \left(\sup_{z\in D(w,r)} |f(z) - f(w)| \right) |\xi - w| \le \varepsilon|\xi - w|; \end{aligned}$$

where we employed (4.2.8) in the last inequality. This shows that F is differentiable at w, with $F'(w) = f(w)$. □

Theorem 4.22 (Primitives in Convex Domains)**.** *Let $\Omega \subset \mathbb{C}$ be open and convex, let $z_0 \in \Omega$, and let $f : \Omega \to \mathbb{C}$ be continuous in Ω and holomorphic in $\Omega \setminus \{z_0\}$. Then there exists $F : \Omega \to \mathbb{C}$ holomorphic with $F' = f$ in Ω*

Proof. By Theorem 4.20, $\int_T f = 0$ for every triangle T with $\mathrm{co}(T) \subset \Omega$. Thus, Lemma 4.21 implies the existence of $F : \Omega \to \mathbb{C}$ with $F' = f$ in Ω. □

As a consequence of Theorem 4.22, we can show a more general version of the Cauchy-Gourset Theorem 4.20, where we can replace the triangle with any path contained in a convex domain.

Corollary 4.23 (Cauchy Theorem in a Convex Domain)**.** *Let $\Omega \subset \mathbb{C}$ be open and convex, $z_0 \in \Omega$, and let $f : \Omega \to \mathbb{C}$ be continuous in Ω and holomorphic in $\Omega \setminus \{z_0\}$. Then, for every closed piecewise C^1-path $\gamma : [a, b] \to \Omega$, one has*

$$\int_\gamma f = 0.$$

Proof. By Theorem 4.22, we can find $F \in \mathcal{H}(\Omega)$ with $F' = f$ on Ω. Thus, applying Theorem 4.15 to F and the closed path γ, we obtain that $\int_\gamma f = 0$. □

4.3 The Cauchy Integral Formulae

The main result of this section is the *Cauchy Integral Formula* for a holomorphic function $f \in \mathcal{H}(\Omega)$:

$$f(z) = \frac{1}{2\pi i} \int_{S(z,r)} \frac{f(w)}{w - z} \, \mathrm{d}w, \quad z \in \Omega;$$

where the integral is understood along the path that *travels* the circle $S(z, r)$ only once and with counterclockwise orientation; see Corollary 4.28 for the precise statement. This formula will have numerous implications in holomorphic functions that we will show in the following sections.

4.3.1 The Winding Numbers

The Cauchy Integral formula can be generalized to path-integrals over more general closed paths than the circle. This is done via the *winding numbers.*

Definition 4.24. *Let $\gamma : [a, b] \to \mathbb{C}$ be a closed piecewise C^1-path and let $z \in \mathbb{C} \setminus \gamma^*$. We define the* ***winding number of γ around*** *z by*

$$W(\gamma, z) := \frac{1}{2\pi i} \int_{\gamma} \frac{1}{w - z} \, \mathrm{d}w. \tag{4.3.1}$$

These numbers can be interpreted as the number of times that a path travels counterclockwise around a point. Let us examine an elementary example.

Example 4.25. Given $z_0 \in \mathbb{C}$ and $k \in \mathbb{Z}$, consider the path $\gamma_k : [0, 2\pi] \to \mathbb{C}$, $\gamma_k(t) = z_0 + re^{ikt}$. By Corollary 4.19, for every $z \in D(z_0, r)$, one has

$$W(\gamma, z) = \frac{1}{2\pi i} \int_{\gamma_k} \frac{1}{w - z} \, \mathrm{d}w = \frac{2\pi k i}{2\pi i} = k.$$

We collect some properties of the winding numbers in the following proposition.

Proposition 4.26. *Let $\gamma : [a, b] \to \mathbb{C}$ a closed and piecewise C^1-path, and $z \in \mathbb{C} \setminus \gamma^*$. The following properties hold.*

(i) If $\gamma^- : [a, b] \to \mathbb{C}$ is the reverse path of γ, then $W(\gamma^-, z) = -W(\gamma, z)$.

(ii) If $\sigma : [c, d] \to \mathbb{C}$ is another closed and piecewise C^1-path, with $\gamma(b) = \sigma(c)$, and $z \in \mathbb{C}\setminus\gamma^* \cup \sigma^*$, then

$$W(\gamma \star \sigma, z) = W(\gamma, z) + W(\sigma, z).$$

(iii) $W(\gamma, z) \in \mathbb{Z}$.

(iv) If z, w are in the same connected component of $\mathbb{C} \setminus \gamma^*$, then $W(\gamma, z) = W(\gamma, w)$.

(v) If z is in the unbounded connected component of $\mathbb{C} \setminus \gamma^*$, then $W(\gamma, z) = 0$.

Proof.

(i), (ii) They are immediate from Proposition 4.11 and Definition 4.24 of winding numbers.

(iii) We define the function $h : [a, b] \to \mathbb{C}$ by

$$h(t) := \int_a^t \frac{\gamma'(s)}{\gamma(s) - z} \, \mathrm{d}s, \quad t \in [a, b].$$

Because γ' is continuous (possibly) except at finitely many points, we get (by virtue of the Fundamental Theorem of Calculus) that h is piecewise C^1 in $[a, b]$, and

$$h'(t) = \frac{\gamma'(t)}{\gamma(t) - z}, \quad t \in [a, b] \setminus \{t_1, \ldots, t_N\}.$$

We also define $H : [a,b] \to \mathbb{C}$ by the formula $H(t) := (\gamma(t) - z)e^{-h(t)}$, $t \in [a,b]$. By differentiating we get

$$H'(t) = \gamma'(t)e^{-h(t)} - (\gamma(t) - z)\, e^{-h(t)}h'(t) = 0$$

for all $t \in [a,b] \setminus \{t_1, \ldots, t_N\}$. Since H is continuous in $[a,b]$, this implies that H is constant in $[a,b]$. Thus

$$\gamma(a) - z = (\gamma(a) - z)e^0 = (\gamma(a) - z)e^{-h(a)} = H(a) = H(b) = (\gamma(b) - z)e^{-h(b)};$$

which together with $\gamma(a) = \gamma(b)$ yields that $e^{-h(b)} = 1$. But according to Theorem 2.49, this means that $h(b) \in 2\pi i\mathbb{Z}$, and so we have we have that

$$W(\gamma, z) = \frac{1}{2\pi i}\int_\gamma \frac{1}{w - z}\,\mathrm{d}z = \frac{1}{2\pi i}\int_a^b \frac{\gamma'(s)}{\gamma(s) - z}\,\mathrm{d}s = \frac{h(b)}{2\pi i} \in \mathbb{Z}.$$

(iv) We first claim that the function $\mathbb{C} \setminus \gamma^* \ni z \mapsto W(\gamma, z)$ is continuous. This can be justified for example by considering the function

$$\gamma^* \times \mathbb{C} \setminus \gamma^* \ni (w, z) \mapsto \varphi(w, z) := \frac{1}{w - z},$$

so that, for every $w \in \gamma^*$, the mapping $\mathbb{C} \setminus \gamma^* \ni z \mapsto \varphi(w, z)$ is holomorphic, and $\frac{\partial\varphi}{\partial z}$

$$\gamma^* \times \mathbb{C} \setminus \gamma^* \ni (w, z) \mapsto \frac{\partial\varphi}{\partial z}(w, z) = \frac{1}{(w - z)^2}$$

is continuous in $\gamma^* \times \mathbb{C} \setminus \gamma^*$. By Theorem 4.18, $\mathbb{C} \setminus \gamma^* \mapsto \int_\gamma \frac{1}{w-z}\,\mathrm{d}w$ is holomorphic, and in particular continuous in $\mathbb{C} \setminus \gamma^*$.

Therefore, $\mathbb{C} \setminus \gamma^* \ni z \mapsto W(\gamma, z)$ is continuous. But we saw in (iii) that $W(\gamma, z) \in \mathbb{Z}$ for all $z \in \mathbb{C} \setminus \gamma^*$, so Proposition 2.28 says that $W(\gamma, z)$ must be constant on each connected component of $\mathbb{C} \setminus \gamma^*$.

(v) Since ξ^* is a compact set, there exists $R > 0$ such that $\xi^* \subset \overline{D}(0, R)$. By (iv), we know that $W(\gamma, z) = W(\gamma, n)$ for every $n \in \mathbb{N}$ with $n \geq 2R$. Applying Proposition 4.11(v) we get, for all $n \geq 2R$:

$$|W(\gamma, z)| = |W(\gamma, n)| = \left|\frac{1}{2\pi i}\int_\gamma \frac{1}{w - n}\,\mathrm{d}w\right| \leq \frac{1}{2\pi}\int_\gamma \frac{1}{|w - n|}|\mathrm{d}w| \leq \frac{1}{2\pi}\int_\gamma \frac{1}{n - R}|\mathrm{d}w| \leq \frac{\mathrm{length}(\gamma)}{2\pi(n - R)}.$$

Letting $n \to \infty$ in the last term, we may conclude $W(\gamma, z) = 0$. □

4.3.2 The Cauchy Integral Formula. The Mean Value Property

Theorem 4.27 (Cauchy Integral Formula in Convex Domains)**.** *Let $\Omega \subset \mathbb{C}$ be a convex open set, $\gamma : [a,b] \to \Omega$ a closed piecewise C^1-path and $f : \Omega \to \mathbb{C}$ holomorphic. Then,*

$$W(\gamma, z)f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w - z}\,\mathrm{d}w, \quad \textit{for all} \quad z \in \Omega \setminus \gamma^*. \tag{4.3.2}$$

Proof. Fix $z \in \Omega \setminus \gamma^*$ and define a new function $h : \Omega \to \mathbb{C}$ by

$$h(w) = \begin{cases} \dfrac{f(w) - f(z)}{w - z} & \text{if } w \in \Omega \setminus \{z\}, \\ f'(z) & \text{if } w = z. \end{cases}$$

Because f is holomorhic in Ω, we have that h is continuous in Ω and holomorphic in $\Omega \setminus \{z\}$. Corollary 4.23 tells us that

$$0 = \int_\gamma h(w)\,\mathrm{d}w = \int_\gamma \frac{f(w) - f(z)}{w - z}\,\mathrm{d}w = \int_\gamma \frac{f(w)}{w - z}\,\mathrm{d}w - \int_\gamma \frac{f(z)}{w - z}\,\mathrm{d}w = \int_\gamma \frac{f(w)}{w - z}\,\mathrm{d}w - 2\pi i f(z)W(\gamma, z);$$

which clearly yields (4.3.2). □

A particular case of Theorem 4.27 gives the following corollary.

Corollary 4.28 (Local Cauchy Integral Formula)**.** *Let $\Omega \subset \mathbb{C}$ be an open set, $f : \Omega \to \mathbb{C}$ holomorphic, and $\overline{D}(z_0, r) \subset \Omega$ a closed disk. Then,*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D(z_0,r)} \frac{f(w)}{w - z} \, \mathrm{d}w, \quad \text{for all} \quad z \in D(z_0, r); \tag{4.3.3}$$

where the integral is along the circle $\partial D(z_0, r)$ traveled counterclockwise once.

Proof. Letting $\gamma : [0, 2\pi] \to \mathbb{C}$ be the path $\gamma(t) = z_0 + re^{it}$. We saw in Example 4.25 that $W(\gamma, z) = 1$ for all $z \in D(z_0, r)$. Consequently, Theorem 4.27 implies (4.3.3). □

We can improve a bit Corollary 4.28 as follows.

Corollary 4.29 (Cauchy Integral Formula in a disk)**.** *Let $f : \overline{D}(z_0, r) \to \mathbb{C}$ be continuous in $\overline{D}(z_0, r)$ and holomorphic in $D(z_0, r)$. Then,*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D(z_0,r)} \frac{f(w)}{w - z} \, \mathrm{d}w, \quad \text{for all} \quad z \in D(z_0, r); \tag{4.3.4}$$

where the integral is along the circle $\partial D(z_0, r)$ traveled counterclockwise once.

Proof. Let $\{a_n\}_n \subset (0, 1)$ be a sequence with $a_n \uparrow 1$. Define, for each $n \in \mathbb{N}$, the function

$$f_n : D(z_0, \tfrac{r}{a_n}) \to \mathbb{C}, \quad f_n(z) = f(a_n z), \quad z \in D(z_0, \tfrac{r}{a_n}).$$

Because f is continuous in holomorphic in $D(z_0, r)$, we see that f_n is holomorphic in $D(z_0, \frac{r}{a_n})$. Notice that $\overline{D}(z_0, r) \subset D(z_0, \frac{r}{a_n})$ and we can apply Corollary 4.28 to f_n, thus obtaining

$$f_n(z) = \int_{\partial D(z_0,r)} \frac{f(w)}{w - z} \, \mathrm{d}w \quad \text{for all} \quad z \in D(z_0, r),\ n \in \mathbb{N}. \tag{4.3.5}$$

Let us now show that $\{f_n\}_n$ converges to f uniformly in $\overline{D}(z_0, r)$. Indeed, since $\overline{D}(z_0, r)$, f is uniformly continuous there; see Proposition 2.25. Thus, given $\varepsilon > 0$, we can find $\delta > 0$ such that $|f(\xi) - f(w)| \le \varepsilon$ for all $\xi, w \in \overline{D}(z_0, r)$ with $|\xi - w| \le \delta$. Let $N \in \mathbb{N}$ such that $(1 - a_N)r \le \delta$. We have that $|a_n w - w| = (1 - a_n)|w| \le (1 - a_n)r\delta$, for all $n \ge N$, and consequently

$$\sup_{w \in \overline{D}(z_0,r)} |f_n(w) - f(w)| = \sup_{w \in \overline{D}(z_0,r)} |f(a_n w) - f(w)| \le \varepsilon, \quad \text{for all} \quad n \ge N.$$

This confirms that $\{f_n\}_n$ converges to f uniformly in $\overline{D}(z_0, r)$. Combining Proposition 4.12 with (4.3.5), we can conclude

$$f(z) = \lim_{n\to\infty} f_n(z) = \frac{1}{2\pi i} \int_{\partial D(z_0,r)} \lim_{n\to\infty} \frac{f_n(w)}{w - z} \, \mathrm{d}w = \frac{1}{2\pi i} \int_{\partial D(z_0,r)} \frac{f(w)}{w - z} \, \mathrm{d}w.$$

□

Also, Corollary 4.29 implies the following *identity principle*, giving a bit of a hint of what we will obtain in Section 4.4.3.

Corollary 4.30. *Let $f, g : \overline{D}(z_0, r) \to \mathbb{C}$ continuous in $\overline{D}(z_0, r)$ and holomorphic in $D(z_0, r)$. If $f = g$ on $\partial D(z_0, r)$, then $f = g$ on $\overline{D}(z_0, r)$.*

Proof. It suffices to apply formula (4.3.4) in Corollary 4.29 to the function $f - g$. □

Another consequence of Corollary 4.29 is the following mean value (integral) property for holomorphic functions.

Corollary 4.31 (Mean Value Property). *Let $f : \overline{D}(z_0, r) \to \mathbb{C}$ be continuous in $\overline{D}(z_0, r)$ and holomorphic in $D(z_0, r)$. Then,*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, \mathrm{d}t. \tag{4.3.6}$$

Proof. We apply Corollary 4.29 for $z = z_0$, where $\partial D(z_0, r)$ is parametrized by the path $\gamma : [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = z_0 + re^{it}$, to obtain

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\gamma(t))}{\gamma(t) - z_0} \gamma'(t)\, \mathrm{d}t = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} ire^{it}\, \mathrm{d}t = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, \mathrm{d}t.$$

□

4.4 Differentiability and Analiticity of Holomorphic functions

4.4.1 The Cauchy Formulae and Estimates for the Derivatives

We continue deriving fundamental properties from the Cauchy Integral Formula; Theorem 4.27 or Corollary (4.28). More precisely, derivatives of holomorphic functions are holomorphic, and their derivatives can be written via formulas similar to that of (4.3.3).

Theorem 4.32 (Cauchy Formulas for the Derivatives). *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ holomorphic. Then, for all $n \in \mathbb{N}$, the n^{th} derivative $f^{(n)} : \Omega \to \mathbb{C}$ exists and is holomorphic in Ω. Moreover, for every open disk D with $\overline{D} \subset \Omega$, the following formula holds:*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w - z)^{n+1}}\, \mathrm{d}w \quad \textit{for all} \quad z \in D, \quad n \in \mathbb{N} \cup \{0\}. \tag{4.4.1}$$

Proof. We prove both the existence and holomorphicity of $f^{(n)}$ and (4.4.1) at the same time and by induction on $\mathbb{N} \cup \{0\}$. In the case $n = 0$, then $f^{(n)} = f$ and the claims (the holomorphicity of f is already known from the assumption) follow from Theorem 4.27. Now assume that $f^{(n)} : \Omega \to \mathbb{C}$ exists and is holomorphic and that (4.4.1) holds. Denoting by $\gamma : [0, 2\pi] \to \partial D$ the curve that travels ∂D once and counterclockwise, we define the function

$$\gamma^* \times D \ni (w, z) \longmapsto \varphi(w, z) := \frac{n!}{2\pi i} \frac{f(w)}{(w - z)^{n+1}}.$$

This function is continuous in $\gamma^* \times D$ and for each $w \in \gamma^*$, the function $D \ni z \mapsto \varphi(w, z)$ is holomorphic in D with derivative (with respect to z) equal to

$$\frac{\partial \varphi}{\partial z}(w, z) = \frac{(n+1)!}{2\pi i} \frac{f(w)}{(w - z)^{n+2}}, \quad z \in D;$$

which defines a continuous function in $\gamma^* \times D$. Applying Theorem 4.18, we get that

$$D \ni z \longmapsto f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w - z)^{n+1}}\, \mathrm{d}w = \int_\gamma \varphi(w, z)\, \mathrm{d}w$$

is holomorphic in D, with

$$D \ni z \longmapsto f^{(n+1)} = \left(f^{(n)}\right)'(z) = \int_\gamma \frac{\partial \varphi}{\partial z}(w, z)\, \mathrm{d}w = \frac{(n+1)!}{2\pi i} \int_\gamma \frac{f(w)}{(w - z)^{n+2}}\, \mathrm{d}w.$$

By induction, the claims are proven for all $n \in \mathbb{N}$. □

An extension of The Cauchy Integral Formulas (4.4.1) are also true when the circle path are replaced by any piecewise C^1-path, provided we have our function holomorphic in a convex domain.

Corollary 4.33 (Cauchy Formulas in Convex Domains). *Let $\Omega \subset \mathbb{C}$ be open and convex, $\gamma : [a,b] \to \Omega$ a closed piecewise C^1-path, and $f : \Omega \to \mathbb{C}$ holomorphic. Then, the following formulae holds:*

$$W(\gamma, z) f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{n+1}} \, \mathrm{d}w \quad \textit{for all} \quad z \in \Omega \setminus \gamma^*, \quad n \in \mathbb{N} \cup \{0\}. \tag{4.4.2}$$

Proof. We already know from Theorem 4.32 that all the derivatives of f exist in Ω. Let $\gamma : [a,b] \to \Omega$ be a closed piecewise C^1-path, and let us show (4.4.2). In the case $n = 0$, then $f^{(n)} = f$ and the formula holds by virtue of (4.3.2). For $n \in \mathbb{N}$, we can apply formula (4.3.2) to $f^{(n)}$ and repeatedly Exercise 4.6 (to $f^{(n-1)}, \dots, f', f$) and we get

$$W(\gamma, z) f^{(n)}(z) = \frac{1}{2\pi i} \int_\gamma \frac{f^{(n)}(w)}{w-z} \, \mathrm{d}w = \frac{1}{2\pi i} \int_\gamma \frac{f^{(n-1)}(w)}{(w-z)^2} \, \mathrm{d}w = \frac{2}{2\pi i} \int_\gamma \frac{f^{(n-2)}(w)}{(w-z)^3} \, \mathrm{d}w$$

$$= \dots = \frac{(n-1)!}{2\pi i} \int_\gamma \frac{f'(w)}{(w-z)^n} \, \mathrm{d}w = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{n+1}} \, \mathrm{d}w.$$

□

Another consequence of Theorem 4.32 is the following collection of useful inequalities for the derivatives of holomorphic functions.

Corollary 4.34 (Cauchy Estimates for the Derivatives). *Let $D(z_0, R)$ be an open disk and $f : D(z_0, R) \to \mathbb{C}$ a holomorphic and bounded function. Then*

$$\left| f^{(n)}(z) \right| \le \frac{n! \cdot R \cdot \sup\{|f(w)| \, : \, w \in D(z_0, R)\}}{(R - |z - z_0|)^{n+1}}, \quad \textit{for all} \quad z \in D(z_0, R), \, n \in \mathbb{N} \cup \{0\}. \tag{4.4.3}$$

Also, if $\Omega \subset \mathbb{C}$ is open, $\overline{D}(z_0, R) \subset \Omega$ and $f : \Omega \to \mathbb{C}$ is holomorphic, then

$$\left| f^{(n)}(z_0) \right| \le \frac{n!}{R^n} \sup\{|f(w)| \, : \, w \in \partial D(z_0, R)\}, \quad \textit{for all} \quad n \in \mathbb{N} \cup \{0\}. \tag{4.4.4}$$

Proof. Let $n \in \mathbb{N} \cup \{0\}$ and $z \in D(z_0, R)$. Observe that

$$|w - z| = |w - z_0 - (z - z_0)| \ge |w - z_0| - |z - z_0|, \quad \text{for all} \quad w \in D(z_0, R). \tag{4.4.5}$$

Now, let $0 < r < R$ so that $z \in D(z_0, r)$. Since $\overline{D}(z_0, r) \subset D(z_0, R)$ and f is analytic in the latter disk, we can apply Theorem 4.32 in combination with Proposition 4.11(vi), and then (4.4.5), to derive, for all $z \in D(z_0, r)$:

$$\begin{aligned}
\left| f^{(n)}(z) \right| &\le \left| \frac{n!}{2\pi i} \int_{\partial D(z_0,r)} \frac{f(w)}{(w-z)^{n+1}} \, \mathrm{d}w \right| \le \frac{n!}{2\pi} \int_{\partial D(z_0,r)} \frac{|f(w)|}{|w-z|^{n+1}} \, |\mathrm{d}w| \\
&\le \frac{n!}{2\pi} \int_{\partial D(z_0,r)} \frac{|f(w)|}{(|w - z_0| - |z - z_0|)^{n+1}} \, |\mathrm{d}w| = \frac{n!}{2\pi} \int_{\partial D(z_0,r)} \frac{|f(w)|}{(r - |z - z_0|)^{n+1}} \, |\mathrm{d}w| \\
&\le \frac{n! \sup\{|f(w)| \, : \, w \in \partial D(z_0, r)\}}{2\pi \, (r - |z - z_0|)^{n+1}} \mathrm{length}(\partial D(z_0, r)) = \frac{n! \cdot r \cdot \sup\{|f(w)| \, : \, w \in \partial D(z_0, r)\}}{(r - |z - z_0|)^{n+1}} \qquad (4.4.6) \\
&\le \frac{n! \cdot r \cdot \sup\{|f(w)| \, : \, w \in D(z_0, R)\}}{(r - |z - z_0|)^{n+1}}.
\end{aligned}$$

Letting $r \uparrow R$, we conclude (4.4.3).

For the second part, when $\overline{D}(z_0, R) \subset \Omega$ and $f : \Omega \to \mathbb{C}$ is holomorphic, we can repeat the previous computations replacing r with R and z with z_0 up to line (4.4.6), obtaining,

$$\left|f^{(n)}(z_0)\right| \leq \frac{n! \cdot R \cdot \sup\{|f(w)| \,:\, w \in \partial D(z_0, R)\}}{R^{n+1}} = \frac{n!}{R^n} \sup\{|f(w)| \,:\, w \in \partial D(z_0, R)\}.$$

□

Furthermore, Theorem 4.32 permits to prove the following extension-type property.

Corollary 4.35 (Holomorphic Extension to a Point)**.** *Let $\Omega \subset \mathbb{C}$ be open, $z_0 \in \Omega$, and $f : \Omega \to \mathbb{C}$ be continuous with $f \in \mathcal{H}(\Omega \setminus \{z_0\})$. Then $f \in \mathcal{H}(\Omega)$.*

Proof. For every $z \in \Omega$, we can find an open disk D with $z \in D$. Since D is convex, and f is continuous in D and holomorphic in $D \setminus \{z_0\}$, Theorem 4.22 says that there is $F : D \to \mathbb{C}$ holomorphic in D with $F' = f$. But Theorem 4.32 tells us that $F' : D \to \mathbb{C}$ is holomorphic in D too, implying, in particular, that f is complex-differentiable at z. □

4.4.2 Morera's Theorem. Weierstrass Convergence Theorem

Theorem 4.36 (Morera's Theorem)**.** *Let $\Omega \subset \mathbb{C}$ be open, and $f : \Omega \to \mathbb{C}$ be continuous. Suppose that for every triangle T with $\operatorname{co}(T) \subset \Omega$, we have*

$$\int_T f = 0.$$

Then f is holomorphic in Ω.

Proof. For every $z \in \Omega$, we can find an open disk D with $z \in D \subset \Omega$. Thus D is convex, and by the assumption we have that $\int_T f = 0$ for every triangle T with $\operatorname{co}(T) \subset D$. Since f is continuous in D, Lemma 4.21 yields the existence of $F : D \to \mathbb{C}$ holomorphic with $F' = f$ in D. But Theorem 4.32 then implies that F' is holomorphic in D, and consequently f is differentiable at z.

□

Theorem 4.37 (Weierstrass Theorem)**.** *Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \to \mathbb{C}$ a function, and let $\{f_k : \Omega \to \mathbb{C}\}_k$ be sequence of holomorphic functions in Ω converging locally uniformly to f in Ω. Then,*

(i) f is holomorphic in Ω.

(ii) For every $n \in \mathbb{N}$, the sequence of n^{th}-derivatives $\{f_k^{(n)} : \Omega \to \mathbb{C}\}_k$ converges locally uniformly in Ω to the n^{th}-derivative $f^{(n)}$ of f.

Proof.

(i) By Proposition 3.10, the function $f : \Omega \to \mathbb{C}$ is continuous in Ω. To show that $f \in \mathcal{H}(\Omega)$, let $z \in \Omega$ and $r > 0$ so that $D(z, r) \subset \Omega$ and f_k converges to f uniformly in D. Observe that, for every triangle T with $\operatorname{co}(T) \subset D$, Theorem 4.20 says that

$$\int_T f_k = 0, \quad k \in \mathbb{N};$$

as each f_k is holomorphic in $D(z, r)$. Since the uniform converge $f_k \to f$ holds in the piecewise C^1-path T, Corollary 4.12 gives

$$\int_T f = \lim_{k \to \infty} \int_T f_k = 0.$$

But since T, with $\operatorname{co}(T) \subset D$, we can apply Morera's Theorem 4.36 to f in D, to deduce that f is holomorphic in D.

(ii) By part (i), we know that $g_k := f - f_k : \Omega \to \mathbb{C}$ is holomorphic in Ω, for every $k \in \mathbb{N}$. Given $z_0 \in \Omega$, let $r > 0$ be so that $\overline{D}(z_0, 2r) \subset \Omega$ and f_k converges to f uniformly in $\overline{D}(z_0, 2r)$. That is,

$$\lim_{k\to\infty} \sup_{w\in\overline{D}(z_0,2r)} |g_k(w)| = 0. \tag{4.4.7}$$

Let us show that

$$\lim_{k\to\infty} \sup_{z\in\overline{D}(z_0,r)} |g_k^{(n)}(z)| = 0, \quad \text{for all} \quad n \in \mathbb{N}. \tag{4.4.8}$$

Note that if $z \in \overline{D}(z_0, r)$, then $\overline{D}(z, r) \subset \overline{D}(z_0, 2r) \subset \Omega$. We can then apply inequality (4.4.4) of Corollary 4.34 to g_k and the disk $D(z, r)$ to infer that, for all $n \in \mathbb{N}$,

$$\left|g_k^{(n)}(z)\right| \leq \frac{n!}{r^n} \sup\{|g_k(w)| \ : \ w \in \partial D(z,r)\} \leq \frac{n!}{r^n} \sup_{w\in\overline{D}(z_0,2r)} |g_k(w)|.$$

By (4.4.7), we have, taking limits of supremums in $z \in \overline{D}(z_0, r)$:

$$\lim_{k\to\infty} \sup_{z\in\overline{D}(z_0,r)} |g_k^{(n)}(z)| \leq \lim_{k\to\infty} \frac{n!}{r^n} \sup_{w\in\overline{D}(z_0,2r)} |g_k(w)| = 0,$$

for all $n \in \mathbb{N}$. This implies (4.4.8), and we haved proved (ii). □

Corollary 4.38. *If $\Omega \subset \mathbb{C}$ is open and $\{f_k : \Omega \to \mathbb{C}\}_k$ is a sequence of holomorphic functions in Ω so that $\sum_{k=1}^{\infty} f_k$ converges locally uniformly in Ω, then $\sum_{k=1}^{\infty} f_k$ is a holomorphic function in Ω.*

For instance, consider the series of functions $\sum_{n=1}^{\infty} \frac{1}{n^z}$, for all $z \in \Omega := \{z \in \mathbb{C} \ \ \mathrm{Re}(z) > 1\}$; see Exercise 3.10. Naturally, the functions $\Omega \ni z \mapsto 1/n^z$ are holomorphic for all $n \in \mathbb{N}$, as $1/n^z$ is nothing but

$$\frac{1}{n^z} = \frac{1}{e^{z\log n}} = e^{-z\log n}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges uniformly on each set $\Omega_\varepsilon := \{z \in \mathbb{C} \ : \ \mathrm{Re}(z) \geq 1 + \varepsilon\}$, $\varepsilon > 0$. But for every $z \in \Omega$, we can find $\varepsilon > 0$ and $r > 0$ for which $D(z, r) \subset \Omega_\varepsilon$, and in particular $\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges uniformly on $D(z, r)$. That is, the series converges locally-uniformly in Ω. According to Corollary 4.38, the sum of the series

$$F(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad z \in \Omega,$$

defines a holomorphic function in Ω.

4.4.3 Analyticity of Holomorphic functions

Theorem 4.39 (Analiticity of Holomorphic Functions)**.** *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ a holomorphic function. Then, for every closed disk $\overline{D}(z_0, r)$ contained in Ω, we have that*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \quad \textit{for all} \quad z \in \overline{D}(z_0, r); \tag{4.4.9}$$

and the series converges absolutely-uniformly in $z \in \overline{D}(z_0, r)$. In particular, f is analytic in Ω.

Proof. If $\overline{D}(z_0, r) \subset \Omega$, then by the compactness of this disk, there exists $\varepsilon > 0$ so that $\overline{D}(z_0, r+\varepsilon) \subset \Omega$ as well. Observe that for $w \in \partial D(z_0, r + \varepsilon)$ and $z \in \overline{D}(z_0, r)$, we have the bounds

$$\frac{|z - z_0|}{|w - z_0|} \leq \frac{r}{r + \varepsilon} < 1. \tag{4.4.10}$$

Thus we can express $(w-z)^{-1}$ as a geometric sum:

$$\frac{1}{w-z} = \frac{1}{(w-z_0)\left(1-\dfrac{z-z_0}{w-z_0}\right)} = \frac{1}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n = \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(w-z_0)^{n+1}}.$$

Moreover, bearing in mind (4.4.10), we can apply the Weierstrass M-test (Theorem 3.9) with $M_n = r^n/(r+\varepsilon)^n$ to deduce that the convergence of the series above is uniform in $w \in \partial D(z_0, r+\varepsilon)$. Now, on the disk $\overline{D}(z_0, r+\varepsilon)$, we use first Corollary 4.28 (formula (4.3.3)) and then Theorem 4.32 for the derivatives of f (see (4.4.1)) to write, for all $z \in \overline{D}(z_0, r) \subset D(z_0, r+\varepsilon)$:

$$f(z) = \frac{1}{2\pi i}\int_{\partial D(z_0,r+\varepsilon)} \frac{f(w)}{w-z}\,\mathrm{d}w = \frac{1}{2\pi i}\int_{\partial D(z_0,r+\varepsilon)} f(w)\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(w-z_0)^{n+1}}\,\mathrm{d}w$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\partial D(z_0,r+\varepsilon)}\frac{f(w)}{(w-z_0)^{n+1}}\,\mathrm{d}w\right)(z-z_0)^n = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.$$

Note that in the third equality we used the uniform convergence of the series in $w \in \partial D(z_0, r+\varepsilon)$ and Corollary 4.12 to move the series outside of the integral. We have shown (4.4.9) for all $z \in \overline{D}(z_0, r)$. To show that the converge is absolute-uniform, we apply Corollary 4.34, estimate (4.4.4) at the point z_0 and over the circle $\overline{D}(z_0, r+\varepsilon)$, obtaining the bound

$$\left|\frac{f^{(n)}(z_0)}{n!}\right|^{1/n} \le \frac{(\sup\{|f(w)| \,:\, w \in \partial D(z_0, r+\varepsilon)\})^{1/n}}{r+\varepsilon}, \quad n \in \mathbb{N}.$$

The limit superior of the last term is at most $1/(r+\varepsilon)$, so by Theorem 3.15 (formula (3.3.3)), the radius of convergence R of the power series

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

is at least $r+\varepsilon$. In particular, again by Theorem 3.15, this series converges absolutely-uniformly on $\overline{D}(z_0, r)$. □

Corollary 4.40. *Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \to \mathbb{C}$ holomorphic, and let $R \in (0, +\infty]$ be the radius of convergence of the Taylor series*

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

centered at $z_0 \in \Omega$. Then $R \ge \sup\{r > 0 \,:\, D(z_0, r) \subset \Omega\}$.

According to Corollary 4.40, if we know that an $f : D(z_0, r) \to \mathbb{C}$ is holomorphic in $D(z_0, r)$, then

$$f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n, \quad z \in D(z_0, r),$$

with absolute pointwise convergence in $D(z_0, r)$ and absolute–uniform convergence on each closed subdisk $\overline{D}(z_0, s)$ with $s < r$.

Also, if $f : \mathbb{C} \to \mathbb{C}$ is holomorphic, then for all $z_0 \in \mathbb{C}$ we can write

$$f(z) = \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n, \quad z \in \mathbb{C},$$

with absolute pointwise convergence in $D(z_0, r)$ and absolute–uniform convergence on each closed subdisk $\overline{D}(z_0, r)$ with $r > 0$.

The Identity Principles for analytic functions from Section 3.4.4 are then true for holomorphic functions. They follow as an immediate consequence of Theorems 4.39, 3.32, and 3.33.

Corollary 4.41 (First Identity Principle for Holomorphic Functions)**.** *Let $\Omega \subset \mathbb{C}$ be open and connected, and $f, g : \Omega \to \mathbb{C}$ two holomorphic functions such that there is $z_0 \in \Omega$ with $f^{(n)}(z_0) = g^{(n)}(z_0)$ for all $n \in \mathbb{N} \cup \{0\}$. Then $f = g$ on Ω.*

Observe that in Corollary 4.41, we really need the identity $f^{(n)}(z_0) = g^{(n)}(z_0)$ for **all** $n \in \mathbb{N} \cup \{0\}$ in order to claim that $f = g$ in Ω, and assuming that identity for infinitely many n's is not enough. For example, the function $f(z) = \cos z$ in $\mathbb{C}$ has the property that $f^{(2n-1)}(0) = 0$ for all $n \in \mathbb{N}$, and of course $f \not\equiv 0$ in $\mathbb{C}$.

However, it is natural to wonder about the case where the first $m-1$ derivatives of a holomorphic function are zero, but not the m-th one. Let us discuss this now.

Definition 4.42 (Order of a zero)**.** *Let $\Omega \subset \mathbb{C}$ be open, $z_0 \in \Omega$, $m \in \mathbb{N}$, and $f : \Omega \to \mathbb{C}$ a holomorphic function. We say that f has a **zero of order** m **at** z_0 provided that*

$$f(z_0) = f'(z_0) = \cdots = f^{m-1}(z_0) = 0 \quad \textit{and} \quad f^{(m)}(z_0) \neq 0.$$

A function with a zero of order m at z_0 admits a *factorization* via $(z - z_0)^m$.

Proposition 4.43. *Let $\Omega \subset \mathbb{C}$ be open, $z_0 \in \Omega$, $m \in \mathbb{N}$, and $f : \Omega \to \mathbb{C}$ a holomorphic function. Then f has a zero of order m at z_0 if and only if there exists $g \in \mathcal{H}(\Omega)$ with $g(z_0) \neq 0$ and*

$$f(z) = (z - z_0)^m g(z), \quad \textit{for all} \quad z \in \Omega.$$

Proof. Assume that f has a zero of order $m \in \mathbb{N}$ at z_0. By Theorem 4.39 there exists $r > 0$ with $D(z_0, r) \subset \Omega$ and such that for all $z \in D(z_0, r)$:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n = (z - z_0)^m \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^{n-m}, \tag{4.4.11}$$

with uniform convergence of the series in $D(z_0, r)$. Defining $h : D(z_0, r) \to \mathbb{C}$ by

$$h(z) = \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^{n-m}, \quad z \in D(z_0, r),$$

we notice that h is continuous in $D(z_0, r)$ by the uniform convergence of the series there (recall Proposition 3.10), and that $h(z_0) = f^{(m)}(z_0)/m! \neq 0$. The desired function g is defined by

$$g(z) = \begin{cases} \frac{f(z)}{(z-z_0)^m} & \text{if } z \in \Omega \setminus \{z_0\} \\ h(z_0) & \text{if } z = z_0. \end{cases}$$

We immediately see that $g(z_0) = h(z_0) \neq 0$ and that g is holomorphic in $\Omega \setminus \{z_0\}$. Also, (4.4.11) yields that $g(z) = h(z)$ for all $z \in D(z_0, r)$, so the continuity of h at z_0 implies the continuity of g at z_0. According to Corollary 4.35, $g \in \mathcal{H}(\Omega)$.

Conversely, assume the factorization $f(z) = (z - z_0)^m g(z)$, $z \in \Omega$, for some $m \in \mathbb{N}$ and $g \in \mathcal{H}(\Omega)$ with $g(z_0) \neq 0$. Differentiating the expression $(z - z_0)^m g(z)$ at z_0 up to m times we get that

$$f(z_0) = f'(z_0) = \cdots = f^{m-1}(z_0) = 0 \quad \text{and} \quad f^{(m)}(z_0) = m! g(z_0) \neq 0,$$

and thus f has zero of order m at z_0. □

Finally, as a consequence of Theorems 4.39 and 3.33, we get that the zeros of a holomorphic function are *isolated.*

Corollary 4.44 (Second Identity Principle for Holomorphic Functions)**.** *Let $\Omega \subset \mathbb{C}$ be open and connected, and $f, g : \Omega \to \mathbb{C}$ two holomorphic functions such that there are $z_0 \in \Omega$ and a sequence $\{z_k\}_k \subset \Omega \setminus \{z_0\}$ such that $\lim_{k\to\infty} z_k = z_0$ and $f(z_k) = g(z_k)$ for all $k \in \mathbb{N}$. Then $f = g$ on Ω.*

In other words, if $f = g$ in a set $E \subset \Omega$ with $E' \cap \Omega \neq \emptyset$, then $f = g$ in Ω.

4.4.4 Liouville's Theorem and The Fundamental Theorem of Algebra

Theorem 4.45 (Liouville's Theorem)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and bounded. Then f is constant in $\mathbb{C}$.*

Proof. For every $z \in \mathbb{C}$ and $r > 0$, the estimate (4.4.4) for f' in the disk $D(z, r)$ gives

$$|f'(z)| \leq \frac{\sup\{|f(w)| \,:\, w \in \partial D(z,r)\}}{r} \leq \frac{\sup\{|f(w)| \,:\, w \in \mathbb{C}\}}{r}.$$

Since the last supremum is a finite positive number, letting $r \to \infty$ in the above inequality implies that $f'(z) = 0$. But because $\mathbb{C}$ is connected, Corollary 2.37 says that f is constant in $\mathbb{C}$. □

A consequence is that the image of a non-constant holomorphic map $f : \mathbb{C} \to \mathbb{C}$ is dense in $\mathbb{C}$.

Corollary 4.46. *Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant holomorphic. Then $\overline{f(\mathbb{C})} = \mathbb{C}$.*

Proof. Suppose, for the sake of contradiction, that there is $w_0 \in \mathbb{C} \setminus \overline{f(\mathbb{C})}$. Then there exists $\varepsilon > 0$ so that $D(w_0, \varepsilon) \cap f(\mathbb{C}) = \emptyset$; see (2.1.1) in Proposition 2.6. Therefore, the function $g : \mathbb{C} \to \mathbb{C}$ given by

$$g(z) = \frac{1}{f(z) - w_0}, \quad z \in \mathbb{C};$$

is holomorphic in $\mathbb{C}$, as $|f(z) - w_0| \geq \varepsilon > 0$ for all $z \in \mathbb{C}$. Precisely thanks to this estimate we have that $|g(z)| \leq \frac{1}{\varepsilon}$ for all $z \in \mathbb{C}$. That is, g is bounded, and hence g (and consequently f) is constant in $\mathbb{C}$, a contradiction. □

There is a stronger result due to Picard (called *Picard's Little Theorem*), which shows that a non-constant holomorphic function $\mathbb{C} \to \mathbb{C}$ takes all the values (possibly) except one.

We are finally equipped with the necessary analytic tools to give a proof of the Fundamental Theorem of Algebra, using ingredients from complex analysis.

Theorem 4.47 (Fundamental Theorem of Algebra)**.** *Let $P(z) = a_n z^n + \cdots + a_1 z + a_0$, $z \in \mathbb{C}$, a polynomial of degree $n \in \mathbb{N}$, that is, $a_n \neq 0$. Then there numbers $z_1, \ldots, z_n \in \mathbb{C}$ so that*

$$P(z) = a_n(z - z_1) \cdots (z - z_n), \quad z \in \mathbb{C}. \tag{4.4.12}$$

In particular, P has at least one root.

Proof. We will prove first that every polynomial P of degree $n \in \mathbb{N}$, must have at least one root. Suppose, for the sake of contradiction, that $P(z) \neq 0$ for all $z \in \mathbb{C}$. The polynomial $P(z) = a_n z^n + \cdots + a_1 z + a_0$ is holomorphic in $\mathbb{C}$, and so is the function $f = 1/P : \mathbb{C} \to \mathbb{C}$; see e.g. Proposition 2.34. Observe that, for all $z \neq 0$,

$$|P(z)| = \left|a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0\right| \geq |z|^n \left(|a_n| - \frac{|a_{n-1}|}{|z|} - \cdots - \frac{|a_1|}{|z|^{n-1}} - \frac{|a_0|}{|z|^n}\right);$$

which clearly shows that $\lim_{|z|\to\infty} |P(z)| = \infty$, and so $\lim_{|z|\to\infty} |f(z)| = 0$. Thus there is $r > 0$ so that $|f(z)| \leq 1$ for all $|z| \geq r$. Since of course $|f|$ is also bounded in $\overline{D}(0, r)$,(by continuity; see Proposition 2.25), we get that $f : \mathbb{C} \to \mathbb{C}$ is bounded and holomorphic. By Theorem 4.45, f must be a (non-zero) constant in $\mathbb{C}$, and so P must be constant in $\mathbb{C}$. This contradicts what we proved above $\lim_{|z|\to\infty} |P(z)| = \infty$. Therefore, this shows that there exists some $z \in \mathbb{C}$ with $P(z) = 0$.

Now, let us prove the factorization (4.4.12) for P. Let $z_1 \in \mathbb{C}$ be so that $P(z_1) = 0$. We can manually factorize P in terms of $z - z_1$, for all $z \in \mathbb{C}$, using the identity (1.1.4):

$$P(z) = P(z) - P(z_1) = \sum_{k=1}^{n} a_k z^k - \sum_{k=1}^{n} a_k z_1^k = \sum_{k=1}^{n} a_k (z^k - z_1^k) = \sum_{k=1}^{n} a_k (z - z_1) \sum_{j=0}^{k-1} z_1^{k-1-j} z^j$$

$$= (z - z_1) \sum_{k=1}^{n} a_k \sum_{j=0}^{k-1} z_1^{k-1-j} z^j = (z - z_1) P_1(z), \quad \text{where} \quad P_1(z) = \sum_{k=1}^{n} a_k \sum_{j=0}^{k-1} z_1^{k-1-j} z^j.$$

Now, P_1 is polynomial of degree $n-1$, and the coefficient of the monomial z^{n-1} of P_1 is equal to a_n. By what we have proved already in the current proof, there exists some $z_2 \in \mathbb{C}$ with $P_1(z_2) = 0$. Repeating the factorization above for P_1, we obtain a new polynomial P_2 of degree $n-2$, with coefficient of the term z^{n-2} equal to a_n and such that

$$P_1(z) = (z - z_2)P_2(z), \quad P(z) = a_n(z - z_1)(z - z_2)P_2(z) \quad z \in \mathbb{C}.$$

By repeating this argument, we obtain numbers $z_1, \ldots, z_{n-1} \in \mathbb{C}$ and a polynomial P_{n-1} of degree 1 with coefficient of the monomial z equal to a_n, and such that

$$P(z) = (z - z_1)(z - z_2)\cdots(z - z_{n-1})P_{n-1}(z) \quad z \in \mathbb{C}.$$

Obviously there exists $z_n \in \mathbb{C}$ such that $P_{n-1}(z) = a_n(z - z_n)$ for all $z \in \mathbb{C}$, yielding (4.4.12). □

4.5 The Maximum Modulus Principles

In this section we show that the modulus of holomorphic functions attain their maximum on the boundary of a disk, or more generally, on the boundary of bounded domains. These are the *Maximum Modulus Principles.*

Theorem 4.48 (Maximum Modulus Principle I)**.** *Let $\Omega \subset \mathbb{C}$ be open and connected, $f : \Omega \to \mathbb{C}$ be holomorphic in Ω, and $z_0 \in \Omega$, $r > 0$ so that $\overline{D}(z_0, r) \subset \Omega$. Then*

$$|f(z_0)| \leq \max\{|f(z)| : z \in \partial D(z_0, r)\}. \tag{4.5.1}$$

Moreover, the inequality (4.5.1) *becomes equality if and only if f is constant in Ω.*

Proof. Define $M(r) := \max\{|f(z)| : z \in \partial D(z_0, r)\}$. By Corollary 4.31, we have

$$|f(z_0)| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,\mathrm{d}t\right| \leq \frac{1}{2\pi}\int_0^{2\pi} \left|f(z_0 + re^{it})\right|\,\mathrm{d}t \leq \frac{1}{2\pi}\int_0^{2\pi} M(r)\,\mathrm{d}t = M(r),$$

which proves (4.5.1). To prove the second part, assume that $|f(z_0)| = M(r)$. In the case where $M(r) = 0$, we have that $f = 0$ on $\partial D(z_0, r)$; which by Corollary 4.44 implies that $f = 0$ on Ω. So, let us study the case where $|f(z_0)| = M(r) > 0$. Define the function $g(z) = e^{-i\,\mathrm{Arg}(f(z_0))}f(z)$ for all $z \in \Omega$, and note that $g(z_0) = e^{-i\,\mathrm{Arg}(f(z_0))}f(z_0) = |f(z_0)| > 0$ and $M(r) = \max\{|g(z)| : z \in \partial D(z_0, r)\}$. Applying Corollary 4.31 to g, we get

$$g(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(g(z_0 + re^{it})\right)\mathrm{d}t + i\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Im}\left(g(z_0 + re^{it})\right)\mathrm{d}t;$$

which clearly implies

$$\frac{1}{2\pi}\int_0^{2\pi} \left(\mathrm{Re}(g(z_0 + re^{it})) - g(z_0)\right)\mathrm{d}t = 0. \tag{4.5.2}$$

But on the other hand, we have that

$$\sqrt{\left(\mathrm{Re}\left(g(z_0 + re^{it})\right)\right)^2 + \left(\mathrm{Im}\left(g(z_0 + re^{it})\right)\right)^2} = \left|g(z_0 + re^{it})\right| \leq M(r) = g(z_0) \tag{4.5.3}$$

holds for all $t \in [0, 2\pi]$. Since the function $[0, 2\pi] \ni t \mapsto \mathrm{Re}(g(z_0 + re^{it}))$ is continuous, (4.5.2) and (4.5.3) together give $\mathrm{Re}(g(z_0 + re^{it})) = g(z_0)$ and $\mathrm{Im}(g(z_0 + re^{it})) = 0$ for all $t \in [0, 2\pi]$. Thus g is constantly equal to $g(z_0)$ in the set $\partial D(z_0, r)$. By Corollary 4.44, g and f are constants in Ω. □

As a corollary, we deduce that non-constant holomorphic function cannot attained their maximum in the interior of their domain.

Corollary 4.49. *Let $\Omega \subset \mathbb{C}$ be open and connected, $f : \Omega \to \mathbb{C}$ be holomorphic in Ω, and assume there exists $z_0 \in \Omega$ with $|f(z_0)| \geq |f(z)|$ for all $z \in \Omega$. Then f is constant in Ω.*

Proof. Let $r > 0$ be so that $\overline{D}(z_0, r) \subset \Omega$. Denoting $M(r) := \max\{|f(z)| : z \in \partial D(z_0, r)\}$, we know from Theorem 4.48 that $|f(z_0)| \leq M(r)$. The assumption of the current corollary says that

$$|f(z_0)| \geq \sup\{|f(z)| : z \in \Omega\} \geq \max\{|f(z)| : z \in \partial D(z_0, r)\} = M(r).$$

Thus $|f(z_0)| = M(r)$ and the second part of Theorem 4.48 yields that f is constant in Ω. □

Theorem 4.48 or Corollary 4.49 do not hold for smooth real functions. For example, notice that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = e^{-(x^2+y^2)}$ is of class $C^\infty(\mathbb{R}^2)$ (even real-analytic in $\mathbb{R}^2$), but

$$f(0, 0) = 1 = \max\{|f(x, y)| : (x, y) \in D(0, 1)\} = \max\{|f(x, y)| : (x, y) \in \mathbb{R}^2\}.$$

Theorem 4.50 (Maximum Modulus Principle II)**.** *Let $\Omega \subset \mathbb{C}$ be open, connected, and* ***bounded****. Let $f : \overline{\Omega} \to \mathbb{C}$ be continuous in $\overline{\Omega}$ and holomorphic in Ω. Then, the maximum of f in $\overline{\Omega}$ is attained in the boundary:*

$$\max\{|f(z)| : z \in \overline{\Omega}\} = \max\{|f(z)| : z \in \partial\Omega\}. \tag{4.5.4}$$

Proof. Suppose, seeking a contradiction, that (4.5.4) is false. The set $\overline{\Omega}$, being closed and bounded, is compact; see Theorem 2.14. Since f is continuous in $\overline{\Omega}$, Proposition 2.25 says that there is $z_0 \in \overline{\Omega}$ at which $|f|$ attains the maximum on $\overline{\Omega}$. This information together with the (assumed) falsity of (4.5.4) is gathered as follows:

$$|f(z_0)| = \max\{|f(z)| : z \in \overline{\Omega}\} > \max\{|f(z)| : z \in \partial\Omega\}. \tag{4.5.5}$$

Then necessarily $z_0 \in \text{int}(\Omega) = \Omega$ and $|f(z_0)| \geq |f(z)|$ for all $z \in \Omega$. Corollary 4.49 tells us that $f(z) = f(z_0)$ for all $z \in \Omega$. The continuity of f in $\overline{\Omega}$ implies that also $f(z) = f(z_0)$ for all $z \in \partial\Omega$, contradicting (4.5.5). □

In Theorem 4.50, the assumption that Ω is bounded is really needed. For example, the right half-plane $\Omega = \{z \in \mathbb{C} : \text{Re}(z) > 0\}$ has boundary $\partial\Omega = i\mathbb{R}$, the pure imaginary numbers, and the holomorphic function $f(z) = e^z$ satisfies

$$\max\{|f(z)| : z \in \partial\Omega\} = \max\{e^{\text{Re}(z)} : z \in i\mathbb{R}\} = e^0 = 1.$$

But $|e^z| = e^{\text{Re}(z)}$ is clearly unbounded in Ω, that is, $\sup\{|f(z)| : z \in \Omega\} = \infty$.

4.5.1 The Schwarz Lemma

As a consequence of Theorem 4.48, we show that bounded holomorphic mappings between the unit disk have a quite rigid structure. Let us denote by $\mathbb{D}$ the open unit disk of $\mathbb{C}$, that is,

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}.$$

Theorem 4.51 (Schwarz Lemma)**.** *Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function with $f(0) = 0$ and $\|f\|_\infty := \sup\{|f(z)| : z \in \mathbb{D}\} \leq 1$. Then*

(i) $|f(z)| \leq |z|$ *for all* $z \in \mathbb{D}$.

(ii) $|f'(0)| \leq 1$.

(iii) If either (i) holds with equality for some $z \in \mathbb{D} \setminus \{0\}$ or (ii) holds with equality, then there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ so that

$$f(z) = \lambda z \quad \text{for all} \quad z \in \mathbb{D}.$$

Proof. We define a new function $g : \mathbb{D} \to \mathbb{C}$ by

$$g(z) = \begin{cases} \dfrac{f(z)}{z}, & \text{if } z \in \mathbb{D} \setminus \{0\}, \\ f'(0) & \text{if } z = 0. \end{cases}$$

Since $f \in \mathcal{H}(\mathbb{D})$ and $f(0) = 0$, then g is continuous in $\mathbb{D}$ and $g \in \mathcal{H}(\mathbb{D} \setminus \{0\})$. By Corollary 4.35, $g \in \mathcal{H}(\mathbb{D})$ as well. For every $r \in (0,1)$, we apply Theorem 4.50 to g in the disk $\overline{D}(0,r) \subset \mathbb{D}$, obtaining for all $z \in D(0,r)$ the estimate:

$$|g(z)| \leq \max\{|g(w)| \,:\, w \in \partial D(0,r)\} = \frac{\max\{|f(w)| \,:\, w \in \partial D(0,r)\}}{r} \leq \frac{1}{r};$$

the last inequality being due to the assumption $\|f\|_\infty \leq 1$. Therefore, one has

$$|g(z)| \leq \lim_{r \to 1^-} \frac{1}{r} = 1, \quad \text{for all} \quad z \in \mathbb{D}. \tag{4.5.6}$$

This implies that $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$ and that $|f'(0)| = |g(0)| \leq 1$, proving (i) and (ii).

To show (iii), observe that if either $|f(z)| = |z|$ for some $z \in \mathbb{D} \setminus \{0\}$ or $|f'(0)| = 1$, then we have that $|g(z_0)| = 1$ for some $z_0 \in \mathbb{D}$. By (4.5.6), this yields that $|g(z_0)| \geq \max\{|g(z)| \,:\, z \in \mathbb{D}\}$, and then Corollary 4.49 tells us that g is constant in $\mathbb{D}$. Thus there is $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $f(z) = g(z)z = \lambda z$ for all $z \in \mathbb{D}$. □

4.6 Exercises

Exercise 4.1. *Compute the following path-integrals in the indicated paths, where, for the closed paths, we understand that the paths are traveled once and counterclockwise.*

(a) $\int_\gamma \overline{z}\,\mathrm{d}z$, *for* γ *equal to the circle* $\partial D(0,R)$, $R > 0$.

(b) $\int_\gamma \overline{z}\,\mathrm{d}z$, *for* γ *equal to the boundary of the square* $[-R,R] \times [-R,R]$, *for* $R > 0$.

(c) $\int_\gamma |z|\,\mathrm{d}z$, *for* $\gamma \equiv \{z \in \mathbb{C} \,:\, |z| = 1,\ \mathrm{Arg}(z) \in [0,\pi]\}$ *and* $\gamma \equiv \partial D(0,1)$.

(d) $\int_\gamma z^2\,\mathrm{d}z$, *for* $\gamma \equiv [1+i, 2]$.

(e) $\int_\gamma \mathrm{Im}(z)\,\mathrm{d}z$, *for* γ *equal to the triangle with vertices* $1, i, -i$.

Exercise 4.2. *Prove the following inequalities.*

(a) $\left|\int_\gamma \frac{e^z}{z}\,\mathrm{d}z\right| \leq \pi e$, *where* $\gamma \equiv \{z \in \mathbb{C} \,:\, |z| = 1,\ \mathrm{Arg}(z) \in [0,\pi]\}$.

(b) $\left|\int_\gamma \frac{1}{z^2+1}\,\mathrm{d}z\right| \leq \pi/6$, *where* $\gamma \equiv \{z \in \mathbb{C} \,:\, |z| = 2,\ \mathrm{Arg}(z) \in [0,\pi/2]\}$.

(c) $\left|\int_\gamma \frac{e^{iz}}{z}\,\mathrm{d}z\right| \leq 2\frac{1-e^{-R}}{R}$, *where* $\gamma \equiv [R, R+iR]$, $R > 0$.

(d) $\left|\int_\gamma \frac{1}{z^2+1}\,\mathrm{d}z\right| \leq \frac{\pi R}{|R^2-1|}$, *where* $\gamma \equiv \{z \in \mathbb{C} \,:\, |z| = R,\ \mathrm{Arg}(z) \in [0,\pi]\}$, $R > 0$, $R \neq 1$.

(e) $\left|\int_\gamma \frac{1-e^{2iz}}{z^2(z^2+1)}\,\mathrm{d}z\right| \leq \frac{2\pi}{R(R^2-1)}$, *where* $\gamma \equiv \{z \in \mathbb{C} \,:\, |z| = R,\ \mathrm{Arg}(z) \in [0,\pi]\}$, $R > 1$.

Suggestion: Use appropriately the inequalities from Proposition 4.11(vi).

Exercise 4.3. *Let* $\Omega \subset \mathbb{C}$ *be open,* $f : \Omega \to \mathbb{C}$ *continuous,* $\gamma : [a,b] \to \Omega$ *a* C^1*-path, and* $\{\gamma_n : [a,b] \to \Omega\}_n$ *a sequence of* C^1*-paths so that* $\gamma_n \to \gamma$ *and* $\gamma_n' \to \gamma'$ *uniformly in* $[a,b]$. *Show that*

$$\lim_{n\to\infty} \int_{\gamma_n} f(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z.$$

Exercise 4.4. *Compute the following path-integrals in the indicated sets/paths.*

(a) $\int_\gamma e^z\,\mathrm{d}z$, *where* $\gamma(t) = ie^{it}$ *for* $t \in [0, \pi]$.

(b) $\int_\gamma z^3\,\mathrm{d}z$, *where* $\gamma(t) = e^{it}$ *for* $t \in [0, \pi]$.

(c) $\int_\gamma \cos z\,\mathrm{d}z$, *where* $\gamma(t) = i + e^{it}$ *for* $t \in [0, \pi/4]$.

(d) $\int_\gamma \frac{1}{z^2}\,\mathrm{d}z$, *where* $\gamma(t) = \cos t + 2i\sin t$, $t \in [0, 2\pi]$.

(e) $\int_\gamma \frac{1}{z}\,\mathrm{d}z$, *where* γ *is the segment line* $[1, i]$.

(f) $\int_\gamma (z - z_0)^n\,\mathrm{d}z$, *where* $z_0 \in \mathbb{C}$, $n \in \mathbb{Z}$, *and* $\gamma \equiv \{z \in \mathbb{C} : |z - z_0| = 1,\ \mathrm{Arg}(z - z_0) \in [0, \pi]\}$, *traveled once and with the counterclockwise orientation.*

Suggestion: Theorem 4.15.

Exercise 4.5. *Let* Ω *be open and convex, and* $f : \Omega \to \mathbb{C}$ *holomorphic (with* f' *continuous) in* Ω *with* $\mathrm{Re}(f'(z)) > 0$ *for all* $z \in \Omega$. *Show that* f *is injective in* Ω.

Exercise 4.6. *Let* $\Omega \subset \mathbb{C}$ *open,* $f : \Omega \to \mathbb{C}$ *holomorphic and* $\gamma : [a, b] \to \Omega$ *a closed piecewise* C^1*-path. Prove that if* $n \in \mathbb{N} \cup \{0\}$ *and* $z_0 \notin \gamma^*$, *then*

$$\int_\gamma \frac{f'(z)}{(z - z_0)^n}\,\mathrm{d}z = n \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z.$$

Exercise 4.7. *Let* $\Omega \subset \mathbb{C}$ *open,* $f, g : \Omega \to \mathbb{C}$ *holomorphic (so* f', g' *are continuous), and* $\gamma : [a, b] \to \Omega$ *a closed piecewise* C^1*-path. Then*

$$\int_\gamma f'(z)g(z)\,\mathrm{d}z = f(\gamma(b))g(\gamma(b)) - f(\gamma(a))g(\gamma(a)) - \int_\gamma f(z)g'(z)\,\mathrm{d}z.$$

Exercise 4.8. *Let* $r > 0$, $z_0 \in D(0, r)$ *and* $n \in \mathbb{N}$. *Prove that for every polynomial* P *with* $\deg(P) \leq n$, *one has*

$$\int_{\partial D(z_0, r)} \frac{P(z)}{z^{n+1}(z - z_0)}\,\mathrm{d}z = 0.$$

Suggestion: Use the linearity of the integral and observe the decomposition into simple fractions of the integrands.

Exercise 4.9. *Let* $z_1, z_2 \in \mathbb{C}$ *and* $r > 0$ *so that* $|z_1| < r < |z_2|$. *Show that*

$$\int_{\partial D(0,r)} \frac{\mathrm{d}z}{(z - z_1)(z - z_2)} = \frac{2\pi i}{z_1 - z_2}.$$

Exercise 4.10. *Compute the following path-integrals in the indicated sets/paths.*

(a) $\int_\gamma \frac{z^2+1}{z(z^2+4)}\,\mathrm{d}z$, *where* $\gamma = \partial D(0, r)$ *for* $r \neq 2$.

(b) $\int_\gamma \frac{e^{1/z}}{z^2+z}\,\mathrm{d}z$, *where* $\gamma = \partial D(-1, 1/2)$.

(c) $\int_\gamma \frac{\cos(\pi z)}{|z-2|^2}\,\mathrm{d}z$, *where* $\gamma = \partial D(0, 1)$.

(d) $\int_\gamma \frac{\sin(e^z)}{z}\,\mathrm{d}z$, *where* $\gamma = \partial D(0, 1)$.

(e) $\int_\gamma \frac{e^z}{z^2(z-1)}\,\mathrm{d}z$, *where* $\gamma = \partial D(0, 2)$.

(f) $\int_\gamma z^2 \sin\left(\frac{1}{z}\right)\,\mathrm{d}z$, *where* $\gamma = \partial D(0, 1)$.

Suggestion: In (f), recall the Taylor expansion of the complex sin *function.*

Exercise 4.11. *Prove that:*

(a) $\int_\gamma \frac{3z-1}{(z+1)(z-3)}\,\mathrm{d}z = 6\pi i$, *where* $\gamma = \partial D(0,4)$.

(b) $\int_\gamma \frac{2z}{z^2+1}\,\mathrm{d}z = 4\pi i$, *where* $\gamma = \partial D(0,2)$.

(c) $\int_\gamma \frac{z^3}{z^4-1}\,\mathrm{d}z = 2\pi i$, *where* $\gamma = \partial D(0,2)$.

(d) $\int_\gamma \frac{e^z}{(z-2)^2}\,\mathrm{d}z = 2\pi i e^2$, *where* $\gamma = \partial D(2,1)$.

Exercise 4.12. *Define* $f : D(0,1) \to \mathbb{C}$ *by* $f(z) = \dfrac{z}{|1-z|^2}$, $z \in D(0,1)$. *Prove, for every* $0 < r < 1$, *that*

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{it})\right|\,\mathrm{d}t = \frac{r}{1-r^2}.$$

Suggestion: It is helpful to calculate the integral $\int_{\partial D(0,r)} \frac{\mathrm{d}z}{(1-z)(z-r^2)}$ *first with the Cauchy Integral Formula, and then with the definition of complex path-integral. Here* $\partial D(0,r)$ *is traveled once and counterclockwise.*

Exercise 4.13. *Let* $r > 0$ *and let* P *be a polynomial whose roots are all contained in the open disk* $D(0,r)$. *Prove that*

$$\int_{\partial D(0,r)} \frac{P'(z)}{P(z)}\,\mathrm{d}z = 2\pi i \deg(P).$$

Suggestion: It is very helpful to use Theorem 4.47 to factorize P. *Recall that for a polynomial* $P(z) = a_n z^n + \cdots + a_1 z + a_0$ *with* $a_n \neq 0$, *we have* $\deg(P) = n$.

Exercise 4.14. *Let* $\Omega \subset \mathbb{C}$ *be open so that* $\overline{D}(z_0, r) \subset \Omega$, *and let* $f : \Omega \to \mathbb{C}$ *be holomorphic. Prove that*

$$|f(z_0)|^2 \leq \frac{1}{\pi r^2}\int_0^{2\pi}\int_0^r |f(z_0 + se^{it})|^2\, s\,\mathrm{d}s\,\mathrm{d}t.$$

Exercise 4.15. *Use appropriately the Cauchy Integral Formula to calculate the integral*

$$\int_{\partial D(0,1)} \frac{\mathrm{d}z}{(z-3/2)(z-2/3)}.$$

Then calculate the real integral

$$\int_0^{2\pi} \frac{\mathrm{d}t}{13 - 12\cos t}.$$

Exercise 4.16. *Let* $\xi \in \mathbb{C}$ *with* $|\xi| \neq 1$. *Compute the integral*

$$\int_0^{2\pi} \frac{\mathrm{d}t}{1 - 2\xi\cos t + \xi^2}.$$

Exercise 4.17. *Define, for each* $r > 0$, *the path* $\gamma_r : [0, \pi/4] \to \mathbb{C}$ *by* $\gamma_r(t) = re^{it}$. *Prove that*

$$\lim_{r\to+\infty} \int_{\gamma_r} e^{-z^2}\,\mathrm{d}z = 0.$$

Then, integrate the function e^{-z^2} *over the paths* $\Gamma_r := [0,r] \star \gamma_r \star [re^{\frac{i\pi}{4}}, 0]$, $r > 0$, *to show that*

$$\int_0^\infty \sin(x^2)\,\mathrm{d}x = \int_0^\infty \cos(x^2)\,\mathrm{d}x = \frac{\sqrt{2\pi}}{4}.$$

Suggestions: For the limit part, take into account the inequality $\cos(2t) \geq 1 - \frac{4}{\pi}t$ *for all* $t \in [0, \frac{\pi}{4}]$. *For the second part, what is the value of* $\int_{\Gamma_r} e^{-z^2}\,\mathrm{d}z$ *for all* $r > 0$*?*

Exercise 4.18. *Define, for each* $r > 0$, *the path* $\gamma_r : [0, \pi] \to \mathbb{C}$ *by* $\gamma_r(t) = re^{it}$. *Prove that:*

(a) $\lim\limits_{r\to 0} \int_{\gamma_r} \frac{e^{iz}}{z}\,\mathrm{d}z = \pi i$.

(b) $\lim\limits_{r\to \infty} \int_{\gamma_r} \frac{e^{iz}}{z}\,\mathrm{d}z = 0$.

(c) *Integrate the function* $z \mapsto \frac{e^{iz}}{z}$ *over the paths* $[-R, -r] \star \gamma_r^- \star [r, R] \star \gamma_R$, *with* $R > r > 0$, *and let* $r \to 0$ *and* $R \to \infty$ *to prove the identity*

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}.$$

Suggestions: In (a), estimate the modulus $\int_{\gamma_r} \frac{e^{iz}}{z}\,\mathrm{d}z - \pi i = \int_{\gamma_r} \frac{e^{iz}-1}{z}\,\mathrm{d}z$, *and take into account the Taylor Series of* $e^w - 1$ *centered at* $w = 0$.
In (b), the estimates $\sin t \geq 2t/\pi$ *and* $\cos t \geq 1 - 2t/\pi$ *for* $t \in [0, \pi/2]$ *can be helpful.*
In (c), find the winding number $W(\Gamma_{r,R}, 0)$ *of the path* $\Gamma_{r,R} := [-R, -r] \star \gamma_r^- \star [r, R] \star \gamma_R$ *around* 0, *for all* $r < R$, *and then use appropriately Theorem 4.27.*

Exercise 4.19. *Let* $f : \mathbb{C} \to \mathbb{C}$ *holomorphic such that there constants* $C, a > 0$ *with* $|f(z)| \leq Ce^{a|z|}$ *for all* $z \in \mathbb{C}$. *Prove that* $|f'(z)| \leq Cae^{a|z|+1}$ *for all* $z \in \mathbb{C}$.

Suggestion: Corollary 4.34.

Exercise 4.20. *Let* $f : \mathbb{D} \to \mathbb{C}$ *be holomorphic. Prove that*

$$2|f'(0)| \leq \sup\{|f(w) - f(-w)| \,:\, w \in \mathbb{D}\}.$$

Exercise 4.21. *Let* $\Omega := \{z \in \mathbb{C} \,:\, |\mathrm{Im}(z)| < 1\}$, *and let* $f : \Omega \to \mathbb{C}$ *be holomorphic so that there are constants* $C > 0$ *and* $a \in \mathbb{R}$ *with*

$$|f(z)| \leq C\,(1 + |z|)^a \quad \textit{for all} \quad z \in \Omega.$$

Prove that for every $n \in \mathbb{N}$, *there is a constant* $C_n > 0$ *so that*

$$\left|f^{(n)}(x)\right| \leq C_n\,(1 + |x|)^a \quad \textit{for all} \quad x \in \mathbb{R}.$$

Exercise 4.22. *Let* $f : \mathbb{C} \to \mathbb{C}$ *be holomorphic. Prove that for every* $z \in \mathbb{C}$ *there exists* $M > 0$ *such that* $|f^{(n)}(z)| \leq M \cdot n!$ *for every* $n \in \mathbb{N}$.

Exercise 4.23. *Let* Ω *be open,* $z_0 \in \Omega$, *and* $f : \Omega \to \mathbb{C}$ *holomorphic in* Ω. *Show that the estimates*

$$|f^{(n)}(z_0)| \geq n!\, n^n$$

only can hold for finitely many $n \in \mathbb{N}$.

Exercise 4.24. *Let* Ω *be open and connected, and* $f : \Omega \to \mathbb{C}$ *be holomorphic so that there exists* $z_0 \in \Omega$ *and* $C > 0$ *with* $|f^{(n)}(z_0)| \leq C$ *for every* $n \in \mathbb{N}$. *Prove that there is a function* $g : \mathbb{C} \to \mathbb{C}$ *holomorphic with* $g = f$ *on* Ω.

Suggestion: What is the radius of convergence of the power series $\sum_{n=0}^\infty \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$*? Then, at some point Corollary 4.44 can be helpful.*

Exercise 4.25. *Let* Ω *be open and connected, and* $f, g : \Omega \to \mathbb{C}$ *be two holomorphic functions with* $f(z)g(z) = 0$ *for all* $z \in \mathbb{C}$. *Show that either* $f = 0$ *in* Ω *or* $g = 0$ *in* Ω.

Suggestion: Use appropriately Corollary 4.44.

Exercise 4.26. *Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic, and suppose there are constants $r, C > 0$, $n \in \mathbb{N}$ such that $|f(z)| \leq C|z|^n$ for all $|z| \geq r$. Prove that f is a polynomial with $\deg(f) \leq n$.*

Exercise 4.27. *Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic with $|f'(z)| \leq |z|$ for all $z \in \mathbb{C}$. Show that there exist $w_1, w_2 \in \mathbb{C}$ with $|w_2| \leq 1/2$ so that $f(z) = w_1 + w_2 z^2$ for all $z \in \mathbb{C}$.*

Exercise 4.28. *Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic, and suppose there are constants $C > 0$, $\alpha > 0$ with $\alpha \notin \mathbb{N}$ such that $|f(z)| \leq C|z|^\alpha$ for all $z \in \mathbb{C}$. Prove that f is identically zero in $\mathbb{C}$.*

Exercise 4.29. *Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic with $\lim\limits_{|z|\to\infty} \frac{f(z)}{z} = 0$. Prove that f is constant in $\mathbb{C}$.*

Exercise 4.30. *Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic so that there exist $M > 0$, $\alpha \in (0,1)$ with $|f(z)| \leq M(1 + |z|^\alpha)$ for all $z \in \mathbb{C}$. Prove that f is constant in $\mathbb{C}$.*

Exercise 4.31. *Find the Taylor series of f centered at z_0 in the following cases.*

(i) $f(z) = e^z$, $z_0 = 1$.

(ii) $f(z) = \sin^2 z$, $z_0 = 0$.

(iii) $f(z) = e^z \sin z$, $z_0 = 0$.

Suggestion: In (ii), it is helpful to note that $2\sin^2 z = 1 - \cos(2z)$.

Exercise 4.32. *Show that there exist **no** function $f : D(0,2) \to \mathbb{C}$ holomorphic in $D(0,2)$ with $f(1/n) = -1/n^2$ and $f\left(\frac{n+1}{n}\right) = 1/n$ for all $n \geq 2$.*

Suggestion: Use appropriately Corollary 4.44.

Exercise 4.33. *Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic with $f\left(\frac{1}{n+1}\right) \in \mathbb{R}$ for all $n \in \mathbb{N}$. Show that $f(\overline{z}) = \overline{f(z)}$ for all $z \in \mathbb{D}$.*

Exercise 4.34. *Let $f : D(z_0, R) \to \mathbb{C}$ be holomorphic. Prove that if $0 < r < R$, then*

(i)

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{it})|^2 \,\mathrm{d}t = \sum_{n=0}^{\infty} \left|\frac{f^{(n)}(z_0)}{n!}\right|^2 r^{2n}.$$

(ii) Denoting $M(r) := \max\{|f(z)| : z \in \partial D(z_0, r)\}$, then

$$\sum_{n=0}^{\infty} \left|\frac{f^{(n)}(z_0)}{n!}\right|^2 r^{2n} \leq (M(r))^2.$$

Exercise 4.35. *Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. Prove that f is constant in any of the following cases.*

(i) $|f(z)| \geq 1$ *for all* $z \in \mathbb{C}$.

(ii) Either $\mathrm{Re}(f)(z) \geq 0$ *for all* $z \in \mathbb{C}$ *or* $\mathrm{Im}(f)(z) \geq 0$ *for all* $z \in \mathbb{C}$.

(iii) Either $\mathrm{Re}(f)(z) \neq 0$ *for all* $z \in \mathbb{C}$ *or* $\mathrm{Im}(f)(z) \neq 0$ *for all* $z \in \mathbb{C}$.

Suggestion: Use appropriately Theorem 4.45.

Exercise 4.36. *Let* $P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ *a polynomial with* $a_0 \neq 0$. *Let* $z_1, \ldots, z_n$ *be the roots of* P. *Show that:*

$$\sum_{k=1}^{n} z_k = -\frac{a_1}{a_0} \quad \textit{and} \quad \prod_{k=1}^{n} z_k = (-1)^n \frac{a_n}{a_0}.$$

Exercise 4.37. *For any nonempty set* $A \subset \mathbb{C}$, *denote* $M(A) := \max\{|e^z| : z \in A\}$. *Find* $M(A)$ *in the following cases:*

(i) $A = \{z \in \mathbb{C} : |z - 1 - i| \leq 1\}$.

(ii) $A = \{z \in \mathbb{C} : |\operatorname{Re}(z) - 2| \leq 1 \text{ and } |\operatorname{Im}(z) - 3| \leq 1\}$.

Exercise 4.38. *For any nonempty set* $A \subset \mathbb{C}$, *denote* $M(A) := \max\{|\cos z| : z \in A\}$. *Find* $M(A)$ *in the following cases:*

(i) $A = \{z \in \mathbb{C} : \operatorname{Re}(z), \operatorname{Im}(z) \in [0, 2\pi]\}$.

(ii) $A = \{z \in \mathbb{C} : |z| \leq 1\}$.

Suggestion: Theorem 4.50.

Exercise 4.39. *Let* $\Omega \subset \mathbb{C}$ *be open and connected, with* $\overline{D}(0, r) \subset \Omega$ *for some* $r > 0$. *Let* $f, g : \Omega \to \mathbb{C}$ *be holomorphic with* $|f(z)| = |g(z)|$ *for all* $z \in \partial D(0, r)$ *and* $f(z) \neq 0 \neq g(z)$ *for all* $z \in \overline{D}(0, r)$. *Prove that there exists* $\lambda \in \mathbb{C}$ *with* $|\lambda| = 1$ *so that* $f = \lambda g$ *in* Ω.

Exercise 4.40. *Let* $\Omega \subset \mathbb{C}$ *be open and connected,* $f : \Omega \to \mathbb{C}$ *a non-constant holomorphic function such that* $f(z) \neq 0$ *for all* $z \in \Omega$. *Prove that the function* $|f| : \Omega \to \mathbb{R}$ *has no local minimum in* Ω.

Exercise 4.41. *Let* $\Omega \subset \mathbb{C}$ *be open and connected,* $\{f_n : \Omega \to \mathbb{C}\}_n$ *a sequence of holomorphic functions in* Ω *with* $f_n(z) \neq 0$ *for all* $z \in \Omega$ *and* $n \in \mathbb{N}$. *Prove that if* $\{f_n\}_n$ *converges uniformly to* $f : \Omega \to \mathbb{C}$ *in compact subsets of* Ω, *then either* $f(z) \neq 0$ *for all* $z \in \Omega$ *or* $f = 0$ *in* Ω.

Exercise 4.42. *Let* $\Omega \subset \mathbb{C}$ *be open and connected with* $\overline{\mathbb{D}} \subset \Omega$, *and* $f : \Omega \to \mathbb{C}$ *a non-constant holomorphic function such that* $|f(z)| = 1$ *for all* $z \in \partial\mathbb{D}$. *Prove that* f *has finitely many, and at least one, zeros in* $\mathbb{D}$.

Exercise 4.43. *Let* $\Omega \subset \mathbb{C}$ *be open and connected so that* $\overline{\mathbb{D}} \subset \Omega$ *and let* $f : \Omega \to \mathbb{C}$ *be holomorphic. Prove the following statements.*

(i) If f *is non-constant in* Ω *and* $|f|$ *is constant in* $\partial\mathbb{D}$, *then there exists* $z_0 \in \mathbb{D}$ *with* $f(z_0) = 0$.

(ii) If f *is pure imaginary in* $\partial\mathbb{D}$ *(that is,* $f(\partial\mathbb{D}) \subset i\mathbb{R}$*), then* f *is constant in* Ω.

(iii) If f *is real in* $\partial\mathbb{D}$ *(that is,* $f(\partial\mathbb{D}) \subset \mathbb{R}$*), then* f *is constant in* Ω.

Exercise 4.44. *Let* $\Omega \subset \mathbb{C}$ *be open, connected and bounded, and* $f : \overline{\Omega} \to \mathbb{C}$ *continuous in* $\overline{\Omega}$ *and holomorphic in* Ω. *Prove that if* $|f|$ *is constant in* $\partial\Omega$, *then either* f *is constant in* $\overline{\Omega}$ *or* f *has a zero in* Ω.

Exercise 4.45. *Let* $f : \mathbb{C} \to \mathbb{C}$ *be a non-constant holomorphic function and* $c > 0$. *Prove that*

$$\overline{\{z \in \mathbb{C} : |f(z)| < c\}} = \{z \in \mathbb{C} : |f(z)| \leq c\}.$$

Exercise 4.46. *Let* $f : \mathbb{D} \to \mathbb{C}$ *holomorphic with* $|f(z)| \leq 1$ *for all* $z \in \mathbb{D}$ *and so that* $f(w) = 0$ *for some* $w \in \mathbb{D}$. *Prove that*

$$|f(z)| \leq \left| \frac{z - w}{1 - \overline{w} z} \right| \quad \textit{for all} \quad z \in \mathbb{D}.$$

Chapter 5

Laurent Series and Singularities

In this chapter, we study the behaviour of functions f that are holomorphic in a *punctured disk* $D(z_0, r) \setminus \{z_0\}$. We then say that the function has an *isolated singularity* at z_0. These functions f can still be written as power series around z_0, if we include negative powers $(z - z_0)^{-n}$ in the series. This series is called the *Laurent Series* of f around z_0. We classify the type of singularities depending on the coefficients of the Laurent Series. Our main goal is to prove the *Cauchy Residues Theorem*, which, among other applications, permits to easily compute improper real integrals.

5.1 Laurent Series

Laurent Series are essentially formal power series containing all possible negative powers as well.

Definition 5.1 (Laurent Series). *A* ***Laurent Series*** *centered at a point $z_0 \in \mathbb{C}$ is any series of the form*

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n; \quad \textit{where} \quad \{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}.$$

The series $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ is called the ***principal part*** *of the Laurent Series above.*

Also, we say that the series $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ ***converges at*** *$z \in \mathbb{C}$ if both series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ converge. Note that then*

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n} = \lim_{N\to\infty} \sum_{n=-N}^{N} a_n(z - z_0)^n.$$

Let us make some remarks on the radii of convergence of the series above, as well as the holomorphicity of those.

Remark 5.2. Let $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ be a Laurent series centered at z_0. If we denote by $R \in [0, +\infty]$ the radius of convergence of the power series $\sum_{n=1}^{\infty} a_{-n}\xi^n$, we know that

$$R = \left(\limsup_{n\to+\infty} |a_{-n}|^{1/n}\right)^{-1},$$

by virtue of Theorem 3.15. We actually know from that theorem that $\sum_{n=1}^{\infty} a_{-n}\xi^n$ converges absolutely when $|\xi| < R$, and absolutely-uniformly on $\xi \in \overline{D}(0, r)$, for all $r < R$. Equivalently, $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ converges absolutely when $|z - z_0| > 1/R$ and absolutely-uniformly when $|z - z_0| \geq 1/r$ for all $r < R$. In other words, denoting

$$R_1 = \limsup_{n\to+\infty} |a_{-n}|^{1/n},$$

the principal part $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ of the Laurent Series $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ is absolutely convergent in the set $\{z \in \mathbb{C} : |z-z_0| > R_1\}$, and absolutely-uniformly convergent on each set $\{z \in \mathbb{C} : |z-z_0| \geq r_1\}$ for all $r_1 > R_1$.

On the other hand, the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has radius of converge $R_2 \in [0,+\infty]$ given by the formula

$$R_2 = \left(\limsup_{n\to+\infty} |a_n|^{1/n}\right)^{-1},$$

again by Theorem 3.15. Thus, if $R_1 < R_2$, **the Laurent Series**

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = \lim_{N\to\infty} \sum_{n=-N}^{N} a_n(z-z_0)^n,$$

converges absolutely whenever $R_1 < |z-z_0| < R_2$**, and absolutely–uniformly on each set** $\{z \in \mathbb{C} : r_1 \leq |z-z_0| \leq r_2\}$**, with** $R_1 < r_1 \leq r_2 < R_2$.

Moreover, by Weierstrass Theorem 4.37, **the Laurent series**

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

defines a holomorphic function in the set $\{z \in \mathbb{C} : R_1 < |z-z_0| < R_2\}$.

Throughout the rest of the chapter we will use the following notation, for $z_0 \in \mathbb{C}$, $0 \leq r_1 < r_2 \leq +\infty$:

$$A_{r_1,r_2}(z_0) := D(z_0,r_2) \setminus \overline{D}(z_0,r_1) = \{z \in \mathbb{C} : r_1 < |z-z_0| < r_2\}.$$

That is, $A_{r_1,r_2}(z_0)$ is the annulus centered at z_0 with larger radius r_2 and smaller radius r_1.

The following lemma for holomorphic functions over annuli is crucial.

Lemma 5.3. *Let $R_1 \in [0,+\infty)$ and $R_2 \in (0,+\infty]$ with $R_1 < R_2$, let $z_0 \in \mathbb{C}$, and let $g : A_{R_1,R_2}(z_0) \to \mathbb{C}$ be holomorphic in $A_{R_1,R_2}(z_0)$. Then, the value of the integral $\int_{\partial D(0,r)} g(w)\,\mathrm{d}w$ is the same for all $R_1 < r < R_2$.*

Proof. Let $R_1 < s < r < R_2$ and let $\varepsilon > 0$ be so that $R_1 < s-\varepsilon < s < r < r+\varepsilon < R_2$. We claim first that for every $\theta \in [0,2\pi]$, we have

$$D_\theta := D\left(z_0 + \frac{s+r}{2}e^{i\theta}, \frac{r-s}{2} + \frac{\varepsilon}{2}\right) \subset A_{s-\varepsilon,r+\varepsilon}(z_0).$$

Indeed, if z is in the disk of the left hand side, then the triangle inequality yields

$$s-\varepsilon < \frac{s+r}{2} - \left|z - \left(z_0 + \frac{s+r}{2}e^{i\theta}\right)\right| \leq |z-z_0| \leq \left|z - \left(z_0 + \frac{s+r}{2}e^{i\theta}\right)\right| + \frac{s+r}{2} < r+\varepsilon. \quad (5.1.1)$$

Now, denote by γ_s and γ_r the paths that travel once and counterclockwise the circles $\partial D(0,s)$ and $\partial D(0,r)$ respectively. Using the inclusions (5.1.1), clearly we can find a partition $0 = t_0 < t_1 < \cdots < t_N < t_{N+1} = 2\pi$ of $[0,2\pi]$, and points $\theta_0,\ldots,\theta_{N+1} \in [0,2\pi]$ so that, if we denote $D_n := D_{\theta_n}$ for all $n \in \{0,\ldots,N+1\}$, then

$$D_n \cap D_{n+1} \neq \emptyset, \quad n = 0,\ldots,N+1; \quad \text{and} \quad \gamma_s([t_n,t_{n+1}]) \cup \gamma_r([t_n,t_{n+1}]) \subset D_n, \quad n = 0,\ldots,N. \quad (5.1.2)$$

But by (5.1.1), each D_n is convex subset of $A_{R_1,R_2}(z_0)$; where g is holomorphic. So, Corollary 4.22 provides us with $F_n : D_n \to \mathbb{C}$ holomorphic so that $(F_n)' = g$ on D_n. But then in the set $D_n \cap D_{n+1}$, both F_n and F_{n+1} are primitives of g and so $F_{n+1} - F_n$ is a constant in $D_n \cap D_{n+1}$, and by (5.1.2) this implies

$$F_{n+1}(\gamma_r(t_{n+1})) - F_{n+1}(\gamma_s(t_{n+1})) = F_n(\gamma_r(t_{n+1})) - F_n(\gamma_s(t_{n+1})), \quad n = 0,\ldots,N. \quad (5.1.3)$$

Now, applying Theorem 4.1.7 to each f on each path $\gamma_s([t_n, t_{n+1}])$, $\gamma_r([t_n, t_{n+1}])$, and taking into account (5.1.3), we can write

$$\begin{aligned}
\int_{\gamma_r} g(w)\,\mathrm{d}w - \int_{\gamma_s} g(w)\,\mathrm{d}w &= \sum_{n=0}^{N}\left(\int_{\gamma_r([t_n,t_{n+1}])} g(w)\,\mathrm{d}w - \int_{\gamma_s([t_n,t_{n+1}])} g(w)\,\mathrm{d}w\right)\\
&= \sum_{n=0}^{N} F_n(\gamma_r(t_{n+1})) - F_n(\gamma_r(t_n)) - \sum_{n=0}^{N} F_n(\gamma_s(t_{n+1})) - F_n(\gamma_s(t_n))\\
&= \sum_{n=0}^{N} [F_n(\gamma_r(t_{n+1})) - F_n(\gamma_s(t_{n+1})) - (F_n(\gamma_r(t_n)) - F_n(\gamma_s(t_n)))]\\
&= \sum_{n=0}^{N} [F_{n+1}(\gamma_r(t_{n+1})) - F_{n+1}(\gamma_s(t_{n+1})) - (F_n(\gamma_r(t_n)) - F_n(\gamma_s(t_n)))]\\
&= F_{N+1}(\gamma_r(t_{N+1})) - F_{n+1}(\gamma_s(t_{N+1})) - (F_0(\gamma_r(t_0)) - F_0(\gamma_s(t_0)))\\
&= F_{N+1}(\gamma_r(2\pi)) - F_{n+1}(\gamma_s(2\pi)) - (F_0(\gamma_r(0)) - F_0(\gamma_s(0))) = 0 - 0 = 0.
\end{aligned}$$

And the assertion is proven. □

We are now ready to prove a version of the Cauchy Integral Formula on annuli; compare to Corollary 4.28.

Theorem 5.4 (Cauchy Integral Formula in an Annulus)**.** *Let $R_1 \in [0, +\infty)$ and $R_2 \in (0, +\infty]$ with $R_1 < R_2$, let $z_0 \in \mathbb{C}$, and let $f : A_{R_1,R_2}(z_0) \to \mathbb{C}$ be holomorphic in $A_{R_1,R_2}(z_0)$. Then, for every r_1, r_2 with $R_1 < r_1 < r_2 < R_2$, we have*

$$f(z) = \frac{1}{2\pi i}\int_{\partial D(z_0,r_2)} \frac{f(w)}{w-z}\,\mathrm{d}w - \frac{1}{2\pi i}\int_{\partial D(z_0,r_1)} \frac{f(w)}{w-z}\,\mathrm{d}w, \quad \textit{whenever} \quad r_1 < |z - z_0| < r_2. \quad (5.1.4)$$

Proof. Fix r_1, r_2 with $R_1 < r_1 < r_2 < R_2$, and z with $r_1 < |z - z_0| < r_2$. Define the function $g : A_{R_1,R_2}(z_0) \to \mathbb{C}$ by

$$g(w) = \begin{cases} \dfrac{f(w) - f(z)}{w - z} & \text{if } w \in A_{R_1,R_2}(z_0) \setminus \{z\}, \\ f'(z) & \text{if } w = z. \end{cases}$$

It is clear that g is continuous in $A_{R_1,R_2}(z_0)$ and holomorphic in $A_{R_1,R_2}(z_0) \setminus \{z\}$, so Corollary 4.35 implies that actually $g \in \mathcal{H}(A_{R_1,R_2}(z_0))$. Denote by γ_{r_1} and γ_{r_2} the paths describing respectively $\partial D(z_0, r_1)$ and $\partial D(z_0, r_2)$ traveled once and counterclockwise. Then, by Lemma 5.3, we may write

$$\begin{aligned}
0 = \int_{\gamma_{r_1}} g(w)\,\mathrm{d}w - \int_{\gamma_{r_2}} g(w)\,\mathrm{d}w &= \int_{\gamma_{r_1}} \frac{f(w) - f(z)}{w-z}\,\mathrm{d}w - \int_{\gamma_{r_2}} \frac{f(w) - f(z)}{w-z}\,\mathrm{d}w\\
&= \int_{\gamma_{r_1}} \frac{f(w)}{w-z}\,\mathrm{d}w - \int_{\gamma_{r_2}} \frac{f(w)}{w-z}\,\mathrm{d}w + f(z)\left(\int_{\gamma_{r_1}} \frac{\mathrm{d}w}{w-z} - \int_{\gamma_{r_2}} \frac{\mathrm{d}w}{w-z}\right)\\
&= \int_{\gamma_{r_1}} \frac{f(w)}{w-z}\,\mathrm{d}w - \int_{\gamma_{r_2}} \frac{f(w)}{w-z}\,\mathrm{d}w + 2\pi i f(z)\,(W(\gamma_{r_1}, z) - W(\gamma_{r_2}, z))\\
&= \int_{\gamma_{r_1}} \frac{f(w)}{w-z}\,\mathrm{d}w - \int_{\gamma_{r_2}} \frac{f(w)}{w-z}\,\mathrm{d}w + 2\pi i f(z).
\end{aligned}$$

In the last equality we used that $W(\gamma_{r_1}, z) = 1$ and $W(\gamma_{r_2}, z) = 0$; see Proposition 4.26. We have thus shown (5.1.4). □

Finally, we show that holomorphic functions in annuli admit Laurent Series expansions.

Theorem 5.5 (Laurent Series Expansion). *Let $R_1 \in [0, +\infty)$ and $R_2 \in (0, +\infty]$ with $R_1 < R_2$, let $z_0 \in \mathbb{C}$, and let $f : A_{R_1,R_2}(z_0) \to \mathbb{C}$ be holomorphic in $A_{R_1,R_2}(z_0)$. Then there are numbers $\{a_n\}_{n\in\mathbb{N}^*}$, $\{b_n\}_{n\in\mathbb{N}}$ such that*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, \quad \text{for all} \quad z \in A_{R_1,R_2}(z_0), \tag{5.1.5}$$

with absolute convergence for all $z \in A_{R_1,R_2}(z_0)$. More precisely:

- *The series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges absolutely in $D(z_0, R_2)$, and absolutely–uniformly in $\overline{D}(z_0, r_2)$ for all $0 < r_2 < R_2$;*
- *The series $\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ converges absolutely in $\mathbb{C} \setminus \overline{D}(z_0, R_1)$, and absolutely–uniformly on each set $\mathbb{C} \setminus D(z_0, r_1)$ for all $R_1 < r_1 < \infty$.*

In particular, both series in (5.1.5) *(simultaneously) converge absolutely-uniformly on each closed annulus $\overline{A_{r_1,r_2}} = \{z \in \mathbb{C} : r_1 \leq |z - z_0| \leq r_2\}$, with $R_1 < r_1 < r_2 < R_2$. Furthemore, if $R_1 < r < R_2$, we have*

$$a_n = \frac{1}{2\pi i} \int_{\partial D(z_0,r)} \frac{f(w)}{(w - z_0)^{n+1}} \,\mathrm{d}w, \ n \in \mathbb{N}^*, \quad \text{and} \quad b_n = \frac{1}{2\pi i} \int_{\partial D(z_0,r)} f(w)(w - z_0)^{n-1} \,\mathrm{d}w, \ n \in \mathbb{N}; \tag{5.1.6}$$

where $\partial D(z_0, r)$ is traveled once and counterclockwise.

Proof. We will denote by γ_r the path travelling $\partial D(z_0, r)$ once and counterclockwise, for every $r > 0$. Define the functions $f_1 : \mathbb{C} \setminus \overline{D}(z_0, R_1) \to \mathbb{C}$ and $f_2 : D(z_0, R_2) \to \mathbb{C}$ by

$$f_1(z) := \frac{1}{2\pi i} \int_{\gamma_s} \frac{f(w)}{w - z} \,\mathrm{d}w, \quad \text{whenever} \quad |z - z_0| > s, \ R_1 < s < R_2, \tag{5.1.7}$$

$$f_2(z) := \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{w - z} \,\mathrm{d}w, \quad \text{whenever} \quad |z - z_0| < r, \ R_1 < r < R_2. \tag{5.1.8}$$

We need to verify that f_1 (resp. f_2) is well-defined in $\mathbb{C}\setminus\overline{D}(z_0, R_1)$ (resp. $D(z_0, R_2)$), since for each $z \in \mathbb{C} \setminus \overline{D}(z_0, R_1)$ (resp. $z \in D(z_0, R_2)$) there are many $s \in (R_1, R_2)$ with $|z - z_0| > s$ (resp. many $r \in (R_1, R_2)$ with $|z - z_0| < r$). For each $z \in \mathbb{C} \setminus \overline{D}(z_0, R_1)$ and $R_1 < s < s' < \min\{R_2, |z - z_0|\}$, let $\varepsilon > 0$ be so that

$$R_1 < s - \varepsilon < s < s' < s' + \varepsilon < \min\{R_2, |z - z_0|\},$$

and consider $g_z : A_{s-\varepsilon,s'+\varepsilon}(z_0) \to \mathbb{C}$ given by $g_z(w) = \frac{f(w)}{w-z}$, which is holomorphic, as $f \in \mathcal{A}_{\mathcal{R}_\infty,\mathcal{R}_\in}(\ddagger_\prime)$. By Lemma 5.3, we have

$$\int_{\gamma_{s'}} \frac{f(w)}{w - z} \,\mathrm{d}w = \int_{\gamma_s} \frac{f(w)}{w - z} \,\mathrm{d}w.$$

Thus, the value of $f_1(z)$ does not depend on the chosen $s \in (R_1, R_2)$ with $|z - z_0| > s$. Similarly, for $z \in D(z_0, R_2)$, and $R_1 < r' < r < R_2$ with $|z - z_0| < r'$, let $\varepsilon > 0$ be so that

$$\max\{|z - z_0|, R_1\} < r' - \varepsilon < r' < r < r + \varepsilon < R_2,$$

and define $h_z : A_{r'-\varepsilon,r+\varepsilon}(z_0) \to \mathbb{C}$ given by $h_z(w) = \frac{f(w)}{w-z}$. By Lemma 5.3, we have that

$$\int_{\gamma_r} \frac{f(w)}{w - z} \,\mathrm{d}w = \int_{\gamma_{r'}} \frac{f(w)}{w - z} \,\mathrm{d}w,$$

and $f_2(z)$ is well-defined. Also, we claim that

$$\lim_{|z|\to\infty} f_1(z) = 0. \tag{5.1.9}$$

Indeed, if we fix some $s \in (R_1, R_2)$, by (5.1.7), we have

$$\lim_{|z|\to\infty} |f_1(z)| = \lim_{|z|\to\infty} \left| \frac{1}{2\pi i} \int_{\gamma_s} \frac{f(w)}{w-z}\,\mathrm{d}w \right| \leq \lim_{|z|\to\infty} \frac{1}{2\pi} \int_{\gamma_s} \frac{|f(w)|}{|w-z|} |\mathrm{d}w|$$

$$\leq \lim_{|z|\to\infty} \frac{2\pi s}{2\pi} \frac{\sup\{|f(w)| \,:\, |w-z_0| = s\}}{|z-z_0| - s} = 0;$$

where we used that f is bounded in the compact set γ_s; recall Proposition 2.25.

We now check that $f_1 \in \mathcal{H}\left(\mathbb{C} \setminus \overline{D}(z_0, R_1)\right)$ and $f_2 \in \mathcal{H}\left(D(z_0, R_2)\right)$. If $z \in \mathbb{C} \setminus \overline{D}(z_0, R_1)$ there exists $\varepsilon > 0$ and $s \in (R_1, R_2)$ for which $D(z, \varepsilon) \subset \mathbb{C} \setminus \overline{D}(z_0, s)$. Then

$$f_1(\xi) = \frac{1}{2\pi i} \int_{\gamma_s} \frac{f(w)}{w - \xi}\,\mathrm{d}w, \quad \xi \in D(z, \varepsilon),$$

where the last integral defines a holomorphic function in $\xi \in D(z, \varepsilon)$, bu virtue Theorem 4.18. Thus f_1 is holomorphic in a disk containing z, and we conclude that $f_1 \in \mathcal{H}\left(\mathbb{C} \setminus \overline{D}(z_0, R_1)\right)$. An identical argument permits to show that $f_2 \in \mathcal{H}\left(D(z_0, R_2)\right)$.

Now, define $g : D(0, R_1) \to \mathbb{C}$ by the formula

$$g(\xi) := \begin{cases} f_1\left(z_0 + \frac{1}{\xi}\right) & \text{if } \xi \in D(0, R_1) \setminus \{0\} \\ 0 & \text{if } \xi = 0 \end{cases}.$$

Because $f_1 \in \mathcal{H}\left(\mathbb{C} \setminus \overline{D}(z_0, R_1)\right)$ it is clear that $g \in \mathcal{H}\left(D(0, R_1) \setminus \{0\}\right)$, and also that g is continuous in all of $D(0, R_1)$ due to (5.1.9). By Corollary 4.35, we get that $g \in \mathcal{H}(D(0, R_1))$. By Theorem 4.39 and Corollary 4.40, g_1 can be written as a power series with radius of convergence R_1 around 0, that is

$$g_1(\xi) = \sum_{n=1}^{\infty} \frac{g_1^{(n)}(0)}{n!} \xi^n, \quad \xi \in D(0, R_1);$$

where the convergence is uniform in disks $\overline{D}(0, t)$, with $t < R_1$. Also notice that the series begins at $n = 1$, as $g_1(0) = 0$, Denoting $b_n = -g_1^{(n)}(0)/n!$ for all $n \in \mathbb{N}$, the above implies that

$$f_1(z) = -\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, \quad z \in \mathbb{C} \setminus \overline{D}(z_0, R_1), \tag{5.1.10}$$

where the convergence is uniform in sets of the form $\mathbb{C} \setminus D(z_0, s)$ with $s > R_1$. On the other hand, because $f_2 \in \mathcal{H}\left(D(z_0, R_2)\right)$, Theorem 4.39 and Corollary 4.40 says that

$$f_2(z) = \sum_{n=0} a_n (z - z_0)^n, \quad z \in D(z_0, R_2),\ a_n = \frac{f_2^{(n)}(z_0)}{n!},\ n \in \mathbb{N}^*, \tag{5.1.11}$$

and the convergence is uniform on each closed disk $\overline{D}(z_0, r)$ with $r < R_2$. If we combine Theorem 5.4 with the definitions (5.1.7)–(5.1.8) of f_1 and f_2, and the power series expansions (5.1.10)–(5.1.11), we can conclude that

$$f(z) = f_2(z) - f_1(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, \quad z \in A_{R_1,R_2}(z_0), \tag{5.1.12}$$

with absolute-uniform convergence of both series in each closed sub-annuli of the form $\overline{A_{r_1,r_2}(z_0)} = \overline{D}(z_0, r_2) \setminus D(z_0, r_1)$, with $R_1 < r_1 < r_2 < R_2$. It only remains to show formulae (5.1.5) for the coefficients a_n, b_n. Let $r \in (R_1, R_2)$ and $n \in \mathbb{Z}$. We can use (5.1.12) and the fact that the convergence of both series is uniform in the circle γ_r to write the equalities

$$\int_{\gamma_r} \frac{f(w)}{(w-z_0)^{n+1}}\,\mathrm{d}w = \sum_{k=0}^{\infty} a_k \int_{\gamma_r} (w-z_0)^{k-n-1}\,\mathrm{d}w + \sum_{k=1}^{\infty} b_k \int_{\gamma_r} (w-z_0)^{-k-n-1}\,\mathrm{d}w$$

$$= \begin{cases} 2\pi i a_n & \text{if } n \geq 0 \\ 2\pi i b_n & \text{if } n < 0 \end{cases}.$$

This shows the validity of formulae (5.1.5). □

Definition 5.6 (Laurent Series and Principal Part)**.** *Let $R_1 \in [0, +\infty)$ and $R_2 \in (0, +\infty]$ with $R_1 < R_2$, let $z_0 \in \mathbb{C}$, and let $f : A_{R_1,R_2}(z_0) \to \mathbb{C}$ be holomorphic in $A_{R_1,R_2}(z_0)$. The Laurent Series*

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}, \qquad z \in A_{R_1,R_2}(z_0),$$

from (5.1.5) *in Theorem 5.5 is called* ***the Laurent Series of*** f ***at*** z_0*. This series is unique, as the coefficients* $\{a_n\}_{n\in\mathbb{N}^*}$, $\{b_n\}_{n\in\mathbb{N}}$ *are uniquely determined by the formulae* (5.1.6)*. Also, the series* $\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ *is called* ***the principal part of the Laurent Series of*** f ***at*** z_0*.*

5.2 Isolated Singularities

The definition of an isolated singularity is as follows.

Definition 5.7 (Isolated Singularity)**.** *We say that $z_0 \in \mathbb{C}$ is* ***an isolated singularity of*** f *provided that $f : D(z_0, r) \setminus \{z_0\} \to \mathbb{C}$ is holomorphic in $D(z_0, r) \setminus \{z_0\}$.*

We will refer to $D(z_0, r) \setminus \{z_0\}$ as the *punctured disk* of center z_0 and radius r.

Example 5.8. Here are some examples of isolated singularities.

(1) The function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $f(z) = 1/z$ has an isolated singularity at 0, as f is holomorphic in $\mathbb{C} \setminus \{0\}$.

(2) The function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $f(z) = \sin(1/z)$ has an isolated singularity at 0 as well.

(3) The function $f(z) = 1/\sin(1/z)$ has an isolated singularity at every point of the form $z_k = 1/k\pi$, $k \in \mathbb{Z} \setminus \{0\}$. However, f does **not** have an isolated singularity at $z_0 = 0$. The reason is that every punctured disk $D(0, r) \setminus \{0\}$ contains (infinitely many) points z_k, at which the function is not defined.

To classify the various types of isolated singularities, we look at the Laurent Series expansion of the function; see Definition 5.6. A particular case of Theorem 5.5 gives the following remark.

Remark 5.9. Let $f : D(z_0, r) \setminus \{z_0\} \to \mathbb{C}$ be holomorphic, that is, with an isolated singularity at z_0. By Theorem 5.5 there are coefficients $\{a_n\}_{n\in\mathbb{Z}}$ so that

$$f(z) = \sum_{n\in\mathbb{Z}} a_n (z-z_0)^n, \quad 0 < |z-z_0| < r; \tag{5.2.1}$$

with absolute-uniform convergence in annuli $\{z \in \mathbb{C} : t \leq |z-z_0| \leq s\}$, with $0 < t < s < r$. More precisely, defining $b_n := a_{-n}$ for all $n \in \mathbb{N}$, we can write

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}, \tag{5.2.2}$$

where $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely in $D(z_0,r)$, and absolutely–uniformly in $\overline{D}(z_0,s)$ for all $0<s<r$, and the **principal part of the Laurent series** $\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ converge absolutely in $\mathbb{C}\setminus\{z_0\}$ and absolutely–uniformly on $\mathbb{C}\setminus D(z_0,\varepsilon)$ for all $\varepsilon>0$.

Definition 5.10 (Types of Isolated Singularity)**.** *Let $f: D(z_0,r)\setminus\{z_0\}\to\mathbb{C}$ be holomorphic in $D(z_0,r)\setminus\{z_0\}$, that is, with an isolated singularity at z_0. Let $\{a_n\}_{n\in\mathbb{Z}}\subset\mathbb{C}$ as in the Laurent expansion* (5.2.1) *of f at z_0, in the punctured disk $D(z_0,r)\setminus\{z_0\}$. We say that*

- *f has **a removable singularity at** z_0 if $a_n=0$ for all $n<0$.*
- *f has **a pole at** z_0 if there exists $N\in\mathbb{N}$ with $a_{-N}\neq 0$ and $a_n=0$ for all $n<-N$. More precisely, in this case we say that f has **a pole of order** N **at** z_0. Sometimes, poles of order 1 are called **simple poles**.*
- *f has **an essential singulartiy at** z_0 if $a_n\neq 0$ for infinitely many $n<0$.*

Let us make some clarifications concerning the aspect of f depending on the various singularities.

Remark 5.11. Let $f: D(z_0,r)\setminus\{z_0\}\to\mathbb{C}$ with an isolated singularity at z_0; that is $f\in\mathcal{H}\left(D(z_0,r)\setminus\{z_0\}\right)$.

(1) If f has a removable singularity at z_0, then by (5.2.1), we have that

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n, \quad 0<|z-z_0|<r.$$

Defining $f(z_0):=a_0$, we get that actually

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n, \quad z\in D(z_0,r);$$

where the series defines a holomorphic function, e.g., by Theorem 3.26. Thus, the definition of the original $f: D(z_0,r)\setminus\{z_0\}\to\mathbb{C}$ can be extended to z_0, obtaining a holomorphic function in all of $D(z_0,r)$.

For example, if $f:\mathbb{C}\setminus\{0\}\to\mathbb{C}$ is defined by $f(z)=\frac{e^z-1}{z}$ for all $z\in\mathbb{C}\setminus\{0\}$, then f has a removable singularity at $z_0=0$. Indeed, a power series expansion of f has the aspect

$$f(z)=\frac{e^z-1}{z}=\frac{\sum_{n=0}^{\infty}\frac{z^n}{n!}-1}{z}=\frac{\sum_{n=1}^{\infty}\frac{z^n}{n!}}{z}=\sum_{n=1}^{\infty}\frac{z^{n-1}}{n!}=\sum_{n=0}^{\infty}\frac{z^n}{(n+1)!}, \quad z\in\mathbb{C}\setminus\{0\}.$$

The coefficients of the powers z^n, for $n<0$, in the Laurent Series are all equal to 0, confirming that f has a removable singularity.

(2) If f has a pole of order $N\in\mathbb{N}$ at z_0, then (5.2.1) becomes

$$f(z)=\frac{a_{-N}}{(z-z_0)^N}+\cdots+\frac{a_{-1}}{z-z_0}+\sum_{n=0}^{\infty} a_n(z-z_0)^n, \quad 0<|z-z_0|<r.$$

And $\frac{a_{-N}}{(z-z_0)^N}+\cdots+\frac{a_{-1}}{z-z_0}$ is the principal part of the Laurent series of f at z_0 in the annulus $0<|z-z_0|<r$. Moreover, the function

$$D(z_0,r)\ni z\mapsto\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

is holomorphic in $D(z_0,r)$.

The simplest example of a function with a pole of order N at 0 is $f(z)=\frac{1}{(z-z_0)^N}$, $z\in\mathbb{C}\setminus\{z_0\}$.

(3) For example, the function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ given by $f(z) = e^{1/z}$ has an essential singularity at $z_0 = 0$. To see this, we can simply write the Laurent series of f at 0 :

$$f(z) = e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^n = 1 + \sum_{n=1}^{\infty} \frac{1}{n! z^n}, \quad z \neq 0,$$

where infinitely many (actually all) of the coefficients $\frac{1}{n!}$ of $\frac{1}{z^n}$ are non-zero.

5.2.1 Removable Singularities. The Riemann Criterion

The following theorem due to Riemann gives a simple characterization of removability of isolated singularities.

Theorem 5.12 (Riemann's Theorem). *If $f : D(z_0, r) \setminus \{z_0\} \to \mathbb{C}$ is holomorphic, then f admits an extension $F : D(z_0, r) \to \mathbb{C}$ holomorphic in all of $D(z_0, r)$ if and only if f is bounded in $D(z_0, r) \setminus \{z_0\}$. In other words, f has a removable singularity at z_0 if and only if f is bounded in the punctured disk $D(z_0, r) \setminus \{z_0\}$.*

Proof. Let $0 < \varepsilon < s < r$ and apply Theorem 5.4 to f in the annulus $A_{0,r}(z_0)$ to obtain

$$f(z) = \frac{1}{2\pi i} \int_{\partial D(z_0,s)} \frac{f(w)}{w - z} \, \mathrm{d}w - \frac{1}{2\pi i} \int_{\partial D(z_0,\varepsilon)} \frac{f(w)}{w - z} \, \mathrm{d}w, \quad \text{whenever} \quad \varepsilon < |z - z_0| < s. \tag{5.2.3}$$

If we fix $z \in D(z_0, s) \setminus \{z_0\}$, let $0 < \varepsilon < |z - z_0|$, and note that then $w \in \partial D(z_0, \varepsilon)$ implies

$$|w - z| \geq |z - z_0| - |w - z_0| = |z - z_0| - \varepsilon.$$

Taking this into account and looking at the second integral of (5.2.3), we see that

$$\left| \int_{\partial D(z_0,\varepsilon)} \frac{f(w)}{w - z} \, \mathrm{d}w \right| \leq \int_{\partial D(z_0,\varepsilon)} \frac{|f(w)|}{|w - z|} |\mathrm{d}w| \leq \frac{\sup\{|f(w)| \, : \, w \in \partial D(z_0, r)\}}{|z - z_0| - \varepsilon} 2\pi\varepsilon.$$

Letting $\varepsilon \to 0^+$, the last term goes to 0, and so (5.2.3) becomes

$$f(z) = \frac{1}{2\pi i} \int_{\partial D(z_0,s)} \frac{f(w)}{w - z} \, \mathrm{d}w, \quad \text{whenever} \quad 0 < |z - z_0| < s. \tag{5.2.4}$$

Defining $g : D(z_0, s) \to \mathbb{C}$ by the formula $g(z) = \frac{1}{2\pi i} \int_{\partial D(z_0,s)} \frac{f(w)}{w-z} \, \mathrm{d}w$, from Theorem 4.18 we get that $g \in \mathcal{H}(D(z_0, s))$. Thus, we can define $F : D(0, r) \to \mathbb{C}$ by

$$F(z) = \begin{cases} f(z) & \text{if } z \in D(z_0, r) \setminus \{z_0\}, \\ g(z_0) & \text{if } z = z_0. \end{cases}$$

We get that $F = f$ in $D(z_0, r) \setminus \{z_0\}$, and so $F \in \mathcal{H}(D(z_0, r) \setminus \{z_0\})$. And also $F = g$ on $D(z_0, s)$; on which g is holomorphic. This shows that $F \in \mathcal{H}(D(z_0, r))$. □

Observe that Theorem 5.12 improves Corollary 4.35, since in that corollary we additional required the function f to be continuous in all of $D(z_0, r)$.

5.2.2 Characterizations of Poles

Recall the Definition 4.42 of a zero of order N of a holomorphic function, and their characterization from Proposition 4.43. There is a similar characterization for poles of order N.

Proposition 5.13. *Let $f : D(z_0, r) \setminus \{z_0\} \to \mathbb{C}$ be holomorphic, and $N \in \mathbb{N}$. The following statements are equivalent.*

(i) f *has a pole of order* N *at* z_0.

(ii) *There exists* $g \in \mathcal{H}(D(z_0, r))$ *with* $g(z_0) \neq 0$ *and so that*

$$g(z) = (z - z_0)^N f(z), \quad z \in D(z_0, r) \setminus \{z_0\}. \tag{5.2.5}$$

(iii) *The function* $1/f$ *admits a holomorphic extension* $\varphi : D(z_0, s) \setminus \{z_0\} \to \mathbb{C}$ *in a disk* $D(z_0, s)$, *so that* φ *has a zero of order* N *at* z_0.

Proof.

$(i) \implies (ii):$ By Remark 5.9 and the Definition 5.10 of pole of order N, there are numbers $\{a_n\}_{n \geq -N} \subset \mathbb{C}$ with $a_{-N} \neq 0$ and so that

$$f(z) = \frac{a_{-N}}{(z - z_0)^N} + \frac{a_{-N+1}}{(z - z_0)^{N-1}} + \cdots + \frac{a_{-1}}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad 0 < |z - z_0| < r.$$

Therefore,

$$(z - z_0)^N f(z) = a_{-N} + a_{-N+1}(z - z_0) + \cdots + a_{-1}(z - z_0) + \sum_{n=0}^{\infty} a_n (z - z_0)^{n+N} = \sum_{n=0}^{\infty} a_{n-N}(z - z_0)^n, \tag{5.2.6}$$

for all $0 < |z - z_0| < r$. Since the last series converges for all $z \in D(z_0, r)$, the function

$$g(z) := \sum_{n=0}^{\infty} a_{n-N}(z - z_0)^n, \quad z \in D(z_0, r)$$

is holomorphic in $D(z_0, r)$ (see e.g. Proposition 3.20). By the definition of g, we have $g(z_0) = a_{-N} \neq 0$. Moreover, by (5.2.6), we have $g(z) = (z - z_0)^N f(z)$ for all $z \in D(z_0, r) \setminus \{z_0\}$.

$(ii) \implies (iii):$ Since $g \in \mathcal{H}(D(z_0, r))$ and $g(z_0) \neq 0$, there exists $0 < s < r$ with $g(z) \neq 0$ for all $z \in D(z_0, s)$. Thus the function

$$\varphi(z) := \frac{(z - z_0)^N}{g(z)}, \quad z \in D(z_0, s);$$

is well-defined and holomorphic in $D(z_0, s)$. Moreover, by Proposition 4.43, φ has a zero of order N at z_0. Also, by the expression (5.2.5), we get that $f(z) \neq 0$ and $\varphi(z) = 1/f(z)$ for all $z \in D(z_0, s) \setminus \{z_0\}$. This shows (iii).

$(iii) \implies (i):$ If φ is as in (iii), then by Proposition 4.43, there exists $h \in \mathcal{H}(D(z_0, s))$ with $g(z_0) \neq 0$ and $\varphi(z) = (z - z_0)^N h(z)$ for all $z \in D(z_0, s)$. Moreover, replacing s with a smaller radius, we can assume that $h(z) \neq 0$ for all $z_0 \in D(z_0, s)$. Thus the function $1/h(z)$ is holomorphic in $D(z_0, s)$ and thus there are coefficients $\{c_n\}_{n \geq 0} \subset \mathbb{C}$ such that

$$\frac{1}{h(z)} = \sum_{n=0}^{\infty} c_n (z - z_0)^n, \quad z \in D(z_0, s);$$

where $c_0 = 1/h(z_0) \neq 0$. But then,

$$f(z) = \frac{1}{\varphi(z)} = \frac{1}{(z - z_0)^N} \frac{1}{h(z)} = \sum_{n=0}^{\infty} c_n (z - z_0)^{n-N} = \frac{c_0}{(z - z_0)^N} + \cdots + \frac{c_{N-1}}{z - z_0} + \sum_{n=0}^{\infty} c_{n+N}(z - z_0)^n,$$

for all $0 < |z - z_0| < s$. By the uniqueness of the Laurent series (of f at the point z_0), and the fact that $c_0 \neq 0$, we deduce that f has a pole of order N at z_0. □

Without specifying the order of the pole, we have a simpler characterization.

Proposition 5.14. *Let $f : D(z_0, r) \setminus \{z_0\} \to \mathbb{C}$ be holomorphic. The following statements are equivalent.*

(i) *f has a pole at z_0.*

(ii) $\lim_{z \to z_0} |f(z)| = \infty$.

Proof.

$(i) \implies (ii)$: Since f has a pole of some order $N \in \mathbb{N}$ at z_0, by Proposition 5.13, there exists $\varphi : D(z_0, s) \setminus \{z_0\} \to \mathbb{C}$ in a disk $D(z_0, s)$, with $\varphi = 1/f$ in $D(z_0, s) \setminus \{z_0\}$ and so that φ has a zero of order N at z_0. In particular φ is continuous at z_0 and $\varphi(z_0)$. Thus

$$\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} \frac{1}{|\varphi(z)|} = \infty.$$

$(ii) \implies (i)$: Since $\lim_{z \to z_0} |f(z)| = \infty$, in particular there exists $0 < s < r$ so that $f(z) \neq 0$ for all $z \in D(z_0, s) \setminus \{z_0\}$. We can then define

$$\varphi(z) = \begin{cases} 1/f(z) & \text{if } z \in D(z_0, s) \setminus \{z_0\} \\ 0 & \text{if } z = z_0. \end{cases}$$

Then, $\varphi \in \mathcal{H}(D(z_0, s) \setminus \{z_0\})$ and φ is continuous in $D(z_0, r)$ (including at $z = z_0$) by the condition $\lim_{z \to z_0} |f(z)| = \infty$. By Corollary 4.35, we get $\varphi \in \mathcal{H}(D(z_0, s))$. Because $\varphi(z_0) = 0$, we have that φ has a zero of order N, for some $N \in \mathbb{N}$. By Proposition 5.13 (see statement (iii) there), we may conclude that f has a pole (of order N) at z_0. □

5.2.3 Essential Singularities: The Casorati-Weierstrass Theorem

As concerns essential singularities, the following theorem provides a characterization.

Theorem 5.15 (Casorati-Weierstrass). *Let $f : D(z_0, r) \setminus \{z_0\} \to \mathbb{C}$ be holomorphic. Then, the following statements are equivalent.*

(i) *f has an essential singularity at z_0.*

(ii) $\overline{f\left(D(z_0, s) \setminus \{z_0\}\right)} = \mathbb{C}$ *for every* $0 < s \leq r$. *That is, for every $w \in \mathbb{C}$ there exists a sequence $\{z_n\}_n$ converging to z_0, with $z_n \neq z_0$ for all $n \in \mathbb{N}$, so that $\{f(z_n)\}_n$ converges to w.*

Proof.

$(i) \implies (ii)$: Suppose, for the sake of contradiction, that there exists $0 < s < r$ so that

$$\overline{f\left(D(z_0, r) \setminus \{z_0\}\right)} \subsetneq \mathbb{C}.$$

Then there exists $w \in \mathbb{C}$ and $\varepsilon > 0$ for which $|f(z) - w| \geq \varepsilon$ for all $z \in D(z_0, s) \setminus \{z_0\}$. We can then define

$$h(z) = \frac{1}{f(z) - w}, \quad z \in D(z_0, s) \setminus \{z_0\}.$$

Then $h \in \mathcal{H}(D(z_0, s) \setminus \{z_0\})$ and

$$|h(z)| \leq \frac{1}{|f(z) - w|} \leq \frac{1}{\varepsilon}, \quad z \in D(z_0, s) \setminus \{z_0\}.$$

That is, h is bounded in $D(z_0, s) \setminus \{z_0\}$. By Theorem 5.12, there exists $g : D(z_0, s) \to \mathbb{C}$ holomorphic in $D(z_0, s)$ with $g = h$ in $D(z_0, s) \setminus \{z_0\}$.

Now note that $f = w + 1/g$ in $D(z_0, s) \setminus \{z_0\}$. If we had $g(z_0) = 0$, then by the continuity of g at z_0 we would get that

$$\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} \left| w + \frac{1}{g(z)} \right| = \infty.$$

And Proposition 5.14 would imply that f has a pole at z_0, contradicting that the singularity of f at z_0 is essential. Therefore, we must have $g(z_0) \neq 0$. But then the continuity of g implies that there exists $\delta > 0$ and $0 < s' < s$ with $|g(z)| \geq \delta$ for all $z \in D(z_0, s')$. Therefore

$$|f(z)| \leq |w| + \frac{1}{|g(z)|} \leq |w| + \frac{1}{\delta}, \quad z \in D(z_0, s') \setminus \{z_0\}.$$

Thus f is bounded in $D(z_0, s')$ and Theorem 5.12 says that f has a removable singularity at z_0, contradicting again that the singularity of f at z_0 is essential. Therefore, (ii) must hold.

$(ii) \implies (i):$ For the sake of contradiction, assume that f does not have a removable singularity at z_0. If the singularity is removable, then there exists $g \in \mathcal{H}(D(z_0, r))$ with $g = f$ on $D(z_0, r) \setminus \{z_0\}$, and then

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} g(z) = g(z_0).$$

This contradicts (ii), taking any $w \in \mathbb{C}$ with $w \neq g(z_0)$. And if the singularity of f at z_0 is a pole, then Proposition 5.14 says that

$$\lim_{z \to z_0} |f(z)| = \infty,$$

and this contradicts (ii) as well. □

5.3 Residues at isolated singularities

5.3.1 Definition and Calculus of Residues

Definition 5.16 (Residue of a function at a point)**.** *Let $z_0 \in \mathbb{C}$, $r > 0$ and $f : D(z_0, r) \setminus \{z_0\} \to \mathbb{C}$ a holomorphic function. Let $\{a_n\}_{n \geq 0}$, $\{b_n\}_{n \in \mathbb{N}}$ be the unique sequences of complex numbers so that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, \quad 0 < |z - z_0| < r;$$

see Theorem 5.5. We define the ***residue of f at z_0*** *as the complex number*

$$\operatorname{Res}(f, z_0) = b_1.$$

Remark 5.17. If $f : D(z_0, r) \setminus \{z_0\} \to \mathbb{C}$ has an isolated singularity at z_0, then Theorem 5.5 says that in fact

$$\operatorname{Res}(f, z_0) = \int_{\partial D(0,s)} f(w)\, dw, \quad \text{for all} \quad 0 < s < r.$$

On the other hand, if f has a removable singularity at z_0, by Remark 5.11, we get that

$$\operatorname{Res}(f, z_0) = 0.$$

Let us now compute residues for some functions whose singularities are poles.

Example 5.18 (Residues of rational functions)**.** Consider the function $f(z) = \frac{P(z)}{Q(z)}$, with P, Q polynomials in $\mathbb{C}$. Naturally, f is holomorphic in $\mathbb{C} \setminus \{z \in \mathbb{C} : Q(z) = 0\}$. Assume that z_0 is a root of Q of multiplicity $N \in \mathbb{N}$ and that $P(z_0) \neq 0$. *Then f has a pole of order N and* $\operatorname{Res}(f, z_0)$ *coincides with the coefficient of the term $\frac{1}{z - z_0}$ in the partial fraction decomposition of f.*

Indeed, f has partial fraction decomposition of the form

$$f(z) = \frac{A_N}{(z - z_0)^N} + \frac{A_{N-1}}{(z - z_0)^{N-1}} + \cdots + \frac{A_1}{z - z_0} + h(z);$$

where $h(z)$ is the sum of the remaining terms of the partial fraction decomposition of f. This expression holds in some punctured disk $D(z_0, r) \setminus \{z_0\}$. The function h consists of a sum of fractions of the form $\frac{a_{j,n}}{(z-z_j)^n}$ (where z_j is a root of Q and $n \in \mathbb{N}$), (possibly) plus a polynomial (only when $\deg(P) \geq \deg(Q)$). In particular, h is holomorphic in a disk $D(z_0, s)$ (with $s \leq r$), and so analytic at the point z_0. Therefore, the expression for f becomes

$$f(z) = \frac{A_N}{(z-z_0)^N} + \frac{A_{N-1}}{(z-z_0)^{N-1}} + \cdots + \frac{A_1}{z-z_0} + \sum_{n \geq 0} a_n (z-z_0)^n; \quad z \in D(z_0, s);$$

for some coefficients $\{a_n\}_{n \geq 0}$. The above sum is therefore the Laurent Series of f around z_0. By Definitions 5.10 and 5.16, we get that f has a pole of order N at z_0, and that

$$\text{Res}(f, z_0) = A_1.$$

Let us look at a concrete rational function. Define

$$f(z) = \frac{z^2+2}{z^3 - z^2 - z + 1}.$$

The roots of $Q(z) = z^3 - z^2 - z + 1$ are 1 (with multiplicity 2) and -1 (with multiplicity 1). So, by the previous discussion, f has a pole of order 2 at $z_0 = 1$ and a pole of order 1 (also called simple pole) at $z_0 = -1$. To calculate the residues at those points, decompose f into partial fraction decomposition

$$f(z) = \frac{z^2+2}{z^3 - z^2 - z + 1} = \frac{z^2+2}{(z-1)^2(z+1)} = \frac{3/2}{(z-1)^2} + \frac{1/4}{z-1} + \frac{3/4}{z+1}.$$

By the previous discussion,

$$\text{Res}(f, 1) = \frac{1}{4}, \quad \text{Res}(f, -1) = \frac{3}{4}.$$

The following two propositions are useful when dealing with functions which have simple poles.

Proposition 5.19. *Let $g, h : \Omega \to \mathbb{C}$ holomorphic functions and $z_0 \in \Omega$. Assume that:*

- $g(z_0) \neq 0$, *and*
- $h(z_0) = 0$ *and* $h'(z_0) \neq 0$.

Then $f := \dfrac{g}{h}$ has a simple pole (pole of order 1) at z_0 and

$$\text{Res}(f, z_0) = \frac{g(z_0)}{h'(z_0)}.$$

Proof. The second condition says that h has a zero 1 at z_0. So, by Proposition 5.13 ($(i) \iff (iii)$) and the fact that $g(z_0) \neq 0$ (and thus $g(z) \neq z_0$ in some disk $D(z_0, s)$), we know that f has a pole of order 1 at z_0. To calculate $\text{Res}(f, z_0)$, we use Proposition 4.43 to write

$$h(z) = (z - z_0)\widetilde{h}(z), \quad z \in D(z_0, s),\ \widetilde{h} \in \mathcal{H}(D(z_0, s)),\ \widetilde{h}(z) \neq 0, \text{ for all } z \in D(z_0, s).$$

Writing $\varphi(z) = \dfrac{g(z)}{\widetilde{h}(z)}$ for all $z \in D(z_0, s)$, we get that $\varphi \in \mathcal{H}(D(z_0, s))$ and hence

$$f(z) = \frac{1}{z-z_0} \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(z_0)}{n!}(z-z_0)^n = \frac{\varphi(z_0)}{z-z_0} + \sum_{n=1}^{\infty} \frac{\varphi^{(n)}(z_0)}{n!}(z-z_0)^{n-1}, \quad z \in D(z_0, s).$$

Therefore $\text{Res}(f, z_0) = \varphi(z_0)$. To compute $\varphi(z_0)$, note first that $h'(z_0) = \widetilde{h}(z_0)$, and then

$$\varphi(z_0) = \frac{g(z_0)}{\widetilde{h}(z_0)} = \frac{g(z_0)}{h'(z_0)}.$$

□

Proposition 5.19 has the following generalization.

Proposition 5.20. *Let $g, h : \Omega \to \mathbb{C}$ holomorphic functions, $z_0 \in \Omega$, and $m \in \mathbb{N}$. Assume the following two conditions:*

- *g has a zero of order $m-1$ at z_0 (in the case $m = 1$, this simply means that $g(z_0) \neq 0$).*
- *h has a zero of order m at z_0.*

Then $f := \dfrac{g}{h}$ has a simple pole (pole of order 1) at z_0 and

$$\operatorname{Res}(f, z_0) = \frac{mg^{(m-1)}(z_0)}{h^{(m)}(z_0)}.$$

Proof. By Proposition 4.43, we can find functions $\psi, \varphi \in \mathcal{H}(\Omega)$ with $\psi(z_0) \neq 0$ and $\varphi(z_0) \neq 0$ and such that

$$g(z) = (z - z_0)^{m-1}\psi(z), \quad h(z) = (z - z_0)^m \varphi(z), \quad z \in \Omega.$$

A simple computation shows that $\psi(z_0) = (m-1)!g^{(m-1)}(z_0)$ and that $\varphi(z_0) = m!h^{(m)}(z_0)$. Then, the formula for f becomes

$$f(z) = \frac{g(z)}{h(z)} = \frac{(z - z_0)^{m-1}\psi(z)}{(z - z_0)^m \varphi(z)} = \frac{\psi(z)}{(z - z_0)\varphi(z)}.$$

The functions ψ and $z \mapsto \widetilde{\varphi}(z) := (z - z_0)\varphi(z)$ satisfy the assumptions of Proposition 5.19, and thus

$$\operatorname{Res}(f, z_0) = \frac{\psi(z_0)}{(\widetilde{\varphi})'(z_0)} = \frac{\psi(z_0)}{\varphi(z_0)} = \frac{(m-1)!g^{(m-1)}(z_0)}{m!h^{(m)}(z_0)} = \frac{mg^{(m-1)}(z_0)}{h^{(m)}(z_0)}.$$

□

Example 5.21. Let us apply Proposition 5.19 to the calculus of residues of two concrete examples.

(1) Consider $f(z) = \dfrac{1}{z^4 + 1}$. The denominator has the roots

$$\left\langle \sqrt[4]{-1} \right\rangle = \{e^{i\frac{\operatorname{Arg}(-1)+2k\pi}{4}} : k = 0, 1, 2, 3\} = \{z_0 := e^{i\frac{\pi}{4}},\ z_1 := e^{i\frac{3\pi}{4}},\ z_2 := e^{i\frac{5\pi}{4}},\ z_3 := e^{i\frac{7\pi}{4}}\}.$$

The function f then has an isolated singularity at each z_k, $k = 0, 1, 2, 3$. Moreover, if $g(z) := 1$ and $h(z) := z^4 + 1$, clearly

$$g(z_k) \neq 0,\ h(z_k) = 0,\ h'(z_k) = 4z_k^3 \neq 0, \quad k = 0, 1, 2, 3.$$

By Proposition 5.19, we get that f has a simple pole at each z_k, with

$$\operatorname{Res}(f, z_k) = \frac{g(z_k)}{h'(z_k)} = \frac{1}{4z_k^3}, \quad k = 0, 1, 2, 3.$$

(2) Consider $f(z) = \dfrac{\sin z}{1 - \cos z}$. The equation $\cos z = 1$ has solutions $\{z_k := 2\pi k : k \in \mathbb{Z}\}$. This is easily checked by observing that $\cos z = 0$ if and only if $e^{2iz} - 2e^{iz} + 1 = 0$; where $e^{2iz} - 2e^{iz} + 1 = (e^{iz} - 1)^2$. And recall from Theorem 2.49, that $e^w = 1$ if and only if $w \in 2\pi\mathbb{Z}$. Therefore f has an isolated singularity at each z_k, $k \in \mathbb{Z}$. Towards applying Proposition 5.20, define $g(z) := z$ and $h(z) = 1 - \cos z$, and compute:

$$g(z_k) = \sin(z_k) = 0,\ g'(z_k) = \cos(z_k) \neq 0, \quad h(z_k) = h'(z_k) = 0,\ h''(z_k) = \cos(z_k) \neq 0,$$

for all $k \in \mathbb{Z}$. Thus we can apply Proposition 5.20 at each z_k to infer that f has a pole of order 1 at z_k, with

$$\operatorname{Res}(f, z_k) = \frac{2g'(z_k)}{h''(z_k)} = \frac{2\cos(z_k)}{\cos(z_k)} = 2, \quad k \in \mathbb{Z}.$$

We next prove criteria for some functions to have a pole of order 2, together with the value of the corresponding residue.

Proposition 5.22. *Let $g, h : \Omega \to \mathbb{C}$ holomorphic functions and $z_0 \in \Omega$. Assume that:*

- $g(z_0) \neq 0$, *and*
- $h(z_0) = h'(z_0) = 0$ *and* $h''(z_0) \neq 0$ *(that is, h has a zero of order 2 at z_0).*

Then $f := \dfrac{g}{h}$ has a pole of order 2 at z_0 and

$$\operatorname{Res}(f, z_0) = \frac{2g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)h'''(z_0)}{(h''(z_0))^2}.$$

Proof. By Proposition 4.43, we can write

$$h(z) = (z - z_0)^2\varphi(z), \quad z \in D(z_0, r), \quad \varphi \in \mathcal{H}(D(z_0, r)), \quad \varphi(z) \neq 0 \text{ for all } z \in D(z_0, r).$$

Therefore the function $\psi = g/\varphi$ is holomorphic in $D(z_0, r)$, and so it coincides with its Taylor series at z_0. Thus we have

$$\begin{aligned} f(z) &= \frac{g(z)}{h(z)} = \frac{1}{(z-z_0)^2}\frac{g(z)}{\varphi(z)} = \frac{1}{(z-z_0)^2}\sum_{n=0}^{\infty}\frac{\psi^{(n)}(z_0)}{n!}(z-z_0)^n \\ &= \frac{\psi(z_0)}{(z-z_0)^2} + \frac{\psi'(z_0)}{z-z_0} + \sum_{n=2}^{\infty}\frac{\psi^{(n)}(z_0)}{n!}(z-z_0)^{n-2} = \frac{\psi(z_0)}{(z-z_0)^2} + \frac{\psi'(z_0)}{z-z_0} + \sum_{n=0}^{\infty}\frac{\psi^{(n+2)}(z_0)}{(n+2)!}(z-z_0)^n, \end{aligned}$$

for all $z \in D(z_0, r)$. This tells us that f has a pole of order 2 at z_0, and that

$$\operatorname{Res}(f, z_0) = \psi'(z_0) = \frac{g'(z_0)\varphi(z_0) - g(z_0)\varphi'(z_0)}{(\varphi(z_0))^2} = \frac{g'(z_0)}{\varphi(z_0)} - \frac{g(z_0)\varphi'(z_0)}{(\varphi(z_0))^2}$$

To express $\varphi(z_0)$ and $\varphi'(z_0)$ in terms of h, we look that the expression $h(z) = (z - z_0)^2\varphi(z)$ and differentiate:

$$\begin{aligned} h''(z) &= 2\varphi(z) + 4(z-z_0)\varphi'(z) + (z-z_0)^2\varphi''(z), \quad \text{and so,} \quad h''(z_0) = 2\varphi(z_0), \\ h'''(z) &= 6\varphi'(z) + 3(z-z_0)\varphi''(z) + (z-z_0)^2\varphi'''(z), \quad \text{and so,} \quad h'''(z_0) = 6\varphi'(z_0). \end{aligned}$$

We conclude

$$\operatorname{Res}(f, z_0) = \frac{g'(z_0)}{\varphi(z_0)} - \frac{g(z_0)\varphi'(z_0)}{(\varphi(z_0))^2} = \frac{2g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)h'''(z_0)}{(h''(z_0))^2}.$$

□

Proposition 5.23. *Let $g, h : \Omega \to \mathbb{C}$ holomorphic functions and $z_0 \in \Omega$. Assume that:*

- $g(z_0) = 0$, $g'(z_0) \neq 0$ *(that is, g has a zero of order 1 at z_0),*
- $h(z_0) = h'(z_0) = h''(z_0) = 0$ *and* $h'''(z_0) \neq 0$ *(that is, h has a zero of order 3 at z_0).*

Then $f := \dfrac{g}{h}$ has a pole of order 2 at z_0 and

$$\operatorname{Res}(f, z_0) = \frac{3g''(z_0)}{h'''(z_0)} - \frac{3}{2}\frac{g'(z_0)h^{(\mathrm{iv})}(z_0)}{(h'''(z_0))^2}.$$

Proof. By Proposition 4.43, we can find functions $\widetilde{g}, \widetilde{h} \in \mathcal{H}(\Omega)$ with $\widetilde{g}(z_0), \widetilde{h}(z_0) \neq 0$ and so that $g(z) = (z - z_0)\widetilde{g}(z)$ and $h(z) = (z - z_0)\widetilde{h}(z)$ for all $z \in \Omega$. Thus, for $z \in \Omega \setminus \{z_0\}$,

$$f(z) = \frac{g(z)}{h(z)} = \frac{(z - z_0)\widetilde{g}(z)}{(z - z_0)^3\widetilde{h}(z)} = \frac{\widetilde{g}(z)}{(z - z_0)^2\widetilde{h}(z)} = \frac{\varphi(z)}{\psi(z)}; \quad \varphi(z) := \widetilde{g}(z),\ \psi(z) := (z - z_0)^2\widetilde{h}(z).$$

We have that $\varphi(z_0) \neq 0$ and that $\psi(z_0) = \psi(z_0) = 0$, $\psi''(z_0) \neq 0$. By Proposition 5.22, we have

$$\operatorname{Res}(f, z_0) = \frac{2\varphi'(z_0)}{\psi''(z_0)} - \frac{2}{3}\frac{\varphi(z_0)\psi'''(z_0)}{(\psi''(z_0))^2}.$$

But noticing that $g(z) = (z-z_0)\varphi(z)$ and $h(z) = (z-z_0)\psi(z)$ for all $z \in \Omega$, we get (by differentiating up to four times or by comparing the Taylor Series at z_0) that

$$g'(z_0) = \varphi(z_0),\ g''(z_0) = 2\varphi'(z_0),\ h''(z_0) = 2\psi'(z_0),\ h'''(z_0) = 3\psi''(z_0),\ h^{(\text{iv})}(z_0) = 4\psi'''(z_0).$$

We then get

$$\operatorname{Res}(f, z_0) = \frac{2\varphi'(z_0)}{\psi''(z_0)} - \frac{2}{3}\frac{\varphi(z_0)\psi'''(z_0)}{(\psi''(z_0))^2} = \frac{3g''(z_0)}{h'''(z_0)} - \frac{3}{2}\frac{g'(z_0)h^{(\text{iv})}(z_0)}{(h'''(z_0))^2}.$$

□

Example 5.24. Let us employ Proposition 5.22 (perhaps in combination with Proposition 5.20) to the calculus of residues.

(1) Consider $f(z) = \dfrac{1+z}{1-\cos z}$. The denominator has zeros $\{z_k := 2\pi k \,:\, k \in \mathbb{Z}\}$. Define the functions $g(z) := 1 + z$ and $h(z) = 1 - \cos z$, and compute:

$$g(z_k) = 1 + z_k \neq 0, \quad h(z_k) = h'(z_k) = 0,\ h''(z_k) = \cos(z_k) \neq 0, \quad \text{for all} \quad k \in \mathbb{Z}.$$

By Proposition 5.22 applied to each z_k, we get that f has a pole of order 2 at z_k, with

$$\operatorname{Res}(f, z_k) = \frac{2g'(z_k)}{h''(z_k)} - \frac{2}{3}\frac{g(z_k)h'''(z_k)}{(h''(z_k))^2} = \frac{2}{\cos(z_k)} - \frac{2}{3}\frac{(1+z_k)(-\sin(z_k))}{\cos^2(z_k)} = 2, \quad k \in \mathbb{Z}.$$

(2) Consider $f(z) = \dfrac{z}{1-\cos z}$. Again, the denominator has zeros $\{z_k := 2\pi k \,:\, k \in \mathbb{Z}\}$. Define the functions $g(z) := z$ and $h(z) = 1 - \cos z$, and compute:

$$g(z_k) = z_k \neq 0, \quad h(z_k) = h'(z_k) = 0,\ h''(z_k) = \cos(z_k) \neq 0, \quad \text{for all} \quad k \in \mathbb{Z} \setminus \{0\}.$$

So, for every $k \in \mathbb{Z} \setminus \{0\}$, Proposition 5.22 implies that f has a pole of order 2 at z_k, with

$$\operatorname{Res}(f, z_k) = \frac{2g'(z_k)}{h''(z_k)} - \frac{2}{3}\frac{g(z_k)h'''(z_k)}{(h''(z_k))^2} = \frac{2}{\cos(z_k)} - \frac{2}{3}\frac{(1+z_k)(-\sin(z_k))}{\cos^2(z_k)} = 2, \quad k \in \mathbb{Z} \setminus \{0\}.$$

However, for $z_0 = 0$, we have that

$$g(z_0) = 0,\ g'(z_0) = 1 \neq 0, \quad h(z_0) = h'(z_0) = 0,\ h''(z_0) = \cos(z_0) \neq 0.$$

By Proposition 5.20 for $m = 2$, we obtain that f has a pole of order 1 at $z_0 = 0$ with

$$\operatorname{Res}(f, 0) = \frac{2g'(0)}{h''(0)} = 2.$$

If we *a priori* know the order of a pole of f, the following proposition gives an alternate way to compute the residue.

Proposition 5.25. *Let $f : D(z_0, r) \setminus \{z_0\} \to \mathbb{C}$ be holomorphic in $D(z_0, r) \setminus \{z_0\}$ with a pole of order $N \in \mathbb{N}$ at z_0. Then*

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(N-1)!} \frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}} \left((z - z_0)^N f(z) \right).$$

Proof. The Laurent Series of f at z_0 is

$$f(z) = \frac{a_{-N}}{(z-z_0)^N} + \frac{a_{-N+1}}{(z-z_0)^N} + \cdots + \frac{a_{-1}}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad z \in D(z_0, r) \setminus \{z_0\}.$$

By the definition of residue at z_0, we have $\operatorname{Res}(f, z_0) = a_{-1}$. Multiplying by $(z - z_0)^N$, we get

$$(z - z_0)^N f(z) = a_{-N} + a_{-N+1}(z - z_0) + \cdots + a_{-1}(z - z_0)^{N-1} + \sum_{n=0}^{\infty} a_n (z - z_0)^{n+N}.$$

The term in the right-hand side defines a holomorphic function g in $D(z_0, r)$, and its $(N-1)$-th derivative at z_0 is $(N-1)!\, a_{-1}$. But since such a function g coincidies with $(z - z_0)^N f(z)$ in $D(z_0, r) \setminus \{z_0\}$, we get that

$$(N-1)!\, a_{-1} = \lim_{z \to z_0} \frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}} (z - z_0)^N f(z).$$

□

In particular, if f has a pole of order 1 at z_0, by Proposition 5.25, one has

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

5.3.2 The Cauchy Global Theorem

The following theorem is known as the *Cauchy Homological Theorem* or the *Cauchy Global Theorem*. The amazingly short and elegant proof we include here is due to John D. Dixon [2], published in 1971.

Theorem 5.26 (Cauchy Global Integral Formula)**.** *Let $\Omega \subset \mathbb{C}$ be open and $\gamma : [a, b] \to \Omega$ a closed and piecewise C^1-path in Ω so that $W(\gamma, z) = 0$ for all $z \notin \Omega$. Then, if $f : \Omega \to \mathbb{C}$ is holomorphic in Ω, one has*

$$W(\gamma, z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} \,\mathrm{d}\xi, \quad \text{for all} \quad z \in \Omega \setminus \gamma^*. \tag{5.3.1}$$

Proof. Denote $U := \{z \in \mathbb{C} \setminus \gamma^* \,:\, W(\gamma, z) = 0\}$. As we saw in the proof of Proposition 4.26(iii) (or actually as a consequence of Proposition 4.26(iii)), the function $\mathbb{C} \setminus \gamma^* \ni z \mapsto N_\gamma(z) := W(\gamma, z)$ is continuous and only takes integer values. Thus

$$U = N_\gamma^{-1}(\{0\}) = N_\gamma^{-1}((-1/2, 1/2))$$

is the preimage of an open interval by a continuous function in the open set $\mathbb{C} \setminus \gamma^*$, and thus U is open; see Proposition 2.20. We next define, for each $w \in \Omega$, the function $g_w \Omega \to \mathbb{C}$ as

$$g_w(\xi) = \begin{cases} \frac{f(w) - f(\xi)}{w - \xi} & \text{if } \xi \neq w \\ f'(w) & \text{if } \xi = w. \end{cases}.$$

Because f is holomorphic in Ω, we immediately get that g_w is continuous in Ω for all $w \in \Omega$. Thus we can define a new function $h : \mathbb{C} \to \mathbb{C}$ by the formula

$$h(w) = \begin{cases} \int_\gamma g_w(\xi) \,\mathrm{d}\xi & \text{if } w \in \Omega \\ \int_\gamma \frac{f(\xi)}{\xi - w} \,\mathrm{d}\xi & \text{if } w \in U. \end{cases} \tag{5.3.2}$$

First of all, we need to verify that h is well defined. Let $w \in \Omega \cap U$. Then, $w \notin \gamma^*$ and $W(\gamma, w) = 0$, and in particular $\xi \neq w$ for all $\xi \in \gamma^*$. Thus, looking at the definitions of g_w, we see that

$$\int_\gamma g_w(\xi) = \int_\gamma \frac{f(w) - f(\xi)}{w - \xi}\,\mathrm{d}\xi = -f(w)2\pi i W(\gamma, w) - \int_\gamma \frac{f(\xi)}{w - \xi}\,\mathrm{d}\xi = \int_\gamma \frac{f(\xi)}{\xi - w}\,\mathrm{d}\xi.$$

Thus the two branches of definition of h agree, and h is well-defined. Also, notice that h is defined in all of $\mathbb{C}$, by the assumption $\mathbb{C} \setminus \Omega \subset U$.

Now, since f is holomorphic in Ω, by the Differentiation Under the Integral Sign Theorem 4.18, we get that h is holomorphic in $\Omega \setminus \gamma$, as well as in U. Therefore, we have that h is holomorphic in $\mathbb{C}$. Let us now show that $\lim_{|w| \to \infty} |h(w)| = 0$. Indeed, since γ^* is a compact set, there exists $r > 0$ so that $\gamma^* \subset \overline{D}(0, r)$, Thus, if $|w| \geq 2r$, then w is in the unbounded connected component of $\mathbb{C} \setminus \gamma^*$. By Proposition 4.26, we get that $W(\gamma, w) = 0$, and so $w \in U$. Thus, for $|w| \geq 2r$ we can estimate

$$|h(w)| = \left| \int_\gamma \frac{f(\xi)}{\xi - w}\,\mathrm{d}\xi \right| \leq \int_\gamma \frac{|f(\xi)|}{|\xi - w|} |\mathrm{d}\xi| \leq \frac{\sup\{|f(\xi)| \,:\, \xi \in \gamma^*\} \cdot \mathrm{length}(\gamma)}{|w| - r}.$$

Since the supremum and the length above are finite, letting $|w| \to \infty$ gives $\lim_{|w| \to \infty} |h(w)| = 0$. By the continuity of h, this implies that h is bounded in $\mathbb{C}$. Hence, Liouville's Theorem 4.45 tells us that h is constant, and actually constantly equal to 0, due to $\lim_{|w| \to \infty} |h(w)| = 0$. Therefore, for any $z \in \Omega \setminus \gamma^*$, we have that

$$\begin{aligned} 0 = h(z) &= \int_\gamma g(z, \xi)\,\mathrm{d}\xi = \int_\gamma \frac{f(z) - f(\xi)}{z - \xi}\,\mathrm{d}\xi \\ &= f(z) \int_\gamma \frac{\mathrm{d}\xi}{z - \xi} + \int_\gamma \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi = -2\pi i f(z) W(\gamma, z) + \int_\gamma \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi; \end{aligned}$$

which yiels (5.3.1). □

Theorem 5.26 should be compared to Theorem 4.27, where we assumed that Ω is convex. It is not difficult to show that all closed piecewise C^1-paths in a convex domain Ω satisfy the property that $W(\gamma, z) = 0$ for all $z \notin \Omega$. Therefore Theorem 5.26 generalizes Theorem 4.27.

The following corollary of Theorem 5.26 is one of the key ingredients in the next subsection.

Corollary 5.27 (Cauchy Global Theorem). *Let $\Omega \subset \mathbb{C}$ be open and $\gamma : [a, b] \to \Omega$ a closed and piecewise C^1-path in Ω so that $W(\gamma, z) = 0$ for all $z \notin \Omega$. Then, if $f : \Omega \to \mathbb{C}$ is holomorphic,*

$$\int_\gamma f(\xi)\,\mathrm{d}\xi = 0.$$

Proof. If $f : \Omega \to \mathbb{C}$ is holomorphic, we fix a point $z_0 \in \Omega \setminus \gamma^*$, and define $g(z) = f(z)(z - z_0)$ for all $z \in \Omega$. Clearly $g \in \mathcal{H}(\Omega)$ and we can apply Theorem 5.26 to g at the point z_0 to obtain

$$0 = W(\gamma, z_0) g(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{g(\xi)}{\xi - z_0}\,\mathrm{d}\xi = \int_\gamma \frac{f(\xi)(\xi - z_0)}{\xi - z_0}\,\mathrm{d}\xi = \int_\gamma f(\xi)\,\mathrm{d}\xi.$$

□

5.3.3 The Cauchy Residues Theorem

In the proof of Cauchy Residues Theorem 5.29 below, the following lemma is crucial.

Lemma 5.28. *Let $f : D(z_0, r) \setminus \{z_0\} \to \mathbb{C}$ be holomorphic in $D(z_0, r) \setminus \{z_0\}$, and denote by $f_0 : \mathbb{C} \setminus \{z_0\} \to \mathbb{C}$ the principal part of f at z_0. Then, if $\gamma : [a, b] \to \mathbb{C}$is a closed and piecewise C^1-path with $z_0 \notin \gamma^*$, we have*

$$\int_\gamma f_0(z)\,\mathrm{d}z = 2\pi i \,\mathrm{Res}(f, z_0) W(\gamma, z_0).$$

Proof. Since γ^* is compact and $z_0 \notin \gamma^*$, there exist $0 < \varepsilon < R < +\infty$ so that $\gamma^* \subset D(z_0, R) \setminus D(z_0, \varepsilon) \setminus \{z_0\}$. By Remark 5.9, we can write

$$f_0(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, \quad z \in \mathbb{C} \setminus \{z_0\};$$

with absolute–uniform convergence in the annulus $\{z \in \mathbb{C} : \varepsilon \le |z - z_0| \le R\}$, which contains γ^*. Here $\{b_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$, and $b_1 = \mathrm{Res}(f, z_0)$. Therefore

$$\int_\gamma f_0(z)\,\mathrm{d}z = \sum_{n=1}^{\infty} b_n \int_\gamma \frac{\mathrm{d}z}{(z - z_0)^n} = b_1 \int_\gamma \frac{\mathrm{d}z}{z - z_0} = 2\pi i\, b_1 W(\gamma, z_0).$$

In the second equality we used Theorem 4.15 to each $z \mapsto (z - z_0)^{-n}$ with $n \ge 2$, as these functions have primitives $\frac{1}{1-n}(z - z_0)^{1-n}$. The last equality is just the Definition 4.24 of winding number. □

Theorem 5.29 (Cauchy Residues Theorem)**.** *Let $f : \Omega \setminus \{z_1, \dots, z_N\} \to \mathbb{C}$ be a holomorphic function in Ω except at N distinct points $\{z_1, \dots, z_N\} \subset \Omega$, at which f has isolated singularities. Let $\gamma : [a, b] \to \Omega \setminus \{z_1, \dots, z_N\}$ be a closed and piecewise C^1-path with $W(\gamma, z) = 0$ for all $z \notin \Omega$. Then, we have*

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{N} \mathrm{Res}(f, z_k) W(\gamma, z_k). \tag{5.3.3}$$

Proof. Since the singularity of f at each z_k is isolated and $\{z_1, \dots, z_N\}$ is finite, by Remark 5.9, $r > 0$ so that we can write

$$f(z) = g_k(z) + f_k(z), \quad \text{for all} \quad z \in D(z_k, r),\ k = 1, \dots, N; \tag{5.3.4}$$

where $g_k \in \mathcal{H}(D(z_k, r))$ and the principal part f_k of the Laurent Series of f at z_k is holomorphic in $\mathbb{C} \setminus \{z_k\}$ for all $k = 1, \dots, N$. Because $\{z_1, \dots, z_N\}$ is finite and γ^* is a compact subset of $\Omega \setminus \{z_1, \dots, z_N\}$, we can assume (by taking a smaller r in (5.3.4) if necessary) that

$$\overline{D}(z_j, 2r) \cap \overline{D}(z_k, 2r) = \emptyset, \quad \text{if } j, k \in \{1, \dots, N\},\ j \ne k; \quad \text{and} \quad \gamma^* \subset \Omega \setminus \bigcup_{k=1}^{N} D(z_k, r). \tag{5.3.5}$$

We define a function $h : \Omega \to \mathbb{C}$ in the following manner

$$h(z) = \begin{cases} g_k(z) - \sum_{j=1, j\ne k}^{N} f_j(z) & \text{if } z \in D(z_k, r),\ k \in \{1, \dots, N\} \\ f(z) - \sum_{j=1}^{N} f_j(z) & \text{if } z \in \Omega \setminus \bigcup_{k=1}^{N} \overline{D}(z_k, r/2). \end{cases} \tag{5.3.6}$$

By the first part of (5.3.5), we get that the first branch of definition of h is consistent. Also, if $z \in \Omega \setminus \bigcup_{k=1}^{N} \overline{D}(z_k, r/2)$ and at the same time z belongs to $D(z_k, r)$ for some (unique) $k \in \{1, \dots, N\}$, then (5.3.4) implies that

$$f(z) - \sum_{j=1}^{N} f_j(z) = g_k(z) + f_k(z) - \sum_{j=1}^{N} f_j(z) = g_k(z) - \sum_{j=1, j\ne k}^{N} f_j(z);$$

showing that h is well defined in Ω. Moreover, since $f_j \in \mathcal{H}(\mathbb{C} \setminus \{z_j\})$ and $g_k \in \mathcal{H}(D(z_k, r))$ for all j, k, it is clear that h is holomorphic in Ω. Furthermore, (5.3.5) says that $\gamma^* \subset \Omega \setminus \bigcup_{k=1}^{N} D(z_k, r) \subset \Omega \setminus \bigcup_{k=1}^{N} \overline{D}(z_k, r/2)$, and so $h = f - \sum_{j=1}^{N} f_j$ in γ^*, according to (5.3.6). Applying Theorem 5.26 to h and Lemma 5.28 to f (and its principal part f_j) at each z_j gives

$$0 = \int_\gamma h(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z - \sum_{j=1}^{N} \int_\gamma f_j(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z - \sum_{j=1}^{N} 2\pi i\, \mathrm{Res}(f, z_j) W(\gamma, z_j);$$

which of course yields (5.3.3). □

Example 5.30. For $f(z) = \cot z = \frac{\cos z}{\sin z}$, we evaluate the integrals

$$\int_{\partial D(0,1)} f(z)\, \mathrm{d}z, \quad \int_{\partial D(0,4)} f(z)\, \mathrm{d}z,$$

using Theorem 5.29. The circles are traveled once and counterclockwise. The function f has an isolated singularity at each $z_k := k\pi$, $k \in \mathbb{Z}$. Defining $g(z) = \cos z$ and $h(z) = \sin z$, we see that $g(z_k) = (-1)^k \neq 0$, and $h(z_k) = 0$, $h'(z_k) = \cos(z_k) = (-1)^k \neq 0$. By Proposition 5.19, f has a pole of order 1 at z_k with

$$\operatorname{Res}(f, z_k) = \frac{g(z_k)}{h'(z_k)} = 1, \quad k \in \mathbb{Z}.$$

For the first integral, note that only the singularity $z_0 = 0$ is contained in the *inside* of $\partial D(0,1)$. Hence, the Cauchy Residues Theorem 5.29 applied to $\partial D(0,1)$ gives

$$\int_{\partial D(0,1)} f(z)\, \mathrm{d}z = 2\pi i \operatorname{Res}(f, 0) W(\gamma, 0) = 2\pi i.$$

For the latter integral, observe that the singularities that are contained in the *inside* of $\partial D(0,4)$ are $\{-\pi, 0, \pi\}$. Theorem 5.29 tells us that

$$\int_{\partial D(0,4)} f(z)\, \mathrm{d}z = 2\pi i \Big(\operatorname{Res}(f, -\pi) W(\gamma, -\pi) + \operatorname{Res}(f, 0) W(\gamma, 0) + \operatorname{Res}(f, \pi) W(\gamma, \pi) \Big) = 6\pi i.$$

5.3.4 Evaluation of Integrals via the Cauchy Residues Theorem

Theorem 5.31 (Evaluation of Trigonometric Integrals)**.** *Let $R(u,v)$ be a rational function of two variables such that the function $[0, 2\pi] \ni \theta \mapsto R(\cos\theta, \sin\theta)$ is bounded in $[0, 2\pi]$. Consider the function*

$$f(z) := \frac{1}{iz} R\left(\frac{1}{2}\left(z + \frac{1}{z} \right), \frac{1}{2i}\left(z - \frac{1}{z} \right) \right),$$

and denote $\operatorname{Poles}(f) := \{z \in \mathbb{C} : f \text{ has a pole at } z\}$ *and $\mathbb{D} = D(0,1)$ the open unit disk. Then,*

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, \mathrm{d}\theta = 2\pi i \sum_{z \in \mathbb{D} \cap \operatorname{Poles}(f)} \operatorname{Res}(f, z).$$

Proof. Let $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$. Then

$$\int_\gamma f(z)\, \mathrm{d}z = \int_0^{2\pi} f(e^{i\theta}) i e^{i\theta}\, \mathrm{d}\theta = \int_0^{2\pi} \frac{1}{ie^{i\theta}} R\left(\frac{1}{2}\left(e^{i\theta} + \frac{1}{e^{i\theta}} \right), \frac{1}{2i}\left(e^{i\theta} - \frac{1}{e^{i\theta}} \right) \right) ie^{i\theta}\, \mathrm{d}\theta$$
$$= \int_0^{2\pi} R\left(\frac{e^{i\theta} + e^{-i\theta}}{2}, \frac{e^{i\theta} - e^{-i\theta}}{2i} \right) \mathrm{d}\theta = \int_0^{2\pi} R(\cos\theta, \sin\theta)\, \mathrm{d}\theta.$$

On the other hand, if $\Omega = \mathbb{C}$, then $\gamma^* \subset \Omega$ and $W(\gamma, z) = 0$ for all $z \notin \Omega$ (vacuously). Because the mapping $[0, 2\pi] \ni \theta \mapsto R(\cos\theta, \sin\theta)$ is bounded, if we denote $R = P/Q$, with P, Q polynomials of two variables, then the possible isolated singularities of f are at $z_0 = 0$ and at those $z \in \mathbb{C}$ for which

$$Q\left(\frac{1}{2}\left(z + z^{-1} \right), \frac{1}{2i}\left(z - z^{-1} \right) \right) = 0.$$

But there are only finitely many possible solutions in $z \in \mathbb{C} \setminus \{0\}$ for this equation. Moreover, since P, Q are polynomials, clearly the singularities of f are all poles; see Example 5.18. But none of these poles are contained in $\gamma^* = \partial D(0,1)$, because if $\theta \in [0, 2\pi]$, then $f(e^{i\theta}) = -ie^{-i\theta} R(\cos\theta, \sin\theta)$, which is continuous in $\theta \in [0, 2\pi]$. In other words, none of the singularities of f are contained in γ^*. By Theorem 5.29, we get that

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{z \in \mathrm{Poles}(f)} \mathrm{Res}(\gamma, z) W(\gamma, z) = 2\pi i \sum_{z \in \mathbb{D} \cap \mathrm{Poles}(f)} \mathrm{Res}(f, z).$$

$\square$

Example 5.32. Let us evaluate the integral

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{2 - \sin\theta},$$

via Theorem 5.31. Consider the rational function $R(u, v) = \frac{1}{2-v}$ and

$$f(z) = \frac{1}{iz} R\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right) = \frac{1}{iz} \frac{1}{2 - \frac{1}{2i}\left(z - \frac{1}{z}\right)} = \frac{-2}{z^2 - 4iz + 1}$$

Writing $z^2 - 4iz + 1 = (z - 2i)^2 + 5 = (z - (2 - \sqrt{5})i)(z - (2 + \sqrt{5})i)$, we see that f has isolated singularities, which are poles of order 1, at the points $(2 - \sqrt{5})i$ and $(2 + \sqrt{5})i$. But the latter pole $(2 + \sqrt{5})i$ is not contained in $D(0, 1)$, as $|(2 + \sqrt{5})i| = 2 + \sqrt{5} > 1$. The first pole $(2 - \sqrt{5})i$ is contained in $D(0, 1)$. By Theorem 5.31,

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{2 - \sin\theta} = 2\pi i \,\mathrm{Res}\left(f, (2 - \sqrt{5})i\right).$$

To calculate $\mathrm{Res}\left(f, (2 - \sqrt{5})i\right)$, we can decompose $\frac{-2}{z^2 - 4iz + 1}$ into partial fractions, and look at the coefficient of the fraction $\frac{1}{z - (2 - \sqrt{5})i}$. Alternatively, we can write

$$f(z) = \frac{-2}{z^2 - 4iz + 1} = \frac{g(z)}{h(z)}, \quad g(z) = -2,\ h(z) = z^2 - 4iz + 1,$$

and apply Proposition 5.19 to deduce

$$\mathrm{Res}\left(f, (2 - \sqrt{5})i\right) = \frac{g((2 - \sqrt{5})i))}{h'((2 - \sqrt{5})i))} = \frac{-2}{2\left((2 - \sqrt{5})i\right) - 4i} = \frac{-2}{-2\sqrt{5}i} = \frac{1}{\sqrt{5}i}.$$

Thus,

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{2 - \sin\theta} = 2\pi i \,\mathrm{Res}\left(f, (2 - \sqrt{5})i\right) \frac{2\pi i}{\sqrt{5}i} = \frac{2\pi}{\sqrt{5}}.$$

We now turn our attention to improper integrals of the type $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x$, where $f : \mathbb{R} \to \mathbb{R}$ is continuous in $\mathbb{R}$. In particular, the holomorphic extension f to an open set containing $\mathbb{R}$ has no singularities in $\mathbb{R}$. Recall that such improper integral (more precisely, its *principal value*) is defined by the limit

$$\mathrm{pv} \int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x := \lim_{R \to +\infty} \int_{-R}^{R} f(x)\,\mathrm{d}x.$$

We next establish some methods that are very helpful in a large number of those cases.

Theorem 5.33 (Integrals of Continuous Functions in $\mathbb{R}$, ver. I)**.** *Denote $\mathbb{H} := \{z \in \Omega : \mathrm{Im}(z) \geq 0\}$ the upper half-plane and let $\Omega \subset \mathbb{C}$ be open with $\mathbb{H} \subset \Omega$. Let f be a function with the following conditions:*

- *f is holomorphic in Ω except at finitely many singularities $z_1, \ldots, z_N \in \Omega$.*

- $z_k \notin \mathbb{R}$ *(that is,* $\text{Im}(z_k) \neq 0$*) for all* $k = 1, \ldots, N$.
- *For the paths* $\gamma_R : [0, \pi] \to \mathbb{C}$ *defined by* $\gamma_R(t) = Re^{it}$, $t \in [0, \pi]$, $R > 0$, *there holds that*

$$\lim_{R \to +\infty} \left| \int_{\gamma_R} f(z) \, \mathrm{d}z \right| = 0.$$

Then, we have

$$\text{pv} \int_{-\infty}^{\infty} f(x) \, \mathrm{d}x = 2\pi i \sum_{\{k \,:\, \text{Im}(z_k) > 0\}} \text{Res}(f, z_k).$$

Proof. Let $R > 0$ be large enough so that $R > \max\{|z_1|, \ldots, |z_N|\}$, and define the paths

$$L_R := [-R, R], \quad \gamma_R(t) = Re^{it}, \, t \in [0, \pi], \quad \Gamma_R := L_R \star \gamma_R.$$

Γ_R is a closed and piecewise C^1-path with $\Gamma_R^* \subset \Omega$ (because $\gamma_R^* \subset \mathbb{H}$) and by the choice of R, it is clear that $z_k \notin \Gamma_R^*$ for all $k = 1, \ldots, N$. Thus, by Theorem 5.29, we have

$$\int_{\Gamma_R} f(z) \, \mathrm{d}z = 2\pi i \sum_{k=1}^{N} \text{Res}(f, z_k) W(\Gamma_R, z_k) = 2\pi i \sum_{k=1}^{N} \text{Res}(f, z_k). \tag{5.3.7}$$

The path-integral of f along Γ_R is

$$\int_{\Gamma_R} f(z) \, \mathrm{d}z = \int_{L_R} f(z) \, \mathrm{d}z + \int_{\gamma_R} f(z) \, \mathrm{d}z = \int_{-R}^{R} f(x) \, \mathrm{d}x + \int_{\gamma_R} f(z) \, \mathrm{d}z; \tag{5.3.8}$$

where, by Proposition 4.11 and the third assumption of this theorem,

$$\left| \int_{\gamma_R} f(z) \, \mathrm{d}z \right| \leq \text{length}(\gamma_R) \sup_{z \in \gamma_R^*} |f(z)| \leq \pi R \sup_{z \in \gamma_R^*} \frac{M}{|z|^p} = \frac{\pi}{MR^{p-1}}.$$

Therefore,

$$\lim_{R \to +\infty} \left| \int_{\gamma_R} f(z) \, \mathrm{d}z \right| = 0.$$

Inserting this back into (5.3.8) and (5.3.7), and letting $R \to \infty$, we get

$$\begin{aligned} \text{pv} \int_{-\infty}^{+\infty} f(x) \, \mathrm{d}x &= \lim_{R \to +\infty} \int_{-R}^{R} f(x) \, \mathrm{d}x = \lim_{R \to +\infty} \left(\int_{\Gamma_R} f(z) \, \mathrm{d}z - \int_{\gamma_R} f(z) \, \mathrm{d}z \right) \\ &= \lim_{R \to +\infty} \int_{\Gamma_R} f(z) \, \mathrm{d}z = \lim_{R \to +\infty} 2\pi i \sum_{k=1}^{N} \text{Res}(f, z_k) = 2\pi i \sum_{k=1}^{N} \text{Res}(f, z_k). \end{aligned}$$

□

Let us look at particular situations where the third condition in Theorem 5.33 is satisfied.

Corollary 5.34. *Denote* $\mathbb{H} := \{z \in \Omega \,:\, \text{Im}(z) \geq 0\}$ *the upper half-plane and let* $\Omega \subset \mathbb{C}$ *be open with* $\mathbb{H} \subset \Omega$. *Let* f *be a function with the following conditions:*

- f *is holomorphic in* Ω *except at finitely many singularities* $z_1, \ldots, z_N \in \Omega$.
- $z_k \notin \mathbb{R}$ *(that is,* $\text{Im}(z_k) \neq 0$*) for all* $k = 1, \ldots, N$.
- *There are constants* $M, R_0 > 0$, $p > 1$ *such that* $|f(z)| \leq \frac{M}{|z|^p}$ *for all* $z \in \Omega$ *with* $|z| > R_0$.

Then, we have

$$\text{pv} \int_{-\infty}^{\infty} f(x) \, \mathrm{d}x = 2\pi i \sum_{\{k \,:\, \text{Im}(z_k) > 0\}} \text{Res}(f, z_k).$$

Proof. By Proposition 4.11 and the third assumption of this corollary,

$$\left|\int_{\gamma_R} f(z)\,\mathrm{d}z\right| \le \text{length}(\gamma_R) \sup_{z\in\gamma_R^*} |f(z)| \le \pi R \sup_{z\in\gamma_R^*} \frac{M}{|z|^p} = \frac{\pi}{MR^{p-1}}.$$

Therefore,

$$\lim_{R\to+\infty} \left|\int_{\gamma_R} f(z)\,\mathrm{d}z\right| = 0.$$

We then have all the assumptions of Theorem 5.33. □

Remark 5.35. Observe that the third condition in Corollary 5.34 is fulfilled for example when f is a rational function $f = P/Q$; with P, Q polynomials satisfying

$$\deg(Q) \ge \deg(P) + 2.$$

Let us see how to apply the above in a concrete example.

Example 5.36. We evaluate

$$\text{pv}\int_{-\infty}^{+\infty} \frac{x^2}{1+x^4}\,\mathrm{d}x,$$

using Theorem 5.33 (actually Corollary 5.34 and Remark 5.35). Define $f(z) = \frac{z^2}{1+z^4}$, which is holomorphic in $\mathbb{C}$ except in the set

$$\left\langle \sqrt[4]{-1}\right\rangle = \{z_1 := e^{i\frac{\pi}{4}},\ z_2 := e^{i\frac{3\pi}{4}},\ z_3 := e^{i\frac{5\pi}{4}},\ z_4 := e^{i\frac{7\pi}{4}}\}.$$

Note that $z_k \notin \mathbb{R}$ for $k = 1, 2, 3, 4$, with $\text{Im}(z_1), \text{Im}(z_2) > 0$ and $\text{Im}(z_2), \text{Im}(z_4) < 0$. By Theorem 5.33, we have

$$\text{pv}\int_{-\infty}^{+\infty} \frac{x^2}{1+x^4}\,\mathrm{d}x = \int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = 2\pi i\left(\text{Res}(f, z_1) + \text{Res}(f, z_2)\right).$$

To calculate $\text{Res}(f, w)$ for $w = z_1, z_2$, we can decompose $\frac{z^2}{1+z^4}$ in partial fractions and look at the coefficient of $\frac{1}{z-w}$. But we can also apply directly Proposition 5.19 for $g(z) = z^2$ and $h(z) = 1 + z^4$, where

$$g(z_1) = z_1^2 \neq 0,\ g(z_2) = z_2^2 \neq 0, \quad h(z_1) = h(z_2) = 0, \quad h'(z_1) = 4z_1^3 \neq 0,\ h'(z_2) = 4z_2^3 \neq 0,$$

and then

$$\text{Res}(f, z_1) = \frac{g(z_1)}{h'(z_1)} = \frac{z_1^2}{4z_1^3} = \frac{1}{4z_1}, \quad \text{Res}(f, z_2) = \frac{g(z_2)}{h'(z_2)} = \frac{z_1^2}{4z_2^3} = \frac{1}{4z_2}.$$

We may conclude

$$\begin{aligned}\text{pv}\int_{-\infty}^{+\infty} \frac{x^2}{1+x^4}\,\mathrm{d}x &= 2\pi i\left(\text{Res}(f, z_1) + \text{Res}(f, z_2)\right) = \frac{\pi i}{4}\left(\frac{1}{z_1} + \frac{1}{z_2}\right) \\ &= \frac{\pi i}{2}\left(e^{-\frac{\pi}{4}i} + e^{-\frac{3\pi}{4}i}\right) = \frac{\pi}{2}\left(e^{\frac{\pi}{4}i} + e^{-\frac{\pi}{4}i}\right) = \frac{\pi}{2} 2\,\text{Re}\left(e^{\frac{\pi}{4}i}\right) = \frac{\pi}{\sqrt{2}}.\end{aligned}$$

To evaluate integrals of mixed rational-trigonometric functions, it is convenient to use a second version of Theorem 5.33. We first need the following estimate due to Jordan.

Lemma 5.37 (Jordan's Lemma)**.** *Let $r > 0$ and $\gamma_r : [0, \pi] \to \mathbb{C}$ the path $\gamma_r(t) = re^{it}$, $t \in [0, \pi]$. Then, we have*

$$\int_{\gamma_r} |e^{iz}||\mathrm{d}z| < \pi.$$

Proof. For each $r > 0$, we can write

$$\int_{\gamma_r} |e^{iz}||\mathrm{d}z| = \int_0^\pi e^{\mathrm{Re}(ire^{it})}|rie^{it}|\,\mathrm{d}t = \int_0^\pi re^{-r\sin t}\,\mathrm{d}t.$$

But since $\sin(t) = \sin(\pi - t)$, we have that $\int_{\pi/2}^{\pi} re^{-r\sin t}\,\mathrm{d}t = \int_0^{\pi/2} re^{-r\sin t}\,\mathrm{d}t$, and so the last integral above equals

$$2\int_0^{\pi/2} re^{-r\sin t}\,\mathrm{d}t \leq 2\int_0^{\pi/2} re^{-\frac{2r}{\pi}t}\,\mathrm{d}t = \pi\left(1 - e^{-\frac{2r}{\pi}\frac{\pi}{2}}\right) = \pi\left(1 - e^{-r}\right) < \pi.$$

We have used that $\sin t \geq 2t/\pi$ for all $t \in [0, \pi/2]$, that is, $t \mapsto \sin t$ is concave in the interval $[0, \pi/2]$. □

Theorem 5.38 (Integrals of Continuous Functions in $\mathbb{R}$, ver. II)**.** *Denote* $\mathbb{H} := \{z \in \Omega : \mathrm{Im}(z) \geq 0\}$ *the upper half-plane and let* $\Omega \subset \mathbb{C}$ *be open with* $\mathbb{H} \subset \Omega$*. Let* f *be a function with the following conditions*

- f *is holomorphic in* Ω *except at finitely many singularities* $z_1, \ldots, z_N \in \Omega$.
- $z_k \notin \mathbb{R}$ *(that is,* $\mathrm{Im}(z_k) \neq 0$*) for all* $k = 1, \ldots, N$.
- *There are constants* $M, R_0 > 0$, $p > 0$ *such that* $|f(z)| \leq \frac{M}{|z|^p}$ *for all* $z \in \Omega$ *with* $|z| > R_0$.

Then, for all $a > 0$*, if* $g(z) := f(z)e^{iaz}$, $z \in \Omega$, *we have*

$$\mathrm{pv}\int_{-\infty}^{+\infty} f(x)e^{iax}\,\mathrm{d}x = 2\pi i \sum_{\{k\,:\,\mathrm{Im}(z_k)>0\}} \mathrm{Res}(g, z_k).$$

Proof. Let us verify the assumptions of Theorem 5.33 for g (instead of f). The first two bullet conditions of Theorem 5.33 are immediate from the first two of the current theorem. Now, let $\gamma_R(t) := Re^{it}$, $t \in [0, \pi]$, with $R > R_0$, and use Lemma 5.37 to estimate

$$\left|\int_{\gamma_R} g(z)\,\mathrm{d}z\right| \leq \int_{\gamma_R} |f(z)||e^{iaz}||\mathrm{d}z| \leq \frac{M}{R^p}\int_{\gamma_R} |e^{iaz}||\mathrm{d}z| = \frac{M}{aR^p}\int_{\gamma_{aR}} |e^{iz}||\mathrm{d}z| \leq \frac{\pi M}{aR^p},$$

and the last term tends to 0 as $R \to \infty$. Thus, the third condition of Theorem 5.33 holds, and our statement follows from that theorem. □

Remark 5.39. Theorem 5.38 can be also employed when evaluating integrals of the type

$$\mathrm{pv}\int_{-\infty}^{+\infty} f(x)\cos(ax)\,\mathrm{d}x, \quad \mathrm{pv}\int_{-\infty}^{+\infty} f(x)\sin(ax)\,\mathrm{d}x,$$

assuming that $f(\mathbb{R}) \subset \mathbb{R}$. Indeed, in this case, those integrals are respectively **the real and imaginary part** of the integral

$$\mathrm{pv}\int_{-\infty}^{+\infty} f(x)e^{iax}\,\mathrm{d}x.$$

Also notice that the third condition of Theorem 5.38 is fulfilled for example when $f = P/Q$, where P, Q are polynomials satisfying

$$\deg(Q) \geq \deg(P) + 1;$$

compare to Remark 5.35.

Example 5.40. Let us evaluate the integral

$$\operatorname{pv}\int_{-\infty}^{+\infty} \frac{xe^{ix}}{x^2+1}\,\mathrm{d}x.$$

Writing $f(z) = \frac{z}{z^2+1}$ and $g(z) = f(z)e^{iz}$, and, for example, $\Omega = \mathbb{C}$ let us verify the assumptions of Theorem 5.38. The singularities of f are $\pm i$, which are not in $\mathbb{R}$. Also, for $|z| \geq 1$, we have the bounds

$$|f(z)| = \frac{|z|}{|1+z^2|} \leq \frac{|z|}{2|z|^2} = \frac{1}{2|z|}.$$

All the conditions of Theorem 5.38 are satisfied, and so

$$\operatorname{pv}\int_{-\infty}^{+\infty} \frac{xe^{ix}}{x^2+1}\,\mathrm{d}x = 2\pi i \operatorname{Res}(g,i).$$

It only remains to find $\operatorname{Res}(g,i)$. But since ze^{iz} does not vanish at i, $i^2+1=0$ and $2i \neq 0$, by Proposition 5.19, we have that

$$\operatorname{Res}(g,i) = \frac{ie^{i\cdot i}}{2i} = \frac{1}{2e}, \quad \text{and so} \quad \int_{-\infty}^{+\infty} \frac{xe^{ix}}{x^2+1}\,\mathrm{d}x = \frac{\pi i}{e}.$$

The evaluation of improper integrals when the pertinent function is unbounded around a point is a bit more delicate. Let us recall a definition from Calculus I.

Definition 5.41 (Principal Value of an Integral)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be a function that is unbounded on intervals around real points $x_1 < x_2 < \cdots < x_n$. The* ***principal value of*** $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x$ *by*

$$\operatorname{pv}\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x := \lim_{\varepsilon\to 0^+}\left(\int_{-\infty}^{x_1-\varepsilon} f(x)\,\mathrm{d}x + \int_{x_1+\varepsilon}^{x_2-\varepsilon} f(x)\,\mathrm{d}x + \cdots + \int_{x_{n-1}+\varepsilon}^{x_n-\varepsilon} f(x)\,\mathrm{d}x + \int_{x_n+\varepsilon}^{+\infty} f(x)\,\mathrm{d}x\right).$$

Theorem 5.42 (Integrals with Real Singularities)**.** *Denote $\mathbb{H} := \{z \in \Omega : \operatorname{Im}(z) \geq 0\}$ the upper half-plane and let $\Omega \subset \mathbb{C}$ be open with $\mathbb{H} \subset \Omega$. Let f be a function with the following conditions*

- *f is holomorphic in Ω except at a finite set of singularities $\mathcal{S}$.*
- *If $z \in \mathbb{R} \cap \mathcal{S}$ (that is, $\operatorname{Im}(z) = 0$), then f has a pole of order 1 at z;*
- *For the paths $\gamma_R : [0,\pi] \to \mathbb{C}$ defined by $\gamma_R(t) = Re^{it}$, $t \in [0,\pi]$, $R > 0$, there holds that*

$$\lim_{R\to+\infty}\left|\int_{\gamma_R} f(z)\,\mathrm{d}z\right| = 0.$$

Then, we have

$$\operatorname{pv}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \sum_{\{z\in\mathcal{S}\,:\,\operatorname{Im}(z)>0\}} \operatorname{Res}(f,z) + \pi i \sum_{\{z\in\mathcal{S}\,:\,z\in\mathbb{R}\}} \operatorname{Res}(f,z).$$

Proof. The proof is almost the same as that of Theorem 5.33, except that we need to additionally estimate some integrals along certain small semi-circles. Denote by $\mathcal{S}$ the set of all singularities of f in Ω, and let $\{x_1 < x_2 < \cdots < x_n\}$ be those that are contained in $\mathbb{R}$. Since the singularity of f at each x_j is a pole of order 1, and there are only finitely-many, by (for example) Remark 5.11, we can write

$$f(z) = \frac{\operatorname{Res}(f,x_j)}{z-x_j} + f_j(z), \quad z \in \overline{D}(x_j,\delta)\setminus\{x_j\}, \quad j = 1,\ldots,n; \tag{5.3.9}$$

where each f_j is holomorphic in an open disk containing $\overline{D}(x_j,\delta)$. In particular, simply by continuity, there exists $M>0$ so that

$$\sup\{|f(w)| \,:\, w\in\overline{D}(x_j,\delta)\}\le M, \quad \text{for all} \quad j=1,\dots,n. \tag{5.3.10}$$

Let $R>0$ be large enough to that $R>\max\{1+|z| \,:\, z\in\mathcal{S}\}$ and $\varepsilon>0$ small enough so that

$$\varepsilon<\delta, \quad -R<x_1-\varepsilon, \quad x_n+\varepsilon<R, \quad x_{j-1}+\varepsilon<x_j-\varepsilon, \quad j=2,\dots,n.$$

Define the paths

$$\gamma_R(t):=Re^{it},\ t\in[0,\pi], \quad \sigma_{j,\varepsilon}(t)=x_j-\varepsilon e^{-it},\ t\in[0,\pi], \quad j=1,\dots,n,$$

then

$$\eta_{R,\varepsilon}:=[-R,x_1-\varepsilon]\star\sigma_{1,\varepsilon}\star[x_1+\varepsilon,x_2-\varepsilon]\star\sigma_{2,\varepsilon}\star\cdots\star[x_{n-1}+\varepsilon,x_n-\varepsilon]\star\sigma_{n,\varepsilon}\star[x_n+\varepsilon,R],$$

and finally $\Gamma_{R,\varepsilon}:=\eta_{R,\varepsilon}\star\gamma_R$. We get that $\Gamma_{R,\varepsilon}$ is a closed and piecewise C^1-path with $\Gamma^*_{R,\varepsilon}\subset\Omega$, and by the choice of R and $\varepsilon>0$, it is clear that $\mathcal{S}\cap\Gamma^*_{R,\varepsilon}=\emptyset$. By Theorem 5.29, we have

$$\int_{\Gamma_{R,\varepsilon}} f(z)\,\mathrm{d}z = 2\pi i\sum_{z\in\mathcal{S}}\operatorname{Res}(f,z)W(\Gamma_{R,\varepsilon},z) = 2\pi i\sum_{\{z\in\mathcal{S}\,:\,\operatorname{Im}(z)>0\}}\operatorname{Res}(f,z). \tag{5.3.11}$$

Note that we used that $W(\Gamma_{R,\varepsilon},x_j)=0$ for all $j=1,\dots,n$, as these singularities are in *the outside* of $\Gamma_{R,\varepsilon}$. But the path-integral of f along $\Gamma_{R,\varepsilon}$ is

$$\begin{aligned}\int_{\Gamma_{R,\varepsilon}} f(z)\,\mathrm{d}z &= \int_{\gamma_R} f(z)\,\mathrm{d}z + \int_{\eta_{R,\varepsilon}} f(z)\,\mathrm{d}z\\ &= \int_{\gamma_R} f(z)\,\mathrm{d}z + \int_{-R}^{x_1-\varepsilon} f(x)\,\mathrm{d}x + \sum_{j=2}^{n}\int_{x_{j-1}+\varepsilon}^{x_j-\varepsilon} f(x)\,\mathrm{d}x + \int_{x_n+\varepsilon}^{R} f(x)\,\mathrm{d}x + \sum_{j=1}^{n}\int_{\sigma_{j,\varepsilon}} f(z)\,\mathrm{d}z.\end{aligned}$$

If $\varepsilon>0$ is fixed (but satisfying the original conditions), and we let $R\to+\infty$ in the above, the third bullet point condition implies that

$$\lim_{R\to+\infty}\int_{\Gamma_{R,\varepsilon}} f(z)\,\mathrm{d}z = \int_{-\infty}^{x_1-\varepsilon} f(x)\,\mathrm{d}x + \sum_{j=2}^{n}\int_{x_{j-1}+\varepsilon}^{x_j-\varepsilon} f(x)\,\mathrm{d}x + \int_{x_n+\varepsilon}^{+\infty} f(x)\,\mathrm{d}x + \sum_{j=1}^{n}\int_{\sigma_{j,\varepsilon}} f(z)\,\mathrm{d}z. \tag{5.3.12}$$

Now, let us find $\lim\limits_{\varepsilon\to0^+}\int_{\sigma_{j,\varepsilon}} f(z)\,\mathrm{d}z$. Note first that

$$\int_{\sigma_{j,\varepsilon}}\frac{\operatorname{Res}(f,x_j)}{z-x_j}\,\mathrm{d}z = \operatorname{Res}(f,x_j)\int_0^{\pi}\frac{\varepsilon i e^{-it}}{x_j-\varepsilon e^{-it}-x_j}\,\mathrm{d}t = -\pi i\operatorname{Res}(f,x_j).$$

Then we can use (5.3.9) and (5.3.10) to obtain

$$\left|\int_{\sigma_{j,\varepsilon}} f(z)\,\mathrm{d}z + \pi i\operatorname{Res}(f,z_j)\right| = \left|\int_{\sigma_{j,\varepsilon}} f_j(z)\,\mathrm{d}z\right| \le \sup\{|f_j(w)| \,:\, w\in\overline{D}(z_j,\varepsilon)\}\mathrm{length}(\sigma_{j,\varepsilon}) \le M\pi\varepsilon;$$

which tends to 0 as $\varepsilon\to0$. Combining this together with (5.3.12) and (5.3.11) yields

$$\begin{aligned}&\lim_{\varepsilon\to0^+}\left(\int_{-\infty}^{x_1-\varepsilon} f(x)\,\mathrm{d}x + \int_{x_1+\varepsilon}^{x_2-\varepsilon} f(x)\,\mathrm{d}x + \cdots + \int_{x_{n-1}+\varepsilon}^{x_n-\varepsilon} f(x)\,\mathrm{d}x + \int_{x_n+\varepsilon}^{+\infty} f(x)\,\mathrm{d}x\right)\\ &\qquad = \lim_{\varepsilon\to0^+}\left(\lim_{R\to+\infty}\int_{\Gamma_{R,\varepsilon}} f(z)\,\mathrm{d}z - \sum_{j=1}^{n}\int_{\sigma_{j,\varepsilon}} f(z)\,\mathrm{d}z\right)\\ &\qquad = 2\pi i\sum_{\{z\in\mathcal{S}\,:\,\operatorname{Im}(z)>0\}}\operatorname{Res}(f,z) + \pi i\sum_{j=1}^{n}\operatorname{Res}(f,x_j).\end{aligned}$$

□

Remark 5.43. For example, we can use Theorem 5.42 in the particular case where $f = P/Q$ with P, Q polynomials satisfying

$$\deg(Q) \geq \deg(P) + 2.$$

And, as we did in Theorem 5.38, we can consider integrals of the form

$$\text{pv} \int_{-\infty}^{+\infty} f(x)e^{aix}\,\mathrm{d}x, \quad a > 0;$$

with f possibly having poles of order 1 in the real line.

Let us see a concrete example.

Example 5.44. We want to evaluate

$$\text{pv} \int_{-\infty}^{+\infty} \frac{x}{x^3+1}\,\mathrm{d}x,$$

via Theorem 5.42. To do so, define $f(z) = \frac{z}{z^3+1}$, where f is holomorphic in $\mathbb{C}$ except in

$$\langle \sqrt[3]{-1} \rangle = \{e^{-i\frac{\pi}{3}}, e^{i\frac{\pi}{3}}, -1\}.$$

These singularities satisfy $\operatorname{Im}(e^{-i\frac{\pi}{3}}) < 0$, $-1 \in \mathbb{R}$, and $\operatorname{Im}(e^{i\frac{\pi}{3}}) > 0$, so we only need to look at $z_0 = -1$ and $z_1 = e^{i\frac{\pi}{3}}$. At $z_0 = -1$, the numerator and denominator have $z_0 \neq 0$, $z_0^3 + 1 = 0$, and $3z_0^2 \neq 0$. By Proposition 5.19, f has a pole of order 1 at $z_0 = -1$, and

$$\operatorname{Res}(f, z_0) = \frac{z_0}{3z_0^2} = -\frac{1}{3}.$$

For the same reasons, f has a pole of order 1 at $z_1 = e^{i\frac{\pi}{3}}$, with

$$\operatorname{Res}(f, z_1) = \frac{z_1}{3z_1^2} = \frac{1}{3e^{i\frac{2\pi}{3}}}.$$

Since $\deg(z^3 + 1) = 2 + \deg(z)$, we can apply Theorem 5.42 (see Remark 5.43), to conclude

$$\text{pv} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \operatorname{Res}(f, e^{i\frac{\pi}{3}}) + \pi i \operatorname{Res}(f, -1) = \frac{\pi i}{3}\left(2\cos(\tfrac{2\pi}{3}) - 2i\sin(\tfrac{2\pi}{3}) - 1\right) = \frac{\pi}{\sqrt{3}}.$$

5.4 Exercises

Exercise 5.1. *Find the Laurent Series expansions of the following functions in the indicated annuli (and center), and identify the corresponding Principal Part.*

(a) $\frac{z+1}{z}$ *in* $z \in \mathbb{C} \setminus \{0\}$, *at* $z_0 = 0$.

(b) $\frac{z}{z^2+1}$ *in* $\{z \in \mathbb{C} : 0 < |z - i| < 2\}$, *at* $z_0 = i$.

(c) $\sin(\frac{1}{z})$ *in* $z \in \mathbb{C} \setminus \{0\}$, *at* $z_0 = 0$.

(d) $\frac{1}{z(z+1)}$ *in* $\{z \in \mathbb{C} : 0 < |z + 1| < 1\}$, *at* $z_0 = -1$; *and in* $\{z \in \mathbb{C} : |z| > 1\}$, *and in* $\{z \in \mathbb{C} : 0 < |z| < 1\}$ *and at* $z_0 = 0$.

(e) $\frac{z}{z+1}$ *in* $\mathbb{C} \setminus \{-1\}$, *at* $z_0 = -1$.

(f) $\frac{e^z}{z^2}$ *in* $\mathbb{C} \setminus \{0\}$, *at* $z_0 = 0$.

(g) $\frac{1}{z(z-1)(z-2)}$ *in* $\{z \in \mathbb{C} : 0 < |z| < 1\}$ *and in* $\{z \in \mathbb{C} : 1 < |z| < 2\}$, *both at* $z_0 = 0$.

Exercise 5.2. *Find the isolated singularities and classify them (into removable, poles, or essential) for the following functions. In the case of poles, indicate the order of those.*

$$a)\ \frac{\cos z}{z^2} \quad b)\ \frac{e^z-1}{z^2} \quad c)\ \frac{z+1}{z-1} \quad d)\ \frac{e^z}{z} \quad e)\ \frac{\sin z}{z} \quad f)\ \frac{(e^z-1)^2}{z^2} \quad g)\ \frac{1}{z(z+1)} \quad h)\ \frac{e^z}{z^2}$$

$$i)\ \frac{\cos(z-1)}{z^2} \quad j)\ \frac{1}{z^2-1} \quad k)\ \frac{1}{\cos\left(\frac{1}{z}\right)} \quad l)\ \frac{e^z(z-3)}{(z-1)(z-5)} \quad m)\ \frac{\cos z}{1-z} \quad n)\ \frac{z}{(e^z-1)(e^z-2)}.$$

Exercise 5.3. *Calculate the residues of the following functions at the indicated points* $z_0 \in \mathbb{C}$ *:*

$$a)\ \frac{e^{z^2}}{z-1},\ z_0=1 \quad b)\ \frac{e^z}{(z^2-1)^2},\ z_0=1 \quad c)\ \left(\frac{\cos z-1}{z}\right)^2,\ z_0=0 \quad d)\ \frac{z^2}{z^4-1},\ z_0=i$$

$$e)\ \frac{e^z-1}{\sin z},\ z_0=0 \quad f)\ \frac{1}{e^z-1},\ z_0=0 \quad g)\ \frac{z+2}{z^2-2z},\ z_0=0 \quad h)\ \frac{e^z+1}{z^4},\ z_0=0.$$

Exercise 5.4. *Find the isolated singularities, classify them (including the order in the case of poles), and calculate the residues at all those singularities.*

$$a)\ \frac{1}{e^z-1} \quad b)\ \frac{1}{z^3(z+4)} \quad c)\ \frac{1}{z^2+2z+1} \quad d)\ \frac{1}{z^3-3}, \quad e)\ \frac{e^z}{z(1-z)^3}.$$

Exercise 5.5. *Let* E *be a set with no accumulation points, that is,* $E' = \emptyset$*. Let* $f : \mathbb{C}\setminus E \to \mathbb{C}$ *be holomorphic and bounded in* $\mathbb{C}\setminus E$*. Prove that* f *is constant.*

Suggestion: Riemann's Criterion Theorem 5.12 is vital. If you are not too familiar with topological concepts, assume first that E *is finite.*

Exercise 5.6. *Let* $f : D(z_0, r)\setminus\{z_0\} \to \mathbb{C}$ *be holomorphic. Prove that* f *and* f^2 *have the same type of singularity (removable, pole, or essential) at* z_0*. This amounts to show that:*

- f *has a removable singularity at* $z_0 \iff f^2$ *has a removable singularity at* z_0*.*
- f *has a pole at* $z_0 \iff f^2$ *has a pole at* z_0*.*
- f *has an essential singularity at* $z_0 \iff f^2$ *has an essential singularity at* z_0*.*

Suggestion: Riemann's Criterion Theorem 5.12 and Proposition 5.14 are very helpful.

Exercise 5.7. *For every* $n \in \mathbb{N}\cup\{0\}$*, evaluate the integral*

$$\int_{\partial D(0,1)} z^n e^{1/z}\,\mathrm{d}z;$$

where $\partial D(0,1)$ *is traveled once and counterclockwise.*

Exercise 5.8. *Prove that*

$$\int_{\partial D(0,1)} e^{z+\frac{1}{z}}\,\mathrm{d}z = 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!\cdot(n+1)!}.$$

Exercise 5.9. *For the ellipse* $\gamma(t) = \{a\cos t + ib\sin t\ :\ t\in[0,2\pi]\}$*,* $a, b > 0$*, evaluate*

$$\int_\gamma \frac{e^{-z^2}}{z^2}\,\mathrm{d}z.$$

Exercise 5.10. *Using Theorem 5.29, evaluate the following path-integrals, always travaled once and with the counterclockwise orientation.*

(a) $\int_\gamma \frac{z^2}{e^{2\pi i z^3}-1}\,\mathrm{d}z$*, where* $\gamma \equiv \partial D(0,r)$*,* $n < r^3 < n+1$*,* $n\in\mathbb{N}$*.*

(b) $\int_\gamma \frac{1}{z^4+1}\,\mathrm{d}z$, *where* γ *is the ellipse* $x^2 - xy + y^2 + x + y = 0$.

(c) $\int_\gamma \frac{\cos(\frac{z}{2})}{z^2-4}\,\mathrm{d}z$, *where* γ *is the ellipse* $x^2/9 + y^2/4 = 1$.

(d) $\int_\gamma \frac{z}{z^2+2z+5}\,\mathrm{d}z$, *where* $\gamma \equiv \partial D(0,1)$.

(e) $\int_\gamma \frac{e^z}{(1-z)^3}\,\mathrm{d}z$, *where* $\gamma \equiv \partial D(1,1/2)$.

(f) $\int_\gamma \frac{e^z-1}{(\sin z)^3}\,\mathrm{d}z$, *where* $\gamma \equiv \partial D(0,4)$.

(g) $\int_\gamma \frac{1}{z(z-1)(z-2)}\,\mathrm{d}z$, *where* $\gamma \equiv \partial D(0,3/2)$.

(h) $\int_\gamma \frac{1+z}{1-\cos z}\,\mathrm{d}z$, *where* $\gamma \equiv \partial D(0,7)$.

Exercise 5.11. *Find a holomorphic function* f *in* $D(0,1)$ *so that*

$$2\pi i = \int_{\partial D(0,1/2)} \frac{f(z)}{z^n}\,\mathrm{d}z, \quad \textit{for all} \quad n \in \mathbb{N}.$$

The circle $\partial D(0,1/2)$ *is traveled once and counterclockwise.*

Exercise 5.12. *Evaluate the integral*

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{5 + 4\cos\theta}.$$

Exercise 5.13. *Find a closed formula (in terms of the parameter* a*) for the following the integral*

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + \sin\theta},$$

when $a > 1$, *and also when* $a < -1$.

Exercise 5.14. *Show that*

$$\int_0^{\pi} \frac{\mathrm{d}\theta}{1 + \sin^2\theta} = \frac{\pi}{\sqrt{2}}.$$

Exercise 5.15. *For* $a, b > 0$, *with* $a \neq b$, *evaluate the integral*

$$\mathrm{pv}\int_{-\infty}^{+\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,\mathrm{d}x.$$

Exercise 5.16. *For* $a > 0$, *evaluate the integral*

$$\mathrm{pv}\int_{-\infty}^{+\infty} \frac{\cos x}{(x^2+a^2)^2}\,\mathrm{d}x.$$

Exercise 5.17. *Evaluate the integral*

$$\mathrm{pv}\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{x^2 - 2x + 4}.$$

Exercise 5.18. *Prove the identity*

$$\mathrm{pv}\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{1 + x^{2n}} = \frac{\pi}{n\sin(\frac{\pi}{2n})}, \quad n \in \mathbb{N}.$$

Exercise 5.19. *Evaluate the integral*

$$\mathrm{pv}\int_{-\infty}^{+\infty} \frac{\sin x}{x^2 - 2x + 2}\,\mathrm{d}x.$$

Exercise 5.20. *Evaluate the integral*

$$\mathrm{pv}\int_{-\infty}^{+\infty} \frac{x^3 \sin x}{(x^2+1)^2}\,\mathrm{d}x.$$

Exercise 5.21. *Prove the identity*

$$\mathrm{pv}\int_{-\infty}^{+\infty} \frac{x \sin x}{x^4+1}\,\mathrm{d}x = \pi e^{-\frac{1}{\sqrt{2}}} \sin\left(\tfrac{1}{\sqrt{2}}\right).$$

Exercise 5.22. *Evaluate the integral*

$$\mathrm{pv}\int_{-\infty}^{+\infty} \frac{e^{ix}}{x^3-2x^2+x-2}\,\mathrm{d}x,$$

and then

$$\mathrm{pv}\int_{-\infty}^{+\infty} \frac{\cos x}{x^3-2x^2+x-2}\,\mathrm{d}x,$$

Suggestion: Read carefully Theorem 5.42 and Remark 5.43.

Exercise 5.23. *Evaluate the integral*

$$\mathrm{pv}\int_{-\infty}^{+\infty} \frac{\sin x}{x(x^2-1)}\,\mathrm{d}x.$$

Suggestion: Read carefully Theorem 5.42 and Remark 5.43.

Exercise 5.24. *Evaluate the integral*

$$\mathrm{pv}\int_{-\infty}^{+\infty} \frac{e^{ix}}{x(x-w_0)^2}\,\mathrm{d}x,$$

where $w_0 \in \mathbb{C}$ *is so that* $\mathrm{Im}(w_0) > 0$.

Exercise 5.25. *Evaluate the integral*

$$\int_0^{2\pi} \frac{e^{4i\theta}}{2+\cos\theta}\,\mathrm{d}\theta,$$

and then deduce the value of

$$\int_0^{2\pi} \frac{\cos(4\theta)}{2+\cos\theta}\,\mathrm{d}\theta.$$

Exercise 5.26. *For each* $a > 1$, *evaluate the integral*

$$\int_0^{\pi} \frac{\mathrm{d}\theta}{(a+\cos\theta)^2}\,\mathrm{d}\theta.$$

Exercise 5.27. *Prove the identity*

$$\mathrm{pv}\int_{-\infty}^{+\infty} \left(\frac{\sin x}{x}\right)^2 \mathrm{d}x = \pi.$$

Exercise 5.28. *Prove the identity*

$$\mathrm{pv}\int_{-\infty}^{+\infty} \frac{\cos x}{e^x+e^{-x}}\,\mathrm{d}x = \frac{\pi}{e^{\pi/2}+e^{-\pi/2}}.$$

Exercise 5.29. *Prove the identity*

$$\mathrm{pv}\int_{-\infty}^{+\infty} \frac{e^{-x}}{1+e^{-2\pi x}}\,\mathrm{d}x = \frac{1}{2\sin(\frac{1}{2})}.$$

Chapter 6

Fourier Series and Differential Equations

This chapter serves as a brief introduction to elementary Fourier Analysis, which we will applied to solve some special cases of differential equations. Our main reference is [3, Chapter 13].

6.1 Elements from Fourier Analysis

A **trignometric polynomial** is any function $P : \mathbb{R} \to \mathbb{C}$ of the form

$$P(x) = \sum_{n=-N}^{N} c_n e^{inx}, \quad x \in \mathbb{R};$$

where $\{c_n\}_{n=-N}^{N} \subset \mathbb{C}$ and $N \in \mathbb{N} \cup \{0\}$. We can refer to the largest number $M \in \mathbb{N} \cup \{0\}$ so that one of the coefficients of e^{iMx}, e^{-iMx} are nonzero as the **degree of** P. Notice also that P is 2π-periodic:

$$P(x + 2\pi) = P(x), \quad \text{for all} \quad x \in [0, 2\pi].$$

Moreover, by (1.5.1), one can also express P as

$$P(x) = \sum_{n=-N}^{N} c_n e^{inx} = c_0 + \sum_{n=1}^{N} a_n \cos(nx) + \sum_{n=1}^{N} b_n \sin(nx), \quad a_n := c_n + c_{-n}, b_n := i(c_n - c_{-n}), n \in \mathbb{N}.$$

Our of the main goals in Fourier Analysis is to approximate a sufficiently *regular* function $f : [0, 2\pi] \to \mathbb{C}$ by trigonometric polynomials. Those polynomials associated with f are the *Fourier sums* of f, which we define now.

6.1.1 Fourier Coefficients and Sums. The Bessel's Inequality

We will work from now on with Riemann-integrable functions $h : [0, 2\pi) \to \mathbb{C}$, which we assume to be *extended to all* $\mathbb{R}$ *by* 2π*-periodicity.* This means that

$$h(x + 2\pi) = h(x), \quad \text{for all} \quad x \in \mathbb{R}.$$

Note that then this 2π-periodic function h is continuous in all of $\mathbb{R}$ if and only if $h(0) = h(2\pi)$ and h is continuous in $[0, 2\pi)$. Moreover, it is not difficult to see that

$$\int_0^{2\pi} h(t)\,\mathrm{d}t = \int_a^{a+2\pi} h(t)\,\mathrm{d}t, \quad \text{for all} \quad a \in \mathbb{R}. \tag{6.1.1}$$

Indeed, it suffices to find the unique $k \in \mathbb{Z}$ for which $a \in [2(k-1)\pi, 2k\pi)$, split the integral appropriately and use that h is 2π-periodic. That is, the integral of h is the same on each interval of length 2π. This is very convenient, since sometimes it is easier to examine integrals over $[-\pi, \pi]$ rather than on $[0, 2\pi]$. Below is the key definition of this chapter.

Definition 6.1 (Fourier Coefficients and Sums)**.** *Let $f:[0,2\pi]\to\mathbb{C}$ be a Riemann-integrable function in $[0,2\pi]$, and let $n\in\mathbb{Z}$. We define* ***the n^{th}-Fourier coefficient of*** *f as*

$$\widehat{f}(n) := \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\,\mathrm{d}t. \tag{6.1.2}$$

Also, if $N\in\mathbb{N}\cup\{0\}$, ***the N^{th}-Fourier sum of*** *f is the function $S_N(f):[0,2\pi]\to\mathbb{C}$ given by*

$$S_N(f)(x) := \sum_{n=-N}^{N}\widehat{f}(n)e^{inx}, \quad x\in[0,2\pi]. \tag{6.1.3}$$

Finally, ***the Fourier series of*** *f is the series of functions*

$$S(f)(x) = \sum_{n\in\mathbb{Z}}\widehat{f}(n)e^{inx} := \sum_{n=-\infty}^{+\infty}\widehat{f}(n)e^{inx} := \lim_{N\to\infty}S_N(f)(x) = \lim_{N\to\infty}\sum_{n=-N}^{N}\widehat{f}(n)e^{inx}, \quad x\in[0,2\pi].$$

However, we cannot claim whether or not this series converges for a general function f.

Remark 6.2. Let us make some immediate observations from Definition 6.1.

(1) Define, for each $m\in\mathbb{Z}$, the function $e_m:[0,2\pi]\to\mathbb{C}$ by

$$e_m(x) := e^{imx}, \quad x\in[0,2\pi].$$

Then we have

$$\widehat{e_m}(n) = \delta_{mn} = \begin{cases}1 & \text{if } n=m\\ 0 & \text{if } n\neq m\end{cases}, \quad \text{for all} \quad n\in\mathbb{Z}.$$

Indeed, this follows by just looking at the definition (6.1.2) of Fourier coefficient.

(2) If $\lambda\in\mathbb{C}$, and $f,g:[0,2\pi]\to\mathbb{C}$ are Riemann-integrable, then

$$\widehat{(\lambda f+g)}(n) = \lambda\widehat{f}(n)+\widehat{g}(n), \quad \text{for all} \quad n\in\mathbb{Z}.$$

This means that the operation "taking Fourier coefficients" is a $\mathbb{C}$-linear operation.

(3) Let $P(x)=\sum_{n=-M}^{M}c_ne^{inx}$ be a trigonometric polynomial, where $M\in\mathbb{N}\cup\{0\}$. Then, by the linearity we have seen in (2), and the fact that $\widehat{e_m}(n)=\delta_{mn}$, we get that

$$\widehat{P}(n) = \sum_{m=-M}^{M}c_m\delta_{nm} = \begin{cases}c_n & \text{if } |n|\leq M\\ 0 & \text{if } |n|>M.\end{cases}$$

In particular,

$$S_N(P)(x)=P(x) \quad \text{for all} \quad N\geq M,\, x\in\mathbb{R}, \quad \text{and} \quad S(P)(x)=P(x) \quad \text{for all} \quad x\in\mathbb{R}.$$

(4) Writing down all the terms in the sum defining S_N, we obtain

$$S_N(t) = \sum_{n=-N}^{N}\widehat{f}(n)e^{int} = \sum_{n=-N}^{N}\frac{1}{2\pi}\left(\int_0^{2\pi}f(s)e^{-ins}\,\mathrm{d}s\right)e^{int} = \sum_{n=-N}^{N}\frac{1}{2\pi}\int_0^{2\pi}f(s)e^{in(t-s)}\,\mathrm{d}s.$$

(5) The 0^{th}-Fourier coefficient of an integrable $f:[0,2\pi]\to\mathbb{R}$ is

$$\widehat{f}(0) = \frac{1}{2\pi}\int_0^{2\pi}f(t)\,\mathrm{d}t;$$

that is, the *averaged integral* of f in $[0,2\pi]$.

Example 6.3. Let us now find the Fourier sums of some concrete 2π-periodic functions.

(1) Let $f(x) = x$, $x \in [0, 2\pi]$. We could re-define f at 2π by setting $f(2\pi) = 0(= f(0))$, so that we obtain a function that admits a 2π-periodic extension to all of $\mathbb{R}$. But for the computation of the Fourier coefficients, this will not make any difference. We begin by computing the Fourier coefficients $\widehat{f}(n)$ when $n \in \mathbb{Z} \setminus \{0\}$. We can use Integration by Parts to obtain:

$$\int_0^{2\pi} te^{-int}\,\mathrm{d}t = \left[t\frac{e^{-int}}{-in}\right]_{t=0}^{t=2\pi} - \int_0^{2\pi} \frac{e^{-int}}{-in}\,\mathrm{d}t = \frac{2\pi i}{n},$$

that is $\widehat{f}(n) = \frac{i}{n}$. Now, for $n = 0$, we have that

$$\widehat{f}(0) = \frac{1}{2\pi}\int_0^{2\pi} t\,\mathrm{d}t = \pi.$$

To summarize,

$$\widehat{f}(n) = \begin{cases} \pi & \text{if } n = 0, \\ i/n & \text{if } n \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

The N^{th}-Fourier sum of f is the trigonometric polynomial

$$S_N(f)(x) = \sum_{n=-N}^{N} \widehat{f}(n)e^{inx} = \pi + \sum_{n=-N,\, n\neq 0}^{N} \frac{ie^{inx}}{n} = \pi + 2i\sum_{n=1}^{N} \frac{\sin(nx)}{n},$$

for all $x \in [0, 2\pi]$. The Fourier series of f would be

$$S(f)(x) = \pi + \sum_{n=-\infty,\, n\neq 0}^{+\infty} \frac{ie^{inx}}{n}.$$

We can already see that this series does not converge to $f(0) = 0$ at $x = 0$. For $x \in (0, 2\pi)$, the series converges, for example, by Picard's Criterion 3.18, but we do not know whether it converges to $f(x)$. We will go back to this in Example 6.7 below.

(2) Let $g(x) = x^2 - 2\pi x$, $x \in [0, 2\pi]$. Since we know already the Fourier coefficients an sums of f as in (1), we only need to examine $x \mapsto x^2$, since by Remark 6.2(2), we would obtain the information about g by linearity. So, if $h(x) = x^2$, note that for $n \neq 0$,

$$\int_0^{2\pi} t^2e^{-int}\,\mathrm{d}t = \left[t^2\frac{e^{-int}}{-in}\right]_{t=0}^{t=2\pi} - \int_0^{2\pi} 2t\frac{e^{-int}}{-in}\,\mathrm{d}t.$$

Recycling computations from (1), we know that $\int_0^{2\pi} te^{-int}\,\mathrm{d}t = \frac{2\pi i}{n}$, and consequently

$$\int_0^{2\pi} t^2e^{-int}\,\mathrm{d}t = \frac{4\pi^2 i}{n} + \frac{4\pi}{n^2}.$$

Therefore

$$\widehat{g}(n) = \frac{2\pi i}{n} + \frac{2}{n^2} - 2\pi\frac{i}{n} = \frac{2}{n^2}.$$

And

$$\widehat{g}(0) = \widehat{h}(0) - 2\pi\widehat{f}(0) = \frac{1}{2\pi}\int_0^{2\pi} t^2\,\mathrm{d}t - 2\pi^2 = -\frac{2\pi^2}{3}.$$

The Fourier Series of g is

$$S(g)(x) = -\frac{2\pi^2}{3} + \sum_{n=-\infty,\, n\neq 0}^{+\infty} \frac{2}{n^2}e^{inx} = -\frac{2\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}.$$

(3) Let $h(x) = \cos x$, $x \in \mathbb{R}$. To calculate $\widehat{h}(n)$, we can of course look at the definition, but Remark 6.2(3) tells us that if h is written as a trigonometric polynomial, then the coefficients of the polynomial are precisely the Fourier Coefficients. So, we write

$$h(x) = \cos x = \frac{e^{ix} + e^{-ix}}{2} = \frac{1}{2}e^{-ix} + \frac{1}{2}e^{ix}, \quad x \in \mathbb{R};$$

from which we get that

$$\widehat{h}(n) = \begin{cases} \frac{1}{2} & \text{if } n = -1 \text{ or } n = 1 \\ 0 & \text{if } n \neq -1, 1. \end{cases}$$

Therefore $S_0(h)(x) = 0$ and $S_N(h)(x) = \frac{1}{2}e^{-ix} + \frac{1}{2}e^{ix} = h(x)$ for all $N \in \mathbb{N}$. Then obviously $S(h)(x) = h(x) = \cos x$ for all $x \in \mathbb{R}$.

Theorem 6.4 (Bessel's Inequality)**.** *Let $f : [0, 2\pi] \to \mathbb{C}$ be a function so that f^2 is Riemann-integrable in $[0, 2\pi]$. Then, for every $N \in \mathbb{N}$,*

$$\sum_{n=-N}^{N} \left|\widehat{f}(n)\right|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2 \,\mathrm{d}t.$$

Consequently,

$$\sum_{n=-\infty}^{+\infty} \left|\widehat{f}(n)\right|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2 \,\mathrm{d}t.$$

Proof. For every $N \in \mathbb{N}$, we can write

$$\int_0^{2\pi} |f(t) - S_N(t)|^2 \,\mathrm{d}t = \int_0^{2\pi} \left(f(t) - \sum_{n=-N}^{N} \widehat{f}(n)e^{int}\right)\left(\overline{f(t)} - \sum_{m=-N}^{N} \overline{\widehat{f}(m)}e^{-imt}\right) \mathrm{d}t$$

$$= \int_0^{2\pi} |f(t)|^2 + \sum_{n=-N}^{N}\sum_{m=-N}^{N} \widehat{f}(n)\overline{\widehat{f}(m)} \int_0^{2\pi} e^{int}e^{-imt} \,\mathrm{d}t$$

$$- \sum_{m=-N}^{N} \overline{\widehat{f}(m)} \int_0^{2\pi} f(t)e^{-imt} \,\mathrm{d}t - \sum_{n=-N}^{N} \widehat{f}(n) \int_0^{2\pi} \overline{f(t)}e^{int} \,\mathrm{d}t$$

$$= \int_0^{2\pi} |f(t)|^2 + 2\pi \sum_{n=-N}^{N}\sum_{m=-N}^{N} \widehat{f}(n)\overline{\widehat{f}(m)}\delta_{nm}$$

$$- \sum_{m=-N}^{N} \overline{\widehat{f}(m)} \int_0^{2\pi} f(t)e^{-imt} \,\mathrm{d}t - \overline{\sum_{n=-N}^{N} \overline{\widehat{f}(n)} \int_0^{2\pi} f(t)e^{-int} \,\mathrm{d}t}$$

$$= \int_0^{2\pi} |f(t)|^2 + 2\pi \sum_{n=-N}^{N} \left|\widehat{f}(n)\right|^2 - 2\pi \sum_{m=-N}^{N} \overline{\widehat{f}(m)}\widehat{f}(m) - 2\pi \sum_{n=-N}^{N} \overline{\overline{\widehat{f}(n)}\widehat{f}(n)}$$

$$= \int_0^{2\pi} |f(t)|^2 - 2\pi \sum_{n=-N}^{N} \left|\widehat{f}(n)\right|^2.$$

In the third equality, we used Remark 6.2(1). And in the fourth inequality, simply that $\int_0^{2\pi} \overline{h(t)} \,\mathrm{d}t = \overline{\int_0^{2\pi} h(t) \,\mathrm{d}t}$ for every Riemann-integrable $h : [0, 2\pi] \to \mathbb{C}$; see (4.1.4) in Definition 4.8. But the conclusion from the above chain of equalities is that

$$\int_0^{2\pi} |f(t)|^2 - 2\pi \sum_{n=-N}^{N} \left|\widehat{f}(n)\right|^2 = \int_0^{2\pi} |f(t) - S_N(t)|^2 \,\mathrm{d}t \geq 0,$$

which yields our theorem. ☐

A consequence of Theorem 6.4 is the following corollary.

Corollary 6.5. *Let $f : [0, 2\pi] \to \mathbb{C}$ be a function so that f^2 is Riemann-integrable in $[0, 2\pi]$. Then,*

$$\lim_{|n|\to\infty} \left|\widehat{f}(n)\right| = 0.$$

Proof. By Theorem 6.4, the series

$$\sum_{n\in\mathbb{Z}} \left|\widehat{f}(n)\right|^2 := \lim_{N\to+\infty} \sum_{n=-N}^{N} \left|\widehat{f}(n)\right|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 \,\mathrm{d}t$$

converges, whence $\lim_{|n|\to\infty} \left|\widehat{f}(n)\right| = 0$. □

6.1.2 Convergence of Fourier Series for Lipschitz Functions

In the following theorem, we show that functions with a Lipschitz-type condition at a point have Fourier Series convergent at that point. The proof we give here is due to Paul R. Chernoff [1].

Theorem 6.6. *Let $f : \mathbb{R} \to \mathbb{C}$ a piecewise continuous and 2π-periodic function, and let $x_0 \in [0, 2\pi]$ be a point so that there are $\varepsilon > 0$ and $C > 0$ with*

$$|f(x) - f(x_0)| \leq C|x - x_0|, \quad \textit{for all} \quad x \in (x_0 - \varepsilon, x_0 + \varepsilon).$$

Then, we have that

$$S(f)(x_0) := \lim_{N\to+\infty} S_N(f)(x_0) = f(x_0).$$

In particular, if $f : \mathbb{R} \to \mathbb{C}$ is a piecewise continuous and 2π-periodic function, and $f'(x_0)$ exists at some $x_0 \in [0, 2\pi)$, then $S(f)(x_0) = x_0$.

Proof. We define a new function

$$h(x) = \begin{cases} \dfrac{f(x + x_0) - f(x_0)}{e^{ix} - 1} & \text{if } x \in (0, 2\pi) \\ 1 & \text{if } x = 0; \end{cases}$$

where the value $h(0) = 1$ is playing no role. We can extend h to all of $\mathbb{R}$ with 2π-periodicity. Then h is bounded on an interval around 0. Indeed, first notice that

$$\lim_{x\to 0} \frac{e^{ix} - 1}{x} = i,$$

as the derivative of $t \mapsto e^{it}$ at $t = 0$ is equal to i. Thus, there exists $\delta > 0$ such that $|e^{ix} - 1| \geq \frac{1}{2}|x||i| = \frac{|x|}{2}$ whenever $|x| \leq \delta$. Letting $r = \min\{\delta, \varepsilon\}$; where $\varepsilon > 0$ is the one from the assumption, we have for those $|x| < r$ that

$$|h(x)| = \frac{|f(x + x_0) - f(x_0)|}{|e^{ix} - 1|} \leq \frac{C|x|}{|e^{ix} - 1|} \leq 2C.$$

Since h is bounded and has finitely many discontinuities at $[0, 2\pi]$, we have that h is Riemann-integral. The two functions

$$(0, 2\pi) \ni x \mapsto g_1(x) := f(x + x_0) - f(x_0), \quad (0, 2\pi) \ni x \mapsto g_2(x) := h(x)(e^{ix} - 1)$$

agree on $(0, 2\pi)$, and so they have the same Fourier coefficients. For the first function, those coefficients are, for $n \in \mathbb{Z} \setminus \{0\}$,

$$\widehat{g_1}(n) = \frac{1}{2\pi} \int_0^{2\pi} (f(t + x_0) - f(x_0))\, e^{-int} \,\mathrm{d}t = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-in(t-x_0)} \,\mathrm{d}t = e^{inx_0} \widehat{f}(n),$$

and $\widehat{g_1}(0) = \widehat{f}(0) - f(x_0)$ in the case $n = 0$. And for the second function g_2, the coefficients are

$$\widehat{g_2}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{it}h(t)e^{-int}\,\mathrm{d}t - \widehat{h}(n) = \frac{1}{2\pi}\int_0^{2\pi} h(t)e^{-i(n-1)t}\,\mathrm{d}t - \widehat{h}(n) = \widehat{h}(n-1) - \widehat{h}(n).$$

Thus we can write, for all $N \in \mathbb{N}$,

$$\begin{aligned} S_N(f)(x_0) - f(x_0) &= \sum_{n=-N}^{N} \widehat{f}(n)e^{inx_0} - f(x_0) = \sum_{n=-N}^{N} \widehat{g_1}(n) = \sum_{n=-N}^{N} \widehat{g_2}(n) \\ &= \sum_{n=-N}^{N} \left(\widehat{h}(n-1) - \widehat{h}(n)\right) = \widehat{h}(-N-1) - \widehat{h}(N). \end{aligned}$$

Now h^2 is integrable, because h is bounded and piecewise continuous (and so is h^2). By Corollary 6.5, we know that $\lim_{|N|\to\infty} \left|\widehat{h}(N)\right| = 0$. Thus, the previous chain of equalities says that

$$\lim_{N\to\infty} |S_N(f)(x_0) - f(x_0)| = \lim_{N\to\infty} \left|\widehat{h}(-N-1) - \widehat{h}(N)\right| = 0,$$

as desired. □

We now apply Theorem 6.6 to find some interesting facts.

Example 6.7. Let us get back to the functions from Example 6.3.

(1) Let $f(x) = x$ for $x \in [0, 2\pi)$, and extend it to $\mathbb{R}$ by 2π-periodicity. As we saw in Example 6.3, the Fourier Series $S(f)(0)$ at $x_0 = 0$ does not converge to $f(0) = 0$. However, if $x \in (0, 2\pi)$, the function f is differentiable at x, and, by Theorem 6.6, we get that $S(f)(x)$ converges to $f(x)$. We conclude

$$x = f(x) = \pi + \sum_{n=-N,\, n\neq 0}^{N} \frac{ie^{inx}}{n}, \quad x \in (0, 2\pi).$$

(2) Let $g(x) = x^2 - 2\pi x$, $x \in [0, 2\pi]$. This function satisfies the assumption of Theorem 6.6 at $x_0 = 0$, since

$$|g(x) - g(x_0)| = |x^2 - 2\pi x| = |x||x - 2\pi| \le 3\pi|x| = 3\pi|x - x_0|, \quad \text{for all} \quad x \in (-\pi, \pi).$$

Thus the Fourier Series $S(g)$ of g converges at $x_0 = 0$ and $S(g)(0) = g(0) = 0$. But recall the formula for $S(f)$ we derived in Example 6.3(2):

$$0 = g(0) = S(g)(0) = -\frac{2\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Rearranging the terms, we get the identity:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

An easy consequence of Theorem 6.6 is the following *Identity Principle* for Lipschitz functions. If $E \subset \mathbb{R}$ is a set, and $h : E \to \mathbb{C}$ is a function, we say that h **is Lipschitz in** E if there exists a constant $L > 0$ so that

$$|h(x) - h(y)| \le L|x - y|, \quad \text{for all} \quad x, y \in E.$$

Corollary 6.8. *Let $f, g : \mathbb{R} \to \mathbb{C}$ two piecewise continuous and 2π-periodic functions that are Lipschitz in $(0, 2\pi)$, and so that $\widehat{f}(n) = \widehat{g}(n)$ for all $n \in \mathbb{Z}$. Then $f = g$ in $(0, 2\pi)$.*

Proof. Both f and g satisfy the conditions of Theorem 6.6 for all $x_0 \in (0, 2\pi)$, and therefore, $S(f)(x) = f(x)$ and $S(g)(x) = g(x)$ for all $x \in (0, 2\pi)$. But since f and g have the same Fourier coefficients, obviously $S(f)(x) = S(g)(x)$, and so $f(x) = g(x)$, for all $x \in (0, 2\pi)$. □

In connection with Theorem 6.6, we next examine the coefficients of derivatives functions.

Proposition 6.9. *Let f be a 2π-periodic and continuous function in $\mathbb{R}$, which is differentiable in $(0, 2\pi)$, with f' Riemann-integrable in $[0, 2\pi]$. Then,*

$$\widehat{f'}(n) = in\widehat{f}(n) \quad \text{for all} \quad n \in \mathbb{Z}.$$

Proof. If $n = 0$, note that the Fundamental Theorem of Calculus gives

$$\widehat{f'}(n) = \frac{1}{2\pi}\int_0^{2\pi} f'(t) = f(2\pi) - f(0) = 0 = in\widehat{f}(n).$$

If $n \in \mathbb{Z} \setminus \{0\}$, then Integration by Parts, using that $f(0) = f(2\pi)$, we get

$$\int_0^{2\pi} f'(t)e^{-int}\,\mathrm{d}t = \left[f(t)e^{-int}\right]_{t=0}^{t=2\pi} - \int_0^{2\pi} f(t)(-ine^{-int})\,\mathrm{d}t = in\int_0^{2\pi} f(t)e^{-int}\,\mathrm{d}t;$$

and therefore $\widehat{f'}(n) = in\widehat{f}(n)$.

□

6.1.3 The Dirichlet and the Féjer Kernels

We now define two sequences of functions that are essential to understand the convergence of Fourier Series.

Definition 6.10 (Dirichlet and Féjer Kernels)**.** *If $N \in \mathbb{N} \cup \{0\}$, define the function $D_N : \mathbb{R} \to \mathbb{C}$ by*

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}, \quad x \in \mathbb{R}.$$

*The sequence of functions $\{D_N\}_{N \in \mathbb{N}\cup\{0\}}$ is called the **Dirichlet Kernel**.*

Also, if $N \in \mathbb{N}$, and we define a new function $K_N : \mathbb{R} \to \mathbb{C}$ by the formula

$$K_N(x) := \frac{D_0(x) + D_1(x) + \cdots + D_{N-1}(x)}{N} = \frac{1}{N}\sum_{n=0}^{N-1} D_n(x) = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} e^{ikx}, \quad x \in \mathbb{R};$$

*the sequence of functions $\{K_N\}_{N \in \mathbb{N}}$ is called the **Féjer Kernel**.*

Remark 6.11. Let us make some observations regarding the kernels in Definition 6.10.

(1) Clearly D_N is a trigonometric polynomial of degree N with $D_N(0) = 2N + 1$ for all $N \in \mathbb{N} \cup \{0\}$. In addition, $D_N(x) = D_N(-x)$ for all $x \in \mathbb{R}$. Also, the Fourier coefficients are

$$\widehat{D_N}(n) = \begin{cases} 1 & \text{if } |n| \le N \\ 0 & \text{if } |n| > N. \end{cases}$$

Let us now derive the following useful formula for D_N :

$$D_N(x) = \frac{\sin\left((N + \frac{1}{2})x\right)}{\sin(\frac{x}{2})}, \quad \text{for all} \quad x \in (0, 2\pi). \tag{6.1.4}$$

Indeed, calculating the geometric sum that defines D_N, we get that

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{e^{i(N+1)x} - e^{-iNx}}{e^{ix} - 1} = \frac{e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x}}{e^{ix/2} - e^{-ix/2}} = \frac{\sin\left((N+\frac{1}{2})x\right)}{\sin(\frac{x}{2})}.$$

Also, $\frac{1}{2\pi}\int_0^{2\pi} D_N(t)\,\mathrm{d}t = 1$, because

$$\int_0^{2\pi} D_N(t)\,\mathrm{d}t = \sum_{n=-N}^{N} \int_0^{2\pi} e^{-int}\,\mathrm{d}t = \int_0^{2\pi} \mathrm{d}t = 2\pi.$$

(2) For every integrable function $f : [0, 2\pi] \to \mathbb{C}$,

$$S_N(f)(x) = \frac{1}{2\pi}\int_0^{2\pi} D_N(x-t)f(t)\,\mathrm{d}t, \quad x \in [0, 2\pi],\ N \in \mathbb{N} \cup \{0\}.$$

Indeed, it suffices to observe that

$$\int_0^{2\pi} D_N(x-t)f(t)\,\mathrm{d}t = \int_0^{2\pi} \sum_{n=-N}^{N} e^{in(x-t)}f(t)\,\mathrm{d}t = \sum_{n=-N}^{N} e^{inx} \int_0^{2\pi} e^{-int}f(t)\,\mathrm{d}t = 2\pi \sum_{n=-N}^{N} \widehat{f}(n)e^{inx}.$$

(3) Concerning the Féjer Kernel $\{K_N\}_{N\in N}$, we have that each K_N is a trigonometric polynomial of degree $N-1$, with $K_N(0) = N$. And again, $K_N(x) = K_N(-x)$ for all $x \in \mathbb{R}$. Also,

$$\begin{aligned} N \cdot K_N(x) &= \sum_{n=0}^{N-1} \sum_{k=-n}^{n} e^{ikx} \\ &= N + (N-1)\left(e^{ix} + e^{-ix}\right) + (N-2)\left(e^{2ix} + e^{-2ix}\right) + \cdots + \left(e^{iNx} + e^{-iNx}\right) \\ &= \sum_{n=-N}^{N} (N - |n|)e^{inx}, \end{aligned}$$

and consequently

$$K_N(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{inx}, \quad x \in \mathbb{R}.$$

But since the Fourier coefficients of a trigonometric polynomial are precisely the coefficients of that polynomial, we infer from the above that

$$\widehat{K_N}(n) = \left(1 - \frac{|n|}{N}\right)^{+} := \begin{cases} 1 - \frac{|n|}{N} & \text{if } |n| \le N-1 \\ 0 & \text{if } |n| \ge N. \end{cases}$$

We can derive an expression similar to (6.1.4) for K_N. Indeed, we have

$$K_N(x) = \frac{1}{N}\left(\frac{\sin\left(\frac{Nx}{2}\right)}{\sin(\frac{x}{2})}\right)^2, \quad \text{for all} \quad x \in (0, 2\pi). \tag{6.1.5}$$

To see this, we sum all the geometric series:

$$\begin{aligned} N \cdot K_N(x) &= \sum_{n=0}^{N-1} \sum_{k=-n}^{n} e^{ikx} = \sum_{n=0}^{N-1} \frac{e^{i(n+1)x} - e^{-inx}}{e^{ix} - 1} = \sum_{n=0}^{N-1} \frac{e^{i(n+1)x} - e^{-inx}}{e^{ix} - 1} \\ &= \frac{1}{e^{ix} - 1}\left(\frac{e^{i(N+1)x} - e^{it}}{e^{it} - 1} - \frac{e^{ix} - e^{-i(N-1)x}}{e^{it} - 1}\right) = \frac{\left(e^{i\frac{(N+1)}{2}x} - e^{-i\frac{(N-1)}{2}x}\right)^2}{\left(e^{ix} - 1\right)^2} \\ &= \frac{\left(e^{i\frac{Nx}{2}} - e^{-i\frac{Nx}{2}}\right)^2}{\left(e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}\right)^2} = \left(\frac{\sin\left(\frac{Nx}{2}\right)}{\sin(\frac{x}{2})}\right)^2, \end{aligned}$$

from which (6.1.5) follows.

Finally, we mention that

$$\frac{1}{2\pi}\int_0^{2\pi} K_N(t)\,\mathrm{d}t = \frac{1}{N}\sum_{n=0}^{N-1}\frac{1}{2\pi}\int_0^{2\pi} D_n(t)\,\mathrm{d}t = \frac{1}{N}\sum_{n=0}^{N-1} 1 = 1.$$

6.1.4 Approximation by Trigonometric Polynomials

The Féjer Kernel $\{K_N\}_{N\in\mathbb{N}}$ from Definition 6.10 enables us to approximate continuous functions by trigonometric polynomials. First we need to see the following essential property.

Lemma 6.12. *For every $\delta \in (0,1)$, the sequence of functions $\{K_N\}_{N\in\mathbb{N}}$ converges uniformly to 0 in the set $[-\pi,\pi]\setminus[-\delta,\delta]$.*

Proof. Recall the each $K_N(x) = K_N(-x)$ for all $x \in \mathbb{R}$. Thus, the identity (6.1.5) for points of the interval $[-\pi,\pi]$ becomes

$$K_N(x) = \frac{1}{N}\left(\frac{\sin\left(\frac{Nx}{2}\right)}{\sin(\frac{x}{2})}\right)^2, \quad \text{for all} \quad x \in (-\pi,\pi)\setminus\{0\}.$$

But then, if $0 < \delta < 1$, we can estimate this identity, for all $\delta \le |x| \le \pi$:

$$K_N(x) \le \frac{1}{N}\frac{1}{\sin^2(\frac{x}{2})} \le \frac{1}{N}\frac{1}{\sin^2(\frac{\delta}{2})},$$

and the last term goes to 0 as $N \to \infty$. This proves the lemma. □

Theorem 6.13. *Let $f : \mathbb{R} \to \mathbb{C}$ a 2π-periodic function, integrable in $[0,2\pi]$, and so that f is continuous at some $x \in [0,2\pi]$. Then*

$$\lim_{N\to\infty}\frac{1}{2\pi}\int_0^{2\pi} f(t)K_N(x-t)\,\mathrm{d}t = f(x).$$

And if f is continuous at all points $x \in [0,2\pi]$, then the convergence is uniform in $[0,2\pi]$.

Proof. Let $\varepsilon > 0$. By the continuity of f at x and Lemma 6.12, we can find $\delta \in (0,1)$ and $N_0 \in \mathbb{N}$ (depending on δ and ε) so that

$$|f(x-t) - f(x)| \le \varepsilon \ \text{ whenever } \ |t| \le \delta, \quad \text{and} \quad \sup_{\delta<|t|<\pi} K_N(t) \le \varepsilon, \ \text{ whenever } \ N \ge N_0.$$

Using this estimates, we can write, for all $N \ge N_0$,

$$\begin{aligned}
2\pi\left|\int_0^{2\pi} f(t)K_N(x-t)\,\mathrm{d}t - f(x)\right| &= \left|\int_0^{2\pi} f(t)K_N(x-t)\,\mathrm{d}t - \int_0^{2\pi} f(x)K_N(t)\,\mathrm{d}t\right| \\
&= \left|\int_0^{2\pi}(f(x-t)-f(x))K_N(t)\,\mathrm{d}t\right| = \left|\int_{-\pi}^{\pi}(f(x-t)-f(x))K_N(t)\,\mathrm{d}t\right| \\
&\le \int_{|t|\le\delta}|f(x-t)-f(x)|K_N(t)\,\mathrm{d}t + \int_{\delta<|t|<\pi}|f(x-t)-f(x)|K_N(t)\,\mathrm{d}t \\
&\le \varepsilon\int_{-\pi}^{\pi}K_N(t)\,\mathrm{d}t + \varepsilon\left(2\pi|f(x)| + \int_{-\pi}^{\pi}|f(x-t)|\,\mathrm{d}t\right) \le \left(2\pi + 2\pi|f(x)| + \int_{-\pi}^{\pi}|f(t)|\,\mathrm{d}t\right)\varepsilon.
\end{aligned}$$

The term between parentheses is a real number, and so we have proved the first part of the theorem.

Now, if f is continuous a every $x \in [0,2\pi]$, then f is actually uniformly continuous on $[0,2\pi]$. Thus, in the proof above the parameter $\delta \in (0,1)$ can be taken independent of x, and the final bound with $|f(x)|$ can be replaced with $\max\{|f(x)| : x \in [0,2\pi]\}$. The convergence is therefore uniform in $[0,2\pi]$. □

If f is continuous in $[0, 2\pi]$, then, for every $N \in \mathbb{N}$,

$$P_N(x) := \frac{1}{2\pi}\int_0^{2\pi} f(t)K_N(x-t)\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} f(t)\sum_{n=-(N-1)}^{N-1}\widehat{K_N}(n)e^{in(x-t)}\,\mathrm{d}t$$
$$= \sum_{n=-(N-1)}^{N-1}\widehat{K_N}(n)e^{inx}\frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\,\mathrm{d}t = \sum_{n=-(N-1)}^{N-1}\widehat{f}(n)\widehat{K_N}(n)e^{inx}.$$

Thus $\{P_N\}_{N\in\mathbb{N}}$ is a sequence of trigonometric polynomials which, according to Theorem 6.13, approximates uniformly f in $[0, 2\pi]$.

Moreover, we can make the identity principle (Corollary 6.8) valid for general continuous functions.

Corollary 6.14. *Let $f, g : \mathbb{R} \to \mathbb{C}$ be 2π-periodic and continuous functions with $\widehat{f}(n) = \widehat{g}(n)$ for all $n \in \mathbb{Z}$. Then $f = g$ in $\mathbb{R}$.*

Proof. By Theorem 6.13 and the subsequent comment, we know that

$$f(x) = \lim_{N\to\infty}\sum_{n=-(N-1)}^{N-1}\widehat{f}(n)\widehat{K_N}(n)e^{inx} \quad \text{and} \quad g(x) = \lim_{N\to\infty}\sum_{n=-(N-1)}^{N-1}\widehat{g}(n)\widehat{K_N}(n)e^{inx},$$

for all $x \in \mathbb{R}$. But the two limits are the same because $\widehat{f}(n) = \widehat{g}(n)$ for all $n \in \mathbb{Z}$, and we can conclude that $f = g$ in $\mathbb{R}$. □

6.1.5 The Fourier Transform

In this section we consider *integrable* functions $f : \mathbb{R} \to \mathbb{C}$ and define the Fourier Transform $\widehat{f}$ of f. By saying that f is integrable in $\mathbb{R}$, we mean that

$$\int_{-\infty}^{+\infty} |f(x)|\,\mathrm{d}x := \lim_{R\to+\infty}\int_{-R}^{R}|f(x)|\,\mathrm{d}x < \infty.$$

Definition 6.15 (Fourier Transform)**.** *For every integrable function $f : \mathbb{R} \to \mathbb{C}$,* ***Fourier Transform of*** *f is the function $\widehat{f} : \mathbb{R} \to \mathbb{C}$ given by*

$$\widehat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}\,\mathrm{d}x := \lim_{R\to+\infty}\int_{-R}^{R} f(x)e^{-ix\xi}\,\mathrm{d}x, \quad \xi \in \mathbb{R}.$$

Note that if f is integrable, then, for every $\xi \in \mathbb{R}$:

$$\left|\widehat{f}(\xi)\right| = \left|\lim_{R\to+\infty}\int_{-R}^{R} f(x)e^{-ix\xi}\,\mathrm{d}x\right| \le \lim_{R\to+\infty}\int_{-R}^{R}|f(x)||e^{-ix\xi}|\,\mathrm{d}x = \lim_{R\to+\infty}\int_{-R}^{R}|f(x)| < \infty.$$

That is, for each $\xi \in \mathbb{R}$, the function $\mathbb{R} \ni x \mapsto f(x)e^{-ix\xi}$ is integrable in $\mathbb{R}$, and so $\widehat{f}(\xi) \in \mathbb{C}$.

Let us look at a fundamental example: the Fourier Transform of the Gaussian Functions, which we calculate with the help of Cauchy Theorem; see Corollary 4.23.

Theorem 6.16. *For each $a > 0$, define $h_a : \mathbb{R} \to \mathbb{R}$ by $h_a(x) = e^{-ax^2}$, $x \in \mathbb{R}$. The Fourier transform of h_a is*

$$\widehat{h_a}(\xi) := \sqrt{\frac{\pi}{a}}\,e^{-\frac{\xi^2}{4a}}, \quad \xi \in \mathbb{R}.$$

Therefore, if $g_a(x) := \sqrt{\frac{a}{\pi}}e^{-ax^2}$, we have that $\widehat{g_a}(\xi) = e^{-\frac{\xi^2}{4a}}$ for all $\xi \in \mathbb{R}$.

Proof. For every $a > 0$, h_a is integrable in $\mathbb{R}$. Now, we fix $\xi \in \mathbb{R}$ and begin by writing

$$\widehat{h_a}(\xi) = \int_{-\infty}^{+\infty} e^{-ax^2} e^{-ix\xi}\,\mathrm{d}x = \int_{-\infty}^{+\infty} e^{-a\left(x+i\frac{\xi}{2a}\right)^2} e^{-\frac{\xi^2}{4a}}\,\mathrm{d}x = e^{-\frac{\xi^2}{4a}} \int_{-\infty}^{+\infty} e^{-a\left(x+i\frac{\xi}{2a}\right)^2}\,\mathrm{d}x. \tag{6.1.6}$$

Now we look at the last integral, that is, for every $R > 0$, $\int_{-R}^{R} e^{-a\left(x+i\frac{\xi}{2a}\right)^2}\,\mathrm{d}x$. We define the path

$$\Gamma_R := [-R, R] \star [R, R + \tfrac{i\xi}{2a}] \star [R + \tfrac{i\xi}{2a}, -R + \tfrac{i\xi}{2a}] \star [-R + \tfrac{i\xi}{2a}, -R].$$

This is closed and piecewise C^1-path, and the function $f(z) = e^{-az^2}$, $z \in \mathbb{C}$, is holomorphic in $\mathbb{C}$. For example, by the local Cauchy Theorem (Corollary 4.23), we get that

$$\begin{aligned}
0 &= \int_{\Gamma_R} f(z)\,\mathrm{d}z \\
&= \int_{-R}^{R} e^{-ax^2}\,\mathrm{d}x + \int_0^1 e^{-a\left(R+\frac{i\xi}{2a}t\right)^2} \frac{i\xi}{2a}\,\mathrm{d}t + \int_R^{-R} e^{-a\left(x+\frac{i\xi}{2a}\right)^2}\,\mathrm{d}x + \int_0^1 e^{-a\left(-R+\frac{i\xi}{2a}(1-t)\right)^2} \left(-\frac{i\xi}{2a}\right)\mathrm{d}t \\
&= \int_{-R}^{R} e^{-ax^2}\,\mathrm{d}x + \int_0^1 e^{-a\left(R+\frac{i\xi}{2a}t\right)^2} \frac{i\xi}{2a}\,\mathrm{d}t - \int_{-R}^{R} e^{-a\left(x+\frac{i\xi}{2a}\right)^2}\,\mathrm{d}x - \int_0^1 e^{-a\left(R+\frac{i\xi}{2a}(t-1)\right)^2} \frac{i\xi}{2a}\,\mathrm{d}t.
\end{aligned} \tag{6.1.7}$$

Now, we bound the second and the fourth integral, for which we observe first that if $s \in (0,1)$, and $R > 0$, then

$$\operatorname{Re}\left(\left(R + \tfrac{i\xi}{2a}s\right)^2\right) = \operatorname{Re}\left(R^2 - \tfrac{\xi^2 s^2}{4a^2} + \tfrac{\xi R s}{2a} i\right) = R^2 - \tfrac{\xi^2 s^2}{4a^2}.$$

Thus, we estimate the second integral as follows (recall that $|e^w| = e^{\operatorname{Re}(w)}$ for all $w \in \mathbb{C}$):

$$\left|\int_0^1 e^{-a\left(R+\frac{i\xi}{2a}t\right)^2} \frac{i\xi}{2a}\,\mathrm{d}t\right| \le \int_0^1 \left|e^{-a\left(R+\frac{i\xi}{2a}t\right)^2} \frac{i\xi}{2a}\right|\mathrm{d}t = \frac{|\xi|}{2a}\int_0^1 e^{-aR^2} e^{\frac{\xi^2 t^2}{4a}}\,\mathrm{d}t \le \frac{|\xi|}{2a} e^{\frac{\xi^2}{4a}} e^{-aR^2}.$$

And notice that the last term goes to 0 as $R \to +\infty$ (the numbers $\xi \in \mathbb{R}$ and $a > 0$ are constants in this argument). Similarly the fourth integral of (6.1.7) converges to 0, as $R \to \infty$. We may therefore conclude from (6.1.7) that

$$\lim_{R\to+\infty} \int_{-R}^{R} e^{-a\left(x+\frac{i\xi}{2a}\right)^2}\,\mathrm{d}x = \lim_{R\to+\infty} \int_{-R}^{R} e^{-ax^2}\,\mathrm{d}x.$$

Inserting this into (6.1.6) gives

$$\begin{aligned}
\widehat{h_a}(\xi) &= e^{-\frac{\xi^2}{4a}} \lim_{R\to+\infty} \int_{-R}^{R} e^{-a\left(x+i\frac{\xi}{2a}\right)^2}\,\mathrm{d}x = e^{-\frac{\xi^2}{4a}} \lim_{R\to+\infty} \int_{-R}^{R} e^{-ax^2}\,\mathrm{d}x \\
&= e^{-\frac{\xi^2}{4a}} \int_{-\infty}^{+\infty} e^{-ax^2}\,\mathrm{d}x = \frac{1}{\sqrt{a}} e^{-\frac{\xi^2}{4a}} \int_{-\infty}^{+\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\xi^2}{4a}};
\end{aligned}$$

where the last integral is calculated with standard Calculus II methods. □

Now we prove a version of Corollary 6.9 for the Fourier Transform.

Proposition 6.17. *Let $f : \mathbb{R} \to \mathbb{C}$ be integrable in $\mathbb{R}$, and differentiable in $\mathbb{R}$ with $\lim_{|x|\to+\infty} |f(x)| = 0$ and $f' : \mathbb{R} \to \mathbb{C}$ integrable in $\mathbb{R}$ as well. Then*

$$\widehat{f'}(\xi) = i\xi \widehat{f}(\xi), \quad \textit{for all} \quad \xi \in \mathbb{R}.$$

Proof. For every $\xi \in \mathbb{R}$ and every $R > 0$, we apply integration by parts in the Riemann integral:

$$\int_{-R}^{R} f'(x)e^{-ix\xi}\,\mathrm{d}x = \left[e^{-ix\xi}f(x)\right]_{x=-R}^{x=R} + i\xi\int_{-R}^{R} f(x)e^{-ix\xi}\,\mathrm{d}x. \tag{6.1.8}$$

And observe that

$$\lim_{R\to+\infty}\left|\left[e^{-ix\xi}f(x)\right]_{x=-R}^{x=R}\right| \le \lim_{R\to+\infty}\left(|f(R)|+|f(-R)|\right) = 0,$$

by the assumption. Thus, taking limits as $R \to +\infty$ in (6.1.8), we get that $\widehat{f'}(\xi) = i\xi\widehat{f}(\xi)$. □

Corollary 6.18. *Let $f : \mathbb{R} \to \mathbb{C}$ be of class $C^m(\mathbb{R})$, with each $f^{(k)}$ being integrable and satisfying $\lim_{|x|\to+\infty} |f^{(k)}(x)| = 0$ for all $k = 0, \dots, m$. Then*

$$\widehat{f^{(m)}}(\xi) = (i\xi)^m\widehat{f}(\xi), \quad \textit{for all} \quad \xi \in \mathbb{R}.$$

Proof. It follows from applying Proposition 6.17 m times. □

Corollary 6.18 is useful to treat certain Differential Equations involving derivatives of second order (or higher). For example, let us briefly look at the Heat Equation in the real line:

$$\frac{\partial^2 u}{\partial x^2}(x,t) - \frac{\partial u}{\partial t}(x,t) = 0, \quad x \in \mathbb{R},\, t \in (0,+\infty), \quad \text{and} \quad u(x,0) = h(x), \quad x \in \mathbb{R}.$$

one can denote $f_t(x) := u(x,t)$, for each $t > 0$, and consider the Fourier Transforms $\widehat{f_t}$ of f_t. Assuming that u is sufficiently *good* so as to satisfy

$$\widehat{\frac{\partial f_t}{\partial t}} = \frac{\partial \widehat{f_t}}{\partial t},$$

and so that each f_t satisfy the assumptions of Corollary 6.18 for $m = 2$, the Heat Equation becomes:

$$(ix)^2\widehat{f_t} - \frac{\partial \widehat{f_t}}{\partial t} = 0, \quad \text{that is,} \quad x^2\widehat{f_t} + \frac{\partial \widehat{f_t}}{\partial t} = 0,$$

and also $\widehat{f_0} = \widehat{g}$. This equation is now easier to solve, for example, we can multiply by the *integrating factor* e^{tx^2} in both sides, obtaining,

$$\frac{\partial}{\partial t}\left(e^{tx^2}\widehat{f_t}\right)(x,t) = 0.$$

This shows that then $\widehat{f_t}(x) = e^{-tx^2}\varphi(x)$, for some differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$. Letting $t = 0$, we get $\widehat{h} = \varphi$, and so we have found out that

$$\widehat{f_t}(x) = e^{-tx^2}\widehat{h}(x), \quad (x,t) \in \mathbb{R} \times [0,+\infty).$$

A *Fourier Inversion* procedure would lead us to the solution for f_t in terms of the Fourier Inverses of $\widehat{h}$ and e^{-tx^2}. The first would be simply h, and the second one would be the application of Theorem 6.16 to g_a, with $a = 1/4t$, leading us to the function

$$\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}.$$

We will elaborate more on this function and the Heat Equation in Subsection 6.2.4.

Finally, in the same spirit as in Corollary 6.5 for the Fourier Series, we obtain the following for the Fourier Transform.

Corollary 6.19. *Let $f : \mathbb{R} \to \mathbb{C}$ be integrable in $\mathbb{R}$, and differentiable in $\mathbb{R}$ with $\lim_{|x|\to+\infty} |f(x)| = 0$ and $f' : \mathbb{R} \to \mathbb{C}$ integrable in $\mathbb{R}$ as well. Then*

$$\lim_{|\xi|\to+\infty} |\widehat{f}(\xi)| = 0.$$

Proof. As we observed right after Definition 6.15, one always have that

$$|\widehat{f'}(\xi)| \leq \int_{-\infty}^{+\infty} |f'(x)|\,\mathrm{d}x := C, \quad \text{for all} \quad \xi \in \mathbb{R}.$$

By Corollary 6.17 and this inequality, one has

$$\lim_{|\xi|\to+\infty} |\widehat{f}(\xi)| = \lim_{|\xi|\to+\infty} \frac{|\widehat{f'}(\xi)|}{|\xi|} \leq \lim_{|\xi|\to+\infty} \frac{C}{|\xi|} = 0.$$

□

6.2 Differential Equations

6.2.1 The Dirichlet Problem in the Disk. The Poisson Kernel

Throughout this section we will follow the notation for the unit disk and circle.

$$\mathbb{D} := D(0,1), \quad \mathbb{T} = \partial D(0,1)$$

The **Dirichlet Problem in the disk** with boundary data a continuous function $g : \mathbb{T} \to \mathbb{R}$ consists of finding a function $u : \overline{\mathbb{D}} \to \mathbb{R}$ continuous in $\overline{\mathbb{D}}$ and of class $C^2(\mathbb{D})$ so that

$$\begin{cases} \Delta u = 0 & \text{on } \mathbb{D}, \\ u = g & \text{on } \mathbb{T}. \end{cases} \tag{6.2.1}$$

Recall that the Laplacian $\mathbb{D}(x, y) \ni\mapsto \Delta u(x, y)$ is defined by

$$\Delta u(x,y) := \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y), \quad (x,y) \in \mathbb{D}.$$

Observe that any fuction u satisfying (6.2.1) must be harmonic in $\mathbb{D}$, according to Definition 2.44.

If $g : \mathbb{R} \to \mathbb{R}$ is 2π-periodic and continuous, and the Fourier Series of g at θ

$$\sum_{n\in\mathbb{Z}} \widehat{g}(n)e^{in\theta} = \lim_{N\to+\infty} S_N(g)(\theta) = \lim_{N\to+\infty} \sum_{n=-N}^{N} \widehat{g}(n)e^{in\theta}$$

converges, then, a Theorem due to Abel (see Exercise 3.15) says that then

$$\lim_{r\to1^-} \sum_{n\in\mathbb{Z}} \widehat{g}(n)r^{|n|}e^{in\theta} := \lim_{r\to1^-} \lim_{N\to+\infty} \sum_{n=-N}^{N} \widehat{g}(n)r^{|n|}e^{in\theta} = \sum_{n\in\mathbb{Z}} \widehat{g}(n)e^{in\theta}.$$

In the same way each $S_N(g)$ can written in terms of certain integral formula involving D_N and g (see Remark 6.11(2)), it is natural to also try to write

$$\sum_{n\in\mathbb{Z}} \widehat{g}(n)r^{|n|}e^{in\theta}, \quad r \in [0,1),$$

as integral formulas involving functions of the form $\sum_{n\in\mathbb{Z}} r^{|n|}e^{in\theta}$, $r \in [0,1)$, and g. Those functions form the *Poisson Kernel.*

Definition 6.20 (Poisson Kernel)**.** *For every $r \in [0,1)$ define the function $P_r : \mathbb{R} \to [0,+\infty)$ by the formula*

$$P_r(\theta) := \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta}, \quad \theta \in \mathbb{R}.$$

*The family of functions $\{P_r\}_{r\in[0,1)}$ is called **the Poisson Kernel.***

Among other observations, in the following remark we confirm that the functions $P_r(\theta)$ take only real nonnegative values.

Remark 6.21. For every $r \in [0,1)$ and $\theta \in \mathbb{R}$, we have

$$P_r(\theta) := \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta} = 1 + \sum_{n=1}^{\infty} r^n \left(e^{in\theta} + e^{-in\theta}\right) = 1 + 2\sum_{n=1}^{\infty} r^n \cos(n\theta),$$

and in particular $P_r(\theta) \in \mathbb{R}$. But on the other hand, for every $z = re^{i\theta} \in \mathbb{D}$, we have

$$\frac{1+z}{1-z} = (1+z)\sum_{n=0}^{\infty} z^n = 1 + 2\sum_{n=1}^{\infty} z^n = 1 + 2\sum_{n=1}^{\infty} r^n e^{in\theta},$$

and looking at these two formulas, we deduce that

$$P_r(\theta) = 1 + 2\sum_{n=1}^{\infty} r^n \cos(n\theta) = \mathrm{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = \mathrm{Re}\left(\frac{1+z}{1-z}\right), \quad z = re^{i\theta} \in \mathbb{D}. \tag{6.2.2}$$

Looking at the term of the second equality we find that

$$\frac{1+re^{i\theta}}{1-re^{i\theta}} = \frac{(1+re^{i\theta})(1-re^{-i\theta})}{(1-re^{i\theta})(1-re^{-i\theta})} = \frac{1-r^2+r\left(e^{i\theta}-e^{-i\theta}\right)}{1+r^2-r\left(e^{i\theta}+e^{-i\theta}\right)} = \frac{1-r^2+2i\sin\theta}{1-2r\cos\theta+r^2}.$$

But then this equality and (6.2.2) give

$$P_r(\theta) = \mathrm{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = \mathrm{Re}\left(\frac{1-r^2+2i\sin\theta}{1-2r\cos\theta+r^2}\right) = \frac{1-r^2}{1-2r\cos\theta+r^2}, \quad \text{for all} \quad \theta \in \mathbb{R}. \tag{6.2.3}$$

Since $\cos\theta \geq -1$, we have that

$$P_r(\theta) \geq \frac{1-r^2}{1-2r+r^2} = \frac{(1-r)(1+r)}{(1-r)^2} = \frac{1+r}{1-r} \geq 0.$$

This confirms that P_r takes values only on $[0,+\infty)$.

We continue make observations on the Poisson kernel.

Proposition 6.22. *The Poisson kernel $\{P_r\}_{r\in[0,1)}$ has the following properties.*

(i) Each function $P_r : \mathbb{R} \to \mathbb{R}$ is 2π-periodic with $P_r(\theta) = P_r(-\theta)$ for all $\theta \in \mathbb{R}$.

(ii) $\frac{1}{2\pi}\int_0^{2\pi} P_r(t)\,\mathrm{d}t = 1$ for all $r \in [0,1)$.

(iii) If $0 < \delta < |\theta| \leq \pi$, then $P_r(\theta) < P_r(\delta)$ for all $r \in [0,1)$.

(iv) For every $0 < \delta \leq \pi$, one has

$$\lim_{r\to 1^-} \sup_{\delta\leq|\theta|\leq\pi} P_r(\theta) = 0.$$

Proof.

(i) This follows immediately from the expression (6.2.3) for P_r.

(ii) For every $0 \le r < 1$, the series $\sum_{n\in\mathbb{Z}} r^{|n|}e^{in\theta}$ converges uniformly on $\theta \in \mathbb{R}$, and so

$$\int_0^{2\pi} P_r(t)\,\mathrm{d}t = \int_0^{2\pi} \sum_{n\in\mathbb{Z}} r^{|n|}e^{int}\,\mathrm{d}t = \sum_{n\in\mathbb{Z}} r^{|n|} \int_0^{2\pi} e^{int}\,\mathrm{d}t = r^0 \int_0^{2\pi} \mathrm{d}t = 2\pi.$$

(iii) If $0 < \delta < |\theta| \le \pi$, then formula (6.2.3) gives

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2} < \frac{1-r^2}{1-2r\cos\delta + r^2} = P_r(\delta).$$

(iv) For every $0 < \delta \le \pi$, and $r \in [0,1)$, we can apply (iii) to obtain

$$\lim_{r\to 1^-} \sup_{\delta \le |\theta| \le \pi} P_r(\theta) \le \lim_{r\to 1^-} P_r(\delta) = \lim_{r\to 1^-} \frac{1-r^2}{1-2r\cos\delta + r^2} = 0.$$

□

We are now ready to solve the Dirichlet Problem (6.2.1) in the disk.

Theorem 6.23 (Solution to Dirichlet's Problem)**.** *Let $g : \mathbb{T} \to \mathbb{R}$ be a continuous function. Define the function $u : \overline{\mathbb{D}} \to \mathbb{R}$ in polar coordinates by the formula*

$$u(re^{i\theta}) := \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t)g(e^{it})\,\mathrm{d}t, \quad \textit{for all} \quad r \in [0,1),\ \theta \in \mathbb{R}. \tag{6.2.4}$$

and

$$u(e^{i\theta}) := g(e^{i\theta}), \quad \textit{for all} \quad \theta \in \mathbb{R}.$$

Then u is continuous in $\overline{\mathbb{D}}$ and harmonic in $\mathbb{D}$, with $u = g$ in $\mathbb{T}$, that is, u is a solution to problem (6.2.1).

Proof. We define the complex function $F : \mathbb{D} \to \mathbb{C}$ by the formula

$$F(z) := \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{w+z}{w-z}\cdot\frac{g(w)}{w}\,\mathrm{d}w, \quad \text{for all} \quad z \in \mathbb{D}. \tag{6.2.5}$$

The two-variable function $\mathbb{T}\times\mathbb{D} \ni (w,z) \mapsto \varphi(w,z) := \frac{w+z}{w-z}\cdot\frac{g(w)}{w}$ is continuous and for all $w \in \mathbb{T}$, the function $\mathbb{D} \ni z \mapsto \varphi(w,z)$ is holomorphic in $\mathbb{D}$, and $\mathbb{T}\times\mathbb{D} \ni (w,z) \mapsto \frac{\partial\varphi}{\partial z}(w,z)$ is continuous in $\mathbb{T}\times\mathbb{D}$. By the Differentiation Under the Integral Sign Theorem 4.18, we get that F is holomorphic in $\mathbb{D}$. Since we know that F is then of class $C^\infty(\mathbb{D})$ (see e.g. Theorem 4.32), we have that in particular $\mathrm{Re}(F)$ is $C^2(\mathbb{D})$, and, moreover, $\mathrm{Re}(F)$ is harmonic in $\mathbb{D}$ by Proposition 2.45. Defining $u := \mathrm{Re}(F) : \mathbb{D} \to \mathbb{R}$, we then have that $\Delta u = 0$ in $\mathbb{D}$.
Now, if $z \in \mathbb{D}$ as $z = re^{i\theta}$, with $r \in [0,1)$, $\theta \in [0,2\pi]$, and express $u(z)$ via the complex path-integral:

$$\begin{aligned} u(z) &= \mathrm{Re}\left(\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{w+z}{w-z}\cdot\frac{g(w)}{w}\,\mathrm{d}w\right) = \mathrm{Re}\left(\frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\cdot\frac{g(e^{it})}{e^{it}}\cdot ie^{it}\,\mathrm{d}t\right) \\ &= \mathrm{Re}\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{1+re^{i(\theta-t)}}{1-re^{i(\theta-t)}}\cdot g(e^{it})\,\mathrm{d}t\right) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{1+re^{i(\theta-t)}}{1-re^{i(\theta-t)}}\right) g(e^{it})\,\mathrm{d}t. \end{aligned}$$

But looking at formula (6.2.2), the above shows that

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-t)g(e^{it}), \quad \text{for all} \quad r \in [0,1),\ \theta \in \mathbb{R}. \tag{6.2.6}$$

The next step consists in proving that

$$\lim_{r\to 1^-} \sup_{\theta\in\mathbb{R}} \left| u(re^{i\theta}) - g(e^{i\theta}) \right| = 0. \tag{6.2.7}$$

To do so, first observe that the change of variable $s = \theta - t$ in the integral of (6.2.6) leads us to

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{\theta}^{\theta-2\pi} P_r(-s)g(e^{i(\theta-s)})\,\mathrm{d}s = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)g(e^{i(\theta-t)})\,\mathrm{d}t, \quad \text{for all} \quad r\in[0,1),\ \theta\in\mathbb{R}; \tag{6.2.8}$$

where we used the symmetry of $P_r(-t) = P_r(t)$ from Proposition 6.22, that the functions $\mathbb{R} \ni t \mapsto P_r(t), g(e^{i(\theta-t)})$ are 2π-periodic, and that then the integral is the same over any interval of length 2π; see formula (6.1.1).
Now, since $g : \mathbb{T} \to \mathbb{C}$ is continuous and $\mathbb{T}$ is compact, g is uniformly continuous; see Proposition 2.25. Thus, for every $\varepsilon > 0$, there exists $0 < \delta < \pi$ so that

$$s, s' \in \mathbb{R},\ |s - s'| \le \delta \implies |g(e^{is}) - g(e^{is'})| \le \varepsilon. \tag{6.2.9}$$

And by Proposition 6.22(iv) there exists $r_0 \in (0,1)$ so that

$$\sup_{\delta\le|\theta|\le\pi} P_r(\theta) \le \varepsilon, \quad \text{for all} \quad r\in[r_0,1). \tag{6.2.10}$$

For $r \in [r_0, 1)$, we use the expression (6.2.8) to write, for each $\theta \in \mathbb{R}$:

$$\begin{aligned}
\left|u(re^{i\theta}) - g(e^{i\theta})\right| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)g(e^{i(\theta-t)})\,\mathrm{d}t - g(e^{i\theta})\right| \\
&= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)g(e^{i(\theta-t)})\,\mathrm{d}t - \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)g(e^{i\theta})\,\mathrm{d}t\right| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\left(g(e^{i(\theta-t)}) - g(e^{i\theta})\right)\mathrm{d}t\right| \\
&\le \frac{1}{2\pi}\int_{\{|t|\le\delta\}} P_r(t)|g(e^{i(\theta-t)}) - g(e^{i\theta})|\,\mathrm{d}t + \frac{1}{2\pi}\int_{\{\delta<|t|\le\pi\}} P_r(t)|g(e^{i(\theta-t)}) - g(e^{i\theta})|\,\mathrm{d}t \\
&\le \frac{1}{2\pi}\int_{\{|t|\le\delta\}} P_r(t)\varepsilon\,\mathrm{d}t + 2\max_{w\in\mathbb{T}}|g(w)|\frac{1}{2\pi}\int_{\{\delta<|t|\le\pi\}} \varepsilon\,\mathrm{d}t \le \left(1 + 2\max_{w\in\mathbb{T}}|g(w)|\right)\varepsilon;
\end{aligned}$$

where in the second last inequality we used (6.2.9) for the first integral and (6.2.10) in the second one. This shows (6.2.7).

It is now natural to complete the definition of u up to the boundary of $\mathbb{D}$, by simply defining $u(e^{i\theta}) := g(e^{i\theta})$ for all $\theta \in \mathbb{R}$. Since u is already defined in $\mathbb{D}$; and of class $C^2(\mathbb{D})$, it only remains to show that u is continuous at all points of $\mathbb{T}$; where the key property will be (6.2.7). So, let us fix $\theta_0 \in [0, 2\pi)$ and let $\{z_n\}_n \subset \overline{D}$ a sequence converging to $z_0 := e^{i\theta_0}$. We can assume that $z_n \in \mathbb{D}$ for all $n \in \mathbb{N}$, as otherwise, we would have a subsequence $\{z_{n_k}\}_k$ contained in $\mathbb{T}$ converging to z_0, and then we know already that

$$\lim_{k\to\infty} u(z_{n_k}) = \lim_{k\to\infty} g(z_{n_k}) = g(z_0) = u(z_0),$$

because g is continuous. So, let $z_n = r_n e^{i\theta_n}$, where $r_n \in (0,1)$ and $\lim_{n\to\infty} r_n = 1$ and $\lim_{n\to\infty} e^{i\theta_n} = e^{i\theta_0}$. We can write, using (6.2.7),

$$\begin{aligned}
\lim_{n\to\infty}\left|u(r_n e^{i\theta_n}) - u(e^{i\theta_0})\right| &\le \lim_{n\to\infty}\left|u(r_n e^{i\theta_n}) - u(e^{i\theta_n})\right| + \lim_{n\to\infty}\left|u(e^{i\theta_n}) - u(e^{i\theta_0})\right| \\
&\le \lim_{n\to\infty}\sup_{\theta\in\mathbb{R}}\left|u(r_n e^{i\theta}) - u(e^{i\theta})\right| + \lim_{n\to\infty}\left|g(e^{i\theta_n}) - g(e^{i\theta_0})\right| \\
&= \lim_{n\to\infty}\left|g(e^{i\theta_n}) - g(e^{i\theta_0})\right| = 0.
\end{aligned}$$

□

With some extra work, one can check that formula (6.2.4) in Theorem 6.23 is the **unique** solution to the Dirichlet Problem for a fixed continuous boundary data g.

6.2.2 Fourier Series of Cosines and Sines. Even and Odd Functions

It is sometimes convenient to express the Fourier Series in terms of sines and cosines

$$a_0 + \sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

instead of the original $\sum_{n=0}^{\infty} \widehat{f}(n)e^{inx}$ from Definition 6.1.

Definition 6.24 (Cosine-Sine Fourier Series)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function, Riemann-integrable in $[-\pi, \pi]$. We define the coefficients*

$$a_0 := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t, \quad a_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,\mathrm{d}t, \quad b_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,\mathrm{d}t, \quad n \in \mathbb{Z}\setminus\{0\}. \tag{6.2.11}$$

*The **Fourier Series of Sines and Cosines of** f is the formal series*

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right), \quad \in \mathbb{R}. \tag{6.2.12}$$

Let us compare the original coefficients $\widehat{f}(n)$ with the $\{a_n\}_{n\in\mathbb{Z}}$, $\{b_n\}_{n\in\mathbb{Z}\setminus\{0\}}$, as well as both Fourier Series.

Remark 6.25. Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function, Riemann-integrable in $[-\pi, \pi]$. Let $\{a_n\}_{n\in\mathbb{Z}}$, $\{b_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ be as in (6.2.11) and $\{\widehat{f}(n)\}_{n\in\mathbb{Z}}$ as in 6.1.2. We have that

$$a_{-n} = a_n, \quad b_{-n} = -b_n, \quad \text{for all} \quad n \in \mathbb{Z}\setminus\{0\}.$$

We have that

$$\frac{a_n - ib_n}{2} = \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(t)\cos(nt)\,\mathrm{d}t - i\int_{-\pi}^{\pi} f(t)\sin(nt)\,\mathrm{d}t\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,\mathrm{d}t = \widehat{f}(n),$$

for all $n \in \mathbb{Z}\setminus\{0\}$. In the last equality, we used the integral of 2π-periodic functions in $\mathbb{R}$ is the same over any interval of length 2π; see (6.1.1). Now, the series in (6.2.12) can be written as

$$\begin{aligned}
a_0 &+ \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right) \\
&= a_0 + \sum_{n=1}^{\infty}\left[a_n\left(\frac{e^{inx}+e^{-inx}}{2}\right) + b_n\left(\frac{e^{inx}-e^{-inx}}{2i}\right)\right] \\
&= a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n - ib_n}{2}\cdot e^{inx} + \frac{a_n + ib_n}{2}\cdot e^{-inx}\right) \\
&= a_0 + \sum_{n=1}^{\infty}\left(\widehat{f}(n)e^{inx} + \widehat{f}(-n)e^{-inx}\right) = \sum_{n=0}^{\infty}\widehat{f}(n)e^{inx},
\end{aligned}$$

for every $x \in \mathbb{R}$.

Now, the Fourier Series of even (resp. odd) functions contain only cosines (resp. sines).

Proposition 6.26 (Fourier Series of Even or Odd Functions)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function, Riemann-integrable in $[-\pi, \pi]$. Let $\{a_n\}_{n\in\mathbb{Z}}$, $\{b_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ be as in* (6.2.11)*. The following holds.*

(i) If f is even, that is $f(x) = f(-x)$ for all $x \in \mathbb{R}$, then $b_n = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$,

$$a_0 = \frac{1}{\pi} \int_0^\pi f(t)\,\mathrm{d}t, \quad a_n = \frac{2}{\pi} \int_0^\pi f(t) \cos(nt)\,\mathrm{d}t, \quad \textit{for all} \quad n \in \mathbb{Z} \setminus \{0\},$$

and the Fourier Series (6.2.12) *becomes*

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx), \quad x \in \mathbb{R}.$$

(ii) If f is odd, that is $f(-x) = -f(-x)$ for all $x \in \mathbb{R}$, then $a_n = 0$ for all $n \in \mathbb{Z}$,

$$b_n = \frac{2}{\pi} \int_0^\pi f(t) \sin(nt)\,\mathrm{d}t, \quad \textit{for all} \quad n \in \mathbb{Z} \setminus \{0\},$$

and the Fourier Series (6.2.12) *becomes*

$$\sum_{n=1}^{\infty} a_n \sin(nx), \quad x \in \mathbb{R}.$$

Proof.

(i) Since $t \mapsto \cos(t)$ and $t \mapsto f(t)$ are even functions, we clearly have

$$a_n := \frac{1}{\pi} \int_{-\pi}^\pi f(t) \cos(nt)\,\mathrm{d}t = \frac{2}{\pi} \int_0^\pi f(t) \cos(nt)\,\mathrm{d}t, \quad n \in \mathbb{Z},$$

and

$$b_n := \frac{1}{\pi} \int_{-\pi}^\pi f(t) \sin(nt)\,\mathrm{d}t = \frac{1}{\pi} \left(\int_{-\pi}^0 f(t) \sin(nt)\,\mathrm{d}t + \int_0^\pi f(t) \sin(nt)\,\mathrm{d}t \right) = 0,$$

for all $n \in \mathbb{Z} \setminus \{0\}$.

(ii) The proof is very similar to that of (i). □

Remark 6.27. Regarding the convergence of the Cosine–Sine series we obtained in Proposition 6.26, we observe the following. Let $f : [0, \pi] \to \mathbb{R}$ be a continuous function, which is Riemann-integral in $[0, \pi]$. Let us consider two possibilities.

(1) We are interested in expressing or approximating f by its Fourier Series **of Cosines** in $[0, \pi]$. Then we consider the **even** extension $f : [-\pi, \pi] \to \mathbb{R}$ of f, that is $f(x) := f(-x)$ for all $x \in [0, \pi]$. Then f is continuous in $[-\pi, \pi]$, and moreover we can consider the 2π-periodic extension $f : \mathbb{R} \to \mathbb{R}$ of f, which is **continuous in** $\mathbb{R}$ as well. By Proposition 6.26, we get

$$S(f)(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx), \quad x \in \mathbb{R}, \quad \text{where}$$
$$a_0 := \frac{1}{\pi} \int_0^\pi f(t)\,\mathrm{d}t, \quad a_n := \frac{2}{\pi} \int_0^\pi f(t) \cos(nt)\,\mathrm{d}t, \quad n \in \mathbb{N}.$$

Assume additionally that f **satisfies the assumptions of Theorem 6.6**. This is true for example when f satisfies a Lipschitz condition in $[0, \pi]$, or when $f'(x)$ exists for all $x \in [0, \pi]$ (understanding the one-side derivatives for $x = 0$ or $x = \pi$).

Then, by the extension via symmetry and 2π-periodicity of f, the same properties hold for $f : \mathbb{R} \to \mathbb{R}$ on every interval of length 2π. Applying Theorem 6.6, we get that $S(f)(x)$ converges to $f(x)$ for all $x \in \mathbb{R}$, and in particular we have

$$f(x) = S(f)(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx), \quad \text{for all} \quad x \in [0, \pi].$$

(2) We are interested in expressing or approximating f by its Fourier Series **of Sines** in $[0,\pi]$. Here we need to be slightly more careful. We consider the **odd** extension $f:[-\pi,\pi]\to\mathbb{R}$ of f, that is $f(x):=-f(-x)$ for all $x\in[0,\pi]$. Then f is continuous in $[-\pi,\pi]\setminus\{0\}$, but not necessarily at $x=0$. Moreover, the 2π-periodic extension $f:\mathbb{R}\to\mathbb{R}$ of f, which is **continuous in** $\mathbb{R}\setminus\{k\pi\,:\,k\in\mathbb{Z}\}$, which possible discontinuities at $k\pi$, with $k\in\mathbb{Z}$. By Proposition 6.26, we get that

$$S(f)(x)=\sum_{n=1}^{\infty}b_n\sin(nx),\quad x\in\mathbb{R},\quad b_n:=\frac{2}{\pi}\int_0^{\pi}f(t)\sin(nt)\,\mathrm{d}t,\quad n\in\mathbb{N}.$$

Assume additionally that f **satisfies the assumptions of Theorem 6.6** in $[0,\pi]$. This is true for example when f satisfies a Lipschitz condition in $[0,\pi]$, or when $f'(x)$ exists for all $x\in[0,\pi]$ (understanding the one-side derivatives for $x=0$ or $x=\pi$). Applying Theorem 6.6, we get that $S(f)(x)$ converges to $f(x)$ for all $x\in\mathbb{R}\setminus\{k\pi\,:\,k\in\mathbb{Z}\}$, and in particular we have

$$f(x)=S(f)(x)=\sum_{n=1}^{\infty}b_n\sin(nx),\quad\text{for all}\quad x\in(0,\pi).$$

And **assuming further that for** $x_0=0$ **or** $x_0=\pi$, **one has** $f(x_0)=0$, then, by the definition of the extension of f to all of $\mathbb{R}$, Theorem 6.6 applies at x_0, whence

$$f(x_0)=S(f)(x_0)=\sum_{n=1}^{\infty}b_n\sin(nx_0).$$

6.2.3 The Heat Equation in $[0,\pi]$. Separated Variables and Superposition

Let $f:[0,\pi]\to\mathbb{R}$ be continuous. The **Heat Equation in the interval** $[0,\pi]$ with initial data f is

$$\text{(HEI)}\equiv\begin{cases}\dfrac{\partial^2u}{\partial x^2}(x,t)=\dfrac{\partial u}{\partial t}(x,t); & \text{if}\quad (x,t)\in(0,\pi)\times(0,+\infty)\\ u(0,t)=u(\pi,t)=0 & \text{if}\quad t\in[0,+\infty)\\ u(x,0)=f(x) & \text{if}\quad x\in[0,\pi].\end{cases}\tag{6.2.13}$$

The solutions $u:[0,\pi]\times[0,+\infty)\to\mathbb{R}$ we are interested in should be continuous in $[0,\pi]\times[0,+\infty)$, and so $u(\cdot,t)\in C^2((0,\pi))$ for all $t>0$ and $u(x,\cdot)\in C^1((0,+\infty))$ for all $x\in(0,\pi)$. Here we mean

$$x\mapsto u(\cdot,t)(x):=u(x,t),\quad t\mapsto u(x,\cdot)(t):=u(x,t).$$

We remark that the problem can be formulated in any interval $[0,L]$, and with first equation $u_{xx}=\delta u_t$ for some $\delta>0$, but this is equivalent to (6.2.13) after an appropriate rescalling.

Our first attempt to solve (6.2.13) is to begin with potential **Solutions of Separated Variables.** This is a standard technique to find candidates for solutions to many Differential Equations, which consists in defining

$$u(x,t)=\alpha(x)\cdot\beta(t),\quad (x,t)\in(0,\pi)\times(0,+\infty),\tag{6.2.14}$$

of *separated variables,* where $\alpha:[0,\pi]\to\mathbb{R}$ is of class $C^2((0,\pi))$ and continuous in $[0,\pi]$ and $\alpha(0)=\alpha(\pi)$, and $\beta\in C^1((0,+\infty)$. We also assume that α and β are not identically zero in their domain of the definition. Observe that the function u in (6.2.14) satisfies the first equation of (6.2.13) if and only if

$$\alpha''(x)\beta(t)=\alpha(x)\beta'(t),\quad (x,t)\in(0,\pi)\times(0,+\infty).$$

This is satisfied when

$$\alpha''(x)=\lambda\alpha(x),\quad\text{and}\quad\beta'(t)=\lambda\beta(t)\quad (x,t)\in(0,\pi)\times(0,+\infty),\tag{6.2.15}$$

for any constant $\lambda \in \mathbb{R}$. The solutions on the first equation of (6.2.15) depend on the value of λ. Let us examine what the outcome would be when $\lambda \geq 0$. Then (6.2.15) has solutions for α :

$$\alpha(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}, \quad x \in [0, \pi],$$

for constants $A, B \in \mathbb{R}$. But recall that we must have $\alpha(0) = \alpha(\pi) = 0$, and therefore

$$\begin{aligned} 0 &= \alpha(0) = A + B \implies B = -A \\ 0 &= \alpha(\pi) = Ae^{\sqrt{\lambda}\pi} - Ae^{-\sqrt{\lambda}\pi} \implies \text{either } A = 0 \text{ or } \lambda = 0. \end{aligned}$$

And both of the last clauses lead us to $\alpha \equiv 0$ in $[0, \pi]$, which we are discarding. Thus we are only looking at the case $\lambda > 0$ in (6.2.15). In that case, there are constants $A, B \in \mathbb{R}$ for which

$$\alpha(x) = A\cos\left(\sqrt{-\lambda}\,x\right) + B\sin\left(\sqrt{-\lambda}\,x\right), \quad x \in [0, \pi].$$

But using that $\alpha(0) = \alpha(\pi) = 0$, we obtain that

$$\begin{aligned} 0 &= \alpha(0) = A \implies A = 0 \\ 0 &= \alpha(\pi) = B\sin\left(\sqrt{-\lambda}\,\pi\right) \implies \sqrt{-\lambda} \in \mathbb{N}. \end{aligned}$$

Therefore, we have that $\lambda = -n^2$, $n \in \mathbb{N}$. As concerns the solutions of (6.2.15) for β, we have that

$$\beta(t) = Ce^{-n^2 t}, \quad t \in (0, +\infty).$$

Thus, for every $n \in \mathbb{N}$ and every constant $A_n \in \mathbb{R}$ we obtain a function u_n of the form

$$\alpha_n(x, t) = A_n e^{-n^2 t}\sin(nx), \quad (x, t) \in (0, \pi) \times (0, +\infty), \tag{6.2.16}$$

which satisfies the first two line equations of (6.2.13).

However, clearly α_n of (6.2.16) does not necessarily satisfies the initial condition $\alpha_n(x, 0) = f(x)$ for all $x \in [0, \pi]$. To find a solution that satisfies simultaneously all the conditions of (6.2.13), we note the following. The first two equation lines of (6.2.13) are satisfied for the finite sum

$$\sum_{n=1}^{N} u_n(x, t), \quad N \in \mathbb{N},$$

of the functions of (6.2.16). This sum of solutions is typically called **Superposition of Solutions.** Therefore, the same holds for the infinite sum $\{u_n\}_{n=1}^{\infty}$, provided the series converges. In other words, under the suitable conditions in the sequence $\{A_n\}_{n\in\mathbb{N}}$, the series of functions

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-n^2 t}\sin(nx), \quad (x, t) \in [0, \pi] \times (0, +\infty)$$

seems to solve the first two equations of (6.2.16). In order to for the series to have a chance to converges to $f(x)$, as $t \to 0^+$, and for every $x \in [0, \pi]$, we can take $\{A_n\}_{n=1}^{\infty}$ as the Fourier Sine Coefficients $\{b_n\}_{n=1}^{\infty}$ of the function f, as in Remark 6.27. Assuming that the Fourier Series of f converges to f in $[0, \pi]$, and that the $\{b_n\}_{n=1}^{\infty}$ are *summable*:

$$\sum_{n=1}^{\infty} |b_n| < \infty,$$

the series will converge to f as $t \to 0^+$. Let us make this more formal in the next theorem.

Theorem 6.28. *Let $f : [0,\pi] \to \mathbb{R}$ be a function with $f(0) = f(\pi) = 0$ and so that the Fourier Sine Coefficients*

$$b_n = \frac{2}{\pi}\int_0^\pi f(t)\sin(nt)\,\mathrm{d}t, \quad n \in \mathbb{N},$$

satisfy

$$\sum_{n=1}^{\infty} |b_n| < \infty, \quad \textit{and} \quad f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \quad \textit{for all} \quad x \in [0,\pi].$$

Then the formula

$$u(x,t) := \sum_{n=1}^{\infty} b_n e^{-n^2 t}\sin(nx), \quad (x,t) \in [0,\pi] \times (0,+\infty) \tag{6.2.17}$$

defines a continuous function in $[0,\pi) \times [0,+\infty)$*, with* $u(\cdot,t) \in C^2((0,\pi))$ *for all* $t > 0$*;* $u(x,\cdot) \in C^1((0,+\infty))$ *for all* $x \in (0,\pi)$*, and* u *is a solution of the Heat Equation* (6.2.13) *in* $[0,\pi]$*.*

Proof. Observe that for each $(x_0,t_0) \in (0,\pi) \times (0,+\infty)$, the series (6.2.17) converges uniformly in points $(x,t) \in (x_0 - \varepsilon, x_0 + \varepsilon) \times (t_0 - \varepsilon, t_0 + \varepsilon)$ for $0 < \varepsilon < t_0$, since

$$\sum_{n=1}^{\infty} |b_n e^{-n^2 t}\sin(nx)| \le \sum_{n=1}^{\infty} |b_n| e^{-n^2(t_0-\varepsilon)} < \infty;$$

and Weiertrass M-test (Theorem 3.9) applies. Furthermore, the series of derivatives with respect to t also converge uniformly in the mentioned set, because

$$\sum_{n=1}^{\infty} |b_n(-n^2)e^{-n^2 t}\sin(nx)| \le \sum_{n=1}^{\infty} |b_n| n^2 e^{-n^2(t_0-\varepsilon)} < \infty.$$

And the series of derivatives of order 2 with respect to x converges uniformly in the mentioned set, by the exact same argument and estimate. We have shown that the series (6.2.17) and the corresponding series of derivatives converge locally uniformly in $(x_0,t_0) \in (0,\pi)\times(0,+\infty)$, implying that $u(\cdot,t) \in C^2((0,\pi))$ for all $t > 0$; and $u(x,\cdot) \in C^1((0,+\infty))$ for all $x \in (0,\pi)$. As we saw in the previous discussion, formula (6.2.17) gives a solution for the first two equations of (6.2.13).

It only remains to show that

$$\lim_{t\to 0^+} u(x,t) = f(x) \tag{6.2.18}$$

uniformly on $x \in [0,\pi]$. To see this, we use the assumption that f coincides with its Fourier Series of Sines in $[0,\pi]$, and estimate

$$\sup_{x\in[0,\pi]} |u(x,t) - f(x)| = \sup_{x\in[0,\pi]} \left| \sum_{n=1}^{\infty} b_n e^{-n^2 t}\sin(nx) - \sum_{n=1}^{\infty} b_n \sin(nx) \right| \le \sum_{n=1}^{\infty} |b_n| \big|1 - e^{-n^2 t}\big|. \tag{6.2.19}$$

Since $\sum_{n=1}^{\infty} |b_n| < \infty$, given $\varepsilon > 0$ we can find $N \in \mathbb{N}$ so that

$$\sum_{n=N+1}^{\infty} 2|b_n| \le \frac{\varepsilon}{2}.$$

Moreover, it is clear that

$$\lim_{t\to 0^+} \sum_{n=1}^{N} |b_n| \big|1 - e^{-n^2 t}\big| = 0,$$

and so there exists $\delta > 0$ with the property that

$$\sum_{n=1}^{N} |b_n| \big|1 - e^{-n^2 t}\big| \le \frac{\varepsilon}{2}, \quad \text{for all} \quad 0 \le t \le \delta.$$

Therefore, if $t \in [0, \delta]$, one has that

$$\sum_{n=1}^{\infty} |b_n|\left|1 - e^{-n^2 t}\right| = \sum_{n=1}^{N} |b_n|\left|1 - e^{-n^2 t}\right| + \sum_{n=N+1}^{\infty} |b_n|\left|1 - e^{-n^2 t}\right|$$
$$\leq \sum_{n=1}^{N} |b_n|\left|1 - e^{-n^2 t}\right| + \sum_{n=N+1}^{\infty} 2|b_n| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, the limit of the last term of (6.2.19) equals 0, and therefore the claim (6.2.18) holds □

6.2.4 The Heat Equation in $\mathbb{R}$. The Heat Kernel

We now consider the **Heat Equation in** $\mathbb{R}$ with initial data a continuous function $f : \mathbb{R} \to \mathbb{R}$:

$$(\text{HER}) \equiv \begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,t) = \dfrac{\partial u}{\partial t}(x,t); & \text{if} \quad (x,t) \in \mathbb{R} \times (0, +\infty) \\ u(x,0) = f(x) & \text{if} \quad x \in \mathbb{R}. \end{cases} \tag{6.2.20}$$

The solutions $u : R \times [0, +\infty) \to \mathbb{R}$ we are interested in should be continuous in $\mathbb{R} \times [0, +\infty)$, and so $u(\cdot, t) \in C^2((0, \pi))$ for all $t > 0$ and $u(x, \cdot) \in C^1((0, +\infty))$ for all $x \in \mathbb{R}$. As we found out in the discussion right after Corollary 6.18, the solution should somehow involve a *Fourier Inverse* in the variable x, of the function

$$\mathbb{R} \times (0, +\infty) \ni (\xi, t) \mapsto e^{-t\xi^2}.$$

If we are given a Fourier Transform $\widehat{h}$ of some integrable function h, the **Fourier Inverse of** $\widehat{h}$ is formally defined by

$$\mathcal{F}^{-1}\big(\widehat{h}\big)(x) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widehat{h}(\xi) e^{ix\xi}\, \mathrm{d}\xi, \quad x \in \mathbb{R}^n.$$

It is not true in general that $\mathcal{F}^{-1}\big(\widehat{h}\big) = h$. But looking at Theorem 6.16, and letting $a = 1/(4t)$, one has that the Fourier inverse of $\xi \mapsto e^{-t\xi^2}$ (meaning the function whose Fourier Transform is that one) is the function

$$\mathbb{R} \ni x \mapsto \frac{1}{4\pi t} e^{-\frac{x^2}{4t}}.$$

These functions parametrized by $t > 0$, define the *Heat Kernel*, which is the basis for the solution to the Heat Equation (6.2.20).

Definition 6.29 (The Heat Kernel)**.** *For each $t > 0$, define the function $H_t : \mathbb{R} \to \mathbb{R}$ by the formula*

$$H_t(x) = \frac{1}{4\pi t} e^{-\frac{x^2}{4t}}, \quad \textit{for all} \quad x \in \mathbb{R}.$$

The collection of all functions $\{H_t\}_{t>0}$ is called ***the Heat Kernel in*** $\mathbb{R}$.

And the following theorem due to Weierstrass solves problem (6.2.20) for most of the interesting cases of f. The necessary tools to give a rigorous proof of the theorem are out of the scope of this course.

Theorem 6.30. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function in $\mathbb{R}$. For each $t > 0$ define*

$$f^t(x) := \int_{-\infty}^{+\infty} H_t(x-y) f(y)\, \mathrm{d}y = \frac{1}{4\pi t} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4t}} f(y)\, \mathrm{d}y, \quad x \in \mathbb{R}.$$

Then $f^t \in C^\infty(\mathbb{R})$ and $\lim\limits_{t \to 0^+} f^t = f$ uniformly on each bounded subset of $\mathbb{R}$. If, in addition, $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous, then the convergence is uniform in $\mathbb{R}$.

Moreover, defining

$$u(x,t) := f^t(x), \quad \textit{for all} \quad \textit{for all} \quad (x,t) \in \mathbb{R} \times (0, +\infty), \quad \textit{and} \quad u(0,x) := f(x), \quad \textit{for all} \quad x \in \mathbb{R},$$

we have that u is continuous in $\mathbb{R} \times [0, +\infty)$, with $u(\cdot, t) \in C^\infty(\mathbb{R})$ for all $t > 0$, $u(x, \cdot) \in C^1((0, +\infty))$ for all $x \in \mathbb{R}$, and u solves the equation (6.2.20).

6.2.5 The Wave Equation. D'Alembert's Solution. Separation and Superposition

We consider now the Wave Equation in the interval $[0,\pi]$ with initial data a continuous $f:[0,\pi]\to\mathbb{R}$ with $f(0)=f(\pi)=0$. The equation reads as follows:

$$(\text{WEI}) \equiv \begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,t) = \dfrac{\partial^2 u}{\partial t}(x,t); & \text{if} \quad (x,t)\in(0,\pi)\times(0,+\infty) \\ u(0,t)=u(\pi,t)=0 & \text{if} \quad t\in[0,+\infty) \\ \dfrac{\partial u}{\partial t}(x,0)=0 & \text{if} \quad x\in[0,\pi] \\ u(x,0)=f(x) & \text{if} \quad x\in[0,\pi]. \end{cases} \tag{6.2.21}$$

As in the Heat Equation (6.2.13), the problem can be formulated in $[0,L]$ for every $L>0$ and with $u_{xx}=\delta u_{tt}$ for some $\delta>0$, which after an appropriate rescalling, it is equivalent to (6.2.21).

If the function f has sufficient regularity, say C^2 regularity of an even extension $f:\mathbb{R}\to\mathbb{R}$ to all of $\mathbb{R}$, **a solution due to D'Alembert is**

$$u(x,t) := \frac{f(x+t)+f(x-t)}{2}, \quad (x,t)\in[0,\pi]\times[0,+\infty).$$

In this notes, we are finding potential solutions of **separated variables** (for at least the first three equations of (6.2.21)), whose superposition would converge to D'Alembert's solution under suitable assumptions in f.

So, in the same spirit as for the Heat Equation in $\mathbb{R}$, we write

$$u(x,t)=\alpha(x)\beta(t), \quad (x,t)\in[0,\pi]\times[0,+\infty)$$

for α continuous in $[0,\pi]$, of class $C^2((0,\pi))$, with $\alpha(0)=\alpha(\pi)=0$; and β continuous in $[0,+\infty)$, of class $C^1((0,+\infty))$, with $\beta'(0)=0$. The first equation of (6.2.21) for u is equivalent to

$$\alpha''(x)=\lambda\alpha(x), \quad x\in(0,\pi), \quad \beta''(t)=\lambda\beta(t), \quad t\in(0,+\infty),$$

for some constant $\lambda\in\mathbb{R}$. We learnt from the separation of variables for the Heat Equation in $[0,\pi]$ (see the discussion after (6.2.13), and note that the current α satisfies the same as that α), that $\lambda=-n^2$ for $n\in\mathbb{N}$, and α has the form

$$\alpha(x)=C\sin\big(nx\big), \quad x\in[0,\pi],$$

for $C\in\mathbb{R}$ constant. And the solution of the differential equation $\beta''(t)=\lambda\beta(t)$ has the form

$$\beta(t):=A\sin\big(nt\big)+B\cos(nt), \quad t>0,$$

where $A,B\in\mathbb{R}$ are constants. Now, differentiating in t and using that $\beta'(0)=0$, we get that

$$0=\beta'(0)=An\cos(n\cdot 0)-Bn\sin(n\cdot 0)=A \implies A=0.$$

Therefore $\beta(t)=B\cos(nt)$ for all $t>0$. Thus, for each $n\in\mathbb{N}$, and $D_n\in\mathbb{R}$, the function

$$u_n(x,t)=D_n\sin\big(nx\big)\cos(nt), \quad (x,t)\in[0,\pi]\times[0,+\infty), \tag{6.2.22}$$

solves the first three equations of (6.2.21) and has the desired regularity. The formula for u_n in (6.2.22) provides a solution **of separated variables**, but u_n can be rewritten as

$$u_n(x,t)=D_n\left[\sin\big(n(x+t)\big)+\sin\big(n(x-t)\big)\right], \quad (x,t)\in[0,\pi]\times[0,+\infty), \tag{6.2.23}$$

if we use the well-known formula

$$\sin\theta\cos\gamma=\frac{\sin(\theta+\gamma)+\sin(\theta-\gamma)}{2}.$$

The **superposition** of these solutions $\{u_n\}_{n\in\mathbb{N}}$ is the series of functions

$$u(x,t) = \sum_{n=1}^{\infty} D_n \sin\big(nx\big)\cos(nt) = \sum_{n=1}^{\infty} D_n \cdot \frac{\sin\big(n(x+t)\big) + \sin\big(n(x-t)\big)}{2}, \quad (x,t) \in [0,\pi]\times[0,+\infty), \tag{6.2.24}$$

The series converges locally uniformly in $(x,t) \in [0,\pi] \times [0,+\infty)$ if $\{D_n\}_{n\in\mathbb{N}}$ are chosen so that $\sum_{n=1}^{\infty} |D_n| < \infty$. However, unlike for the Heat Equation, further conditions in $\{D_n\}_{n\in\mathbb{N}}$ are needed if we want the series of derivatives (of order 1 and 2) to converge locally uniformly, namely, that

$$\sum_{n=1}^{\infty} n^2 |D_n| < \infty.$$

On the other hand, if we want the fourth equation to be satisfied for u as in (6.2.24), then we choose $D_n := b_n$, where $\{b_n\}_{n\in\mathbb{N}}$ are the Fourier Sine Coefficients of f :

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin(nt)\,\mathrm{d}t, \quad n \in \mathbb{N}.$$

And if we know that f agrees with its Fourier Series at all points, then

$$u(x,t) = \sum_{n=1}^{\infty} b_n \cdot \frac{\sin\big(n(x+t)\big) + \sin\big(n(x-t)\big)}{2} = \frac{f(x+t)+f(x-t)}{2}, \quad (x,t) \in [0,\pi] \times [0,+\infty),$$

which leads us back to D'Alembert's solution.

6.3 Exercises

Exercise 6.1. *For the following functions: consider their 2π-periodic extensions to all of $\mathbb{R}$, and find their Fourier Coefficients, their Fourier Sums, and their Fourier Series. Then, for all $x \in [0,2\pi)$, study whether or not the Fourier Series $Sf(x)$ converges to $f(x)$.*

(i) The function $f : [0,2\pi) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,\pi] \\ -1 & \text{if } x \in [\pi, 2\pi). \end{cases}$$

(ii) The function $f(x) = |x-\pi|$ for all $x \in [0,2\pi]$.

(iii) The function $f(x) = x + \sin x + \cos x$, for all $x \in [0,2\pi)$.

Suggestion: To study the convergence, look at Theorem 6.6.

Exercise 6.2. *Verify the following properties involving Fourier coefficients, for a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$, Riemann-integrable in $[0,2\pi]$.*

(i) If $g : \mathbb{R} \to \mathbb{C}$ is defined by $g(x) = f(-x)$, $x \in [0,2\pi]$, then $\widehat{g}(n) = \widehat{f}(-n)$ for all $n \in \mathbb{Z}$.

(ii) If $h : \mathbb{R} \to \mathbb{C}$ is defined by $h(x) = \overline{f(x)}$, $x \in [0,2\pi]$, then $\widehat{h}(n) = \overline{\widehat{f}(-n)}$ for all $n \in \mathbb{Z}$.

(iii) Deduce from (ii) that if $f : \mathbb{R} \to \mathbb{R}$ is real-valued, then $\overline{\widehat{f}(n)} = \widehat{f}(-n)$ for all $n \in \mathbb{Z}$, and therefore the Fourier Series $S(f)(x)$ of f at $x \in \mathbb{R}$ (if converges), only takes real values.

(iv) If $y \in \mathbb{R}$, and define $f_y : \mathbb{R} \to \mathbb{C}$ by the formula $f_y(x) = f(x-y)$, $x \in \mathbb{R}$. Then, $\widehat{f_y}(n) = e^{-iyn}\widehat{f}(n)$ for all $n \in \mathbb{Z}$.

(v) *If* $m \in \mathbb{Z}$, *and define* $\varphi_m : \mathbb{R} \to \mathbb{C}$ *by the formula* $\varphi_m(x) = e^{imx} f(x)$, $x \in \mathbb{R}$. *Then,* $\widehat{\varphi_m}(n) = \widehat{f}(n-m)$ *for all* $n \in \mathbb{Z}$.

Suggestion: If you need it, use the identity (6.1.1).

Exercise 6.3. *For the Dirichlet Kernel* $\{D_N\}_{N\in\mathbb{N}\cup\{0\}}$ *from the Definition 6.10, prove the following.*

(i) *There exists a constant* $C > 0$ *so that*

$$\int_{-\pi}^{\pi} |D_N(x)|\, \mathrm{d}x \leq C \log N, \quad \textit{for all} \quad N \in \mathbb{N},\, N \geq 2.$$

(ii) *There exists a constant* $C' > 0$ *so that*

$$\int_{-\pi}^{\pi} |D_N(x)|\, \mathrm{d}x \geq C' \log N, \quad \textit{for all} \quad N \in \mathbb{N}.$$

Exercise 6.4. *Calculate the Fourier Transform of the following functions* $f : \mathbb{R} \to \mathbb{C}$:

(i) *For* $a > 0$,

$$f(x) = \begin{cases} 1 & \textit{if } |x| \leq a \\ 0 & \textit{if } |x| > a. \end{cases}$$

(ii) *For* $a > 0$, $f(x) = e^{-ax} \mathcal{X}_{[0,+\infty)}(x)$, $x \in \mathbb{R}$, *where*

$$\mathcal{X}_{[0,+\infty)}(x) = \begin{cases} 1 & \textit{if } x \geq 0 \\ 0 & \textit{if } x < 0. \end{cases}$$

(iii) *For* $a > 0$, $f(x) = e^{-a|x|}$ *for all* $x \in \mathbb{R}$.

(iv) *For every* $k \in \mathbb{N}$, $f(x) = \frac{x^k}{k!} e^{-ax} \mathcal{X}_{[0,+\infty)}(x)$, *for all* $x \in \mathbb{R}$.

Exercise 6.5. *For a function* $f : [0,\pi] \to \mathbb{R}$, *denote the Fourier coefficients*

$$a_0 := \frac{1}{\pi} \int_0^{\pi} f(t)\, \mathrm{d}t, \quad a_n := \frac{2}{\pi} \int_0^{\pi} f(t)\cos(nt)\, \mathrm{d}t, \quad b_n := \frac{2}{\pi} \int_0^{\pi} f(t)\sin(nt)\, \mathrm{d}t, \quad n \in \mathbb{N}.$$

Taking into account Remark 6.27, solve the following tasks.

(i) *Let* $f(x) = \frac{\pi}{2} - \left|x - \frac{\pi}{2}\right|$, $x \in [0,\pi]$.

  (a) *Compute* b_n *for all* $n \in \mathbb{N}$. *Write down the Fourier Series of Sines at each* $x \in [0,\pi]$.

  (b) *Show that* f *is Lipschitz in* $[0,\pi]$, *and then deduce that the series from (a) converges to* $f(x)$ *for all* $x \in [0,\pi]$.

  (c) *Use (b) to deduce that*

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

(ii) *Let* $f(x) = x^2 - \pi x$, $x \in [0,\pi]$.

  (a) *Compute* b_n *for all* $n \in \mathbb{N}$. *Write down the Fourier Series of Sines at each* $x \in [0,\pi]$.

  (b) *Show that* f *is Lipschitz in* $[0,\pi]$, *and then deduce that the series from (a) converges to* $f(x)$ *for all* $x \in [0,\pi]$.

(c) *Use (b), evaluating at* $x = \pi/2$ *to deduce that*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.$$

(iii) *Let* $f(x) = (x - \frac{\pi}{2})^2$, $x \in [0, \pi]$.

(a) *Compute* a_n *for all* $n \in \mathbb{N} \cup \{0\}$. *Write down the Fourier Series of Cosines at each* $x \in [0, \pi]$.

(b) *Show that the the series from (a) converges to* $f(x)$ *for all* $x \in [0, \pi]$.

(c) *Use (b) to deduce that*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(iv) *Let* $f(x) = x$, $x \in [0, \pi]$.

(a) *Compute* a_n *for all* $n \in \mathbb{N} \cup \{0\}$. *Write down the Fourier Series of Cosines at each* $x \in [0, \pi]$.

(b) *Show that the the series from (a) converges to* $f(x)$ *for all* $x \in [0, \pi]$.

(c) *Use (b) to deduce that*

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

(v) *Let* $f(x) = \cosh x := \frac{e^x + e^{-x}}{2}$, $x \in [0, \pi]$.

(a) *Compute* a_n *for all* $n \in \mathbb{N} \cup \{0\}$. *Write down the Fourier Series of Cosines at each* $x \in [0, \pi]$.

(b) *Show that the the series from (a) converges to* $f(x)$ *for all* $x \in [0, \pi]$.

(c) *Use (b) to deduce that*

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{\pi}{2 \tanh \pi} - \frac{1}{2}.$$

(vi) *Let* $f(x) = \sinh x := \frac{e^x - e^{-x}}{2}$, $x \in [0, \pi]$.

(a) *Compute* b_n *for all* $n \in \mathbb{N}$. *Write down the Fourier Series of Sines at each* $x \in [0, \pi]$.

(b) *Show that the the series from (a) converges to* $f(x)$ *for all* $x \in [0, \pi)$.

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