Complex Analysis

TMA4175 - Kompleks Analyse

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Chapter 1

Cauchy Global Theory

In this chapter, we expand the content of [5, Subsection 5.3.2] on the Cauchy Global Theory, where we consider *cycles* instead of the closed paths. We also establish the corresponding improved version of the Cauchy Residues Theorem. These theorems are formulated for arbitrary open sets, but we however must consider a restriction on the cycle or path along which we integrate, namely, that the cycle should be *null-homologous*. While the proofs are almost identical to those from [5, Subsections 5.3.2, 5.3.3], it will be very convenient to review them in detail, so as to refresh some of the elementary (but fundamental) theorems in complex analysis. We will see that simply connectedness implies the validity of the Cauchy Global Integral Formula/Theorem, without restriction on the pertinent paths. Further implications, such as the existence of primitives or holomorphic logarithms will be studied a well. Another consequence of the Cauchy Homological Theorem is a representation theorem for holomorphic functions in compact sets, via an integral along a cycle consisting only of line segments. Then, we define the crucial class of *symply connected domains*, as those whose closed paths are all *null-homotopic*.

1.1 The Cauchy Local Theorems

We remind the Cauchy Local Theorems and Formulae, and also global theorems for **convex** domains. A fully detailed proof can be found in [5, Sections 4.2–4.3]. We begin with the Cauchy local formulae for the the function and its derivatives.

Theorem 1.1 (Local Cauchy Integral Formulae)**.** *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ holomorphic. Then, for all $n \in \mathbb{N}$, the n^{th} derivative $f^{(n)} : \Omega \to \mathbb{C}$ exists and is holomorphic in Ω. Moreover, for every open disk D with $\overline{D} \subset \Omega$, the following formula holds:*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{n+1}} \,\mathrm{d}w \quad \textit{for all} \quad z \in D, \quad n \in \mathbb{N} \cup \{0\}. \tag{1.1.1}$$

Proof. See [5, Corollary 4.32]. □

In a disk, it is enough to have continuity up to the boundary.

Corollary 1.2 (Cauchy Integral Formula in a disk)**.** *Let $f : \overline{D}(z_0, r) \to \mathbb{C}$ be continuous in $\overline{D}(z_0, r)$ and holomorphic in $D(z_0, r)$. Then,*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D(z_0,r)} \frac{f(w)}{w-z} \,\mathrm{d}w, \quad \textit{for all} \quad z \in D(z_0, r); \tag{1.1.2}$$

where the integral is along the circle $\partial D(z_0, r)$ traveled counterclockwise once.

Proof. See [5, Corollary 4.29]. □

Theorem 1.3 (Cauchy Formulae in Convex Domains). *Let $\Omega \subset \mathbb{C}$ be open and convex, $\gamma : [a,b] \to \Omega$ a closed piecewise C^1-path, and $f : \Omega \to \mathbb{C}$ holomorphic. Then, the following formulae holds:*

$$W(\gamma, z) f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{n+1}} \, \mathrm{d}w \quad \text{for all} \quad z \in \Omega \setminus \gamma^*, \quad n \in \mathbb{N} \cup \{0\}. \tag{1.1.3}$$

Proof. See [5, Corollary 4.33]. □

It is then immediate to obtain the following *Cauchy Global Theorem* for convex domains.

Corollary 1.4 (Cauchy Theorem in a Convex Domain). *Let $\Omega \subset \mathbb{C}$ be open and convex. Then, for every closed piecewise C^1-path $\gamma : [a,b] \to \Omega$, one has*

$$\int_\gamma f = 0.$$

Proof. It suffices to fix $z_0 \in \Omega$, define $g(w) = f(w)(w - z_0)$, and apply formula (1.1.3) with the function g, the point $z = z_0$, and $n = 0$. □

1.2 The Cauchy Homological Theorem

We are instered in formulating a theorem like Theorem 1.3 for arbitrary open sets Ω, and for *unions of closed paths* instead of single paths. These are precisely the cycles.

1.2.1 Cycles and Homology Classes

Definition 1.5 (Cycle). *A* ***cycle*** Γ *is a* ***finite sequence of closed piecewise C^1-paths*** *in $\mathbb{C}$, that is,*

$$\Gamma := \gamma_1 + \cdots + \gamma_n := \{\gamma_j\}_{j=1}^n,$$

where $\gamma_j : [a_j, b_j] \to \mathbb{C}$ is a piecewise C^1-path, for each $j \in \{1, \ldots, n\}$, and $n \in \mathbb{N}$. Also, the ***trace of the cycle*** Γ *is the union*

$$\Gamma^* := \bigcup_{j=1}^n \gamma_j^*,$$

of the traces γ_j^*.

In addition, if $A \subset \mathbb{C}$ is a set, and the trace γ_j^* of γ_j is contained in A for all $j \in \{1, \ldots, n\}$, then we say that Γ above is ***a cycle in A.***

It is important to notice that the paths γ_j forming the cycle must be closed paths. Also, observe that **the set Γ^* is compact**, as the union of finitely many compact subsets of $\mathbb{C}$. We now define the winding numbers and the integration associated with cycles, as the sum of those over each of the paths.

Definition 1.6 (Integration with respect to Cycles). *Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a cycle and let $f : \Gamma^* \to \mathbb{C}$ be a continuous function in Γ^*. The* ***integral of f along*** Γ *is defined by*

$$\int_\Gamma f(z) \, \mathrm{d}z := \sum_{j=1}^n \int_{\gamma_j} f(z) \, \mathrm{d}z. \tag{1.2.1}$$

And if $z \notin \Gamma^*$, the ***winding number of*** Γ ***at*** z ***is defined by***

$$W(\Gamma, z) := \frac{1}{2\pi i} \int_\Gamma \frac{\mathrm{d}w}{w - z} = \frac{1}{2\pi i} \sum_{j=1}^n \int_{\gamma_j} \frac{\mathrm{d}w}{w - z} = \sum_{j=1}^n W(\gamma_j, z). \tag{1.2.2}$$

The second identity is by (1.2.1), and the third one by the definition of winding numbers for closed paths.

Consequently, the winding numbers of cycles satisfy the following properties.

Remark 1.7. Let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ be a cycle. Recall that, for each closed and piecewise C^1-path γ, the winding number

$$\mathbb{C} \setminus \gamma^* \ni z \mapsto W(\gamma, z) \in \mathbb{Z},$$

is a well-defined continuous function, taking only integer values; see [5, Proposition 4.26]. Moreover, $W(\gamma, z) = 0$ for each z lying in the unbounded connected component of γ. The function $W(\Gamma, \cdot)$, being the sum (see (1.2.2)) of integer-valued and continuous functions, satisfy the following:

- $W(\Gamma, z) \in \mathbb{Z}$ for all $z \in \mathbb{C} \setminus \Gamma^*$.
- The function $\mathbb{C} \setminus \Gamma^* \ni z \mapsto W(\Gamma, z)$ is continuous in the open set $\mathbb{C} \setminus \Gamma^*$.
- Consequently, the value $W(\Gamma, \cdot)$ is constant on each connected component of $\mathbb{C} \setminus \Gamma^*$.
- If C_j denotes the unbounded connected component of $\mathbb{C} \setminus \gamma_j^*$, for all $j \in \{1, \dots, n\}$, then $W(\Gamma, z) = 0$ for all $z \in \bigcap_{j=1}^{\infty} C_j$. In particular, we can find $r > 0$ so that $\Gamma^* \subset D(0, r)$ and also $W(\Gamma, z) = 0$ for all $z \in \mathbb{C}$ with $|z| \geq r$.

The third bullet point follows from the fact that if $A \subset \mathbb{C}$ is a connected set, and $h : A \to \mathbb{C}$ is continuous, then $h(A)$ must be connected as well. But the only nonempty connected subsets of $\mathbb{C}$ that are contained in $\mathbb{Z}$ are precisely the singletons. The fourth point is obvious from (1.2.2).

Concerning integration along cycles, the linearity and differentiation under the integral sign theorems work as in the integration over paths.

Remark 1.8. Let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ be a cycle. If $f, g : \gamma^* \to \mathbb{C}$ are continuous on γ^*, and $\lambda \in \mathbb{C}$, one has

$$\int_\Gamma (f + \lambda g) = \int_\Gamma f + \lambda \int_\Gamma g.$$

This follows from the linearity of the complex path-integral, and the definition (1.2.1). Recall that for each closed and piecewise C^1-path $\gamma : [a, b] \to \mathbb{C}$, and $h : \gamma^* \to \mathbb{C}$ continuous, the path-integral of h along γ is

$$\int_\gamma h(z)\, \mathrm{d}z = \int_a^b h(\gamma(t)) \cdot \gamma'(t)\, \mathrm{d}t.$$

On the other hand, recall the differentiation under the integral sign theorem [5, Theorem 4.18] for the path-integral

Theorem 1.9 (Differentiation Under the Integral Sign)**.** *Let $\Omega \subset \mathbb{C}$ be open, γ a piecewise C^1-path (not necessarily closed), and $\varphi : \gamma^* \times \Omega \to \mathbb{C}$ a continuous function such that for every $w \in \gamma^*$ the function $\Omega \ni z \mapsto \varphi(w, z)$ is holomorphic in Ω, and $\gamma^* \times \Omega \ni (w, z) \mapsto \frac{\partial \varphi}{\partial z}(w, z)$ is continuous in $\gamma^* \times \Omega$. Then, the function $F : \Omega \to \mathbb{C}$ given by*

$$F(z) = \int_\gamma \varphi(w, z)\, \mathrm{d}w, \quad z \in \Omega,$$

is holomorphic in Ω and

$$F'(z) = \int_\gamma \frac{\partial \varphi}{\partial z}(w, z)\, \mathrm{d}w, \quad z \in \Omega.$$

The same results holds if we replace a piecewise C^1-path γ with a cycle Γ, due to formula (1.2.1).

We define a fundamental equivalent relationships between cycles.

Definition 1.10 (Homologically Equivalent Cycles)**.** *Let $\Omega \subset \mathbb{C}$ be open and Γ_1, Γ_2 be two cycles in Ω. We say that Γ_1* ***and*** *Γ_2* ***are homologous in*** *Ω, and express it by $\Gamma_1 \simeq \Gamma_2$ in Ω, if*

$$W(\Gamma_1, z) = W(\Gamma_2, z), \quad \textit{for all} \quad z \in \mathbb{C} \setminus \Omega.$$

We will say that a cycle Γ in Ω is ***null-homologous in*** *Ω, and express it by $\Gamma \simeq 0$ in Ω, if*

$$W(\Gamma, z) = 0, \quad \textit{for all} \quad z \in \mathbb{C} \setminus \Omega.$$

Let us see an easy example.

Example 1.11. Denote by $\mathbb{D}$ the open unit disk. Then, for each $r \in (0,1)$, the paths $\gamma_r(t) = re^{it}$, $t \in [0, 2\pi]$ are all null-homologous in $\mathbb{D}$, that is $\gamma_r \simeq 0$ in $\mathbb{D}$.

However, if $\Omega := \mathbb{D} \setminus \{0\}$, then γ_r is **not** null-homologous to 0 in Ω for every $r \in (0,1)$. The reason is that $0 \in \mathbb{C} \setminus \Omega$, but $W(\gamma_r, 0) = 1 \neq 0$, for all $r \in (0,1)$. Nevertheless, for this Ω, one has that γ_r and γ_s are homologous in Ω, that is, $\gamma_r \simeq \gamma_s$ in Ω, for each $r, s \in (0,1)$.

Observe that the relation $\Gamma_1 \simeq \Gamma_2$ is a reformulation of $\Gamma \simeq 0$.

Remark 1.12. Let $\Omega \subset \mathbb{C}$ be open and Γ_1, Γ_2 be two cycles in Ω. If Γ_2 is the sequence of paths $\{\gamma_1, \dots, \gamma_n\}$, we can consider the cycle $\Gamma_2^- = \{\gamma_1^-, \dots, \gamma_n^-\}$ consisting of the reverse paths of Γ_2. Defining a new cycle in Ω by $\Gamma := \Gamma_1 \cup \Gamma_2^-$, the sequence of the paths that form Γ_1 and Γ_2^-, we have, using (1.2.2), that

$$W(\Gamma, z) = W(\Gamma_1, z) + W(\Gamma_2^-, z) = W(\Gamma_1, z) - W(\Gamma_2, z), \quad \text{for all} \quad z \in \mathbb{C} \setminus \Omega.$$

We have also used that $W(\gamma, z) = -W(\gamma^-, z)$ for every closed piecewise C^1-path γ and $z \in \mathbb{C} \setminus \gamma^*$. The above shows that then

$$\Gamma_1 \simeq \Gamma_2 \ \text{ in } \ \Omega \iff \Gamma := \{\Gamma_1, \Gamma_2^-\} \simeq 0 \ \text{ in } \ \Omega.$$

Also notice that if Γ_1, Γ_2, and Γ are as above, then $\Gamma^* = \Gamma_1^* \cup \Gamma_2^*$, and by (1.2.1) we deduce that

$$\int_\Gamma f(w)\,\mathrm{d}w = \int_{\Gamma_1} f(w)\,\mathrm{d}w - \int_{\Gamma_2} f(w)\,\mathrm{d}w,$$

for every continuous function $f : \Gamma^* \to \mathbb{C}$.

1.2.2 The Cauchy Global Formulae and Theorem

We now prove the Cauchy Integral Formula in a general open set, with the proof of John. D. Dixon [4]. A similar formula holds for the derivatives.

Theorem 1.13 (Cauchy Global Integral Formulae)**.** *Let $\Omega \subset \mathbb{C}$ be open, and $f : \Omega \to \mathbb{C}$ be holomorphic. If $\Gamma \simeq 0$ in Ω, then*

$$W(\Gamma, z) f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z}\,\mathrm{d}w, \quad \textit{for all} \quad z \in \Omega \setminus \Gamma^*, \quad \textit{and} \tag{1.2.3}$$

$$W(\Gamma, z) f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(w)}{(w - z)^{n+1}}\,\mathrm{d}w, \quad \textit{for all} \quad z \in \Omega \setminus \Gamma^*, \quad n \in \mathbb{N}. \tag{1.2.4}$$

Proof. Let us begin with (1.2.3). Define $U := \{z \in \mathbb{C} \setminus \Gamma^* \,:\, W(\Gamma, z) = 0\}$. By Remark 1.7, the function $\mathbb{C} \setminus \Gamma^* \ni z \mapsto N_\Gamma(z) := W(\Gamma, z)$ is continuous and only takes integer values. Thus

$$U = N_\Gamma^{-1}(\{0\}) = N_\Gamma^{-1}((-1/2, 1/2))$$

is the preimage of an open interval by a continuous function in the open set $\mathbb{C} \setminus \Gamma^*$, and thus U is open. We define the two-variable function $\varphi : \Omega \times \Omega \to \mathbb{C}$ by the formula

$$\varphi(w,z) = \begin{cases} \dfrac{f(z)-f(w)}{z-w} & \text{if } z \neq w \\ f'(w) & \text{if } z = w. \end{cases} \tag{1.2.5}$$

We will later consider an integration of φ over Γ, and we need to verify the assumptions of Theorem 1.9. Let us first show that φ is continuous in $\Omega \times \Omega$. Indeed, let $(w,z) \in \Omega \times \Omega$ and a sequence $\{(w_n, z_n)\}_n \subset \Omega \times \Omega$ converging to (w,z). In the case where $w \neq z$, then for some $n_0 \in \mathbb{N}$ we have $w_n \neq z_n$ for all $n \geq n_0$, and it is clear from the continuity of f that

$$\lim_{n\to\infty} \varphi(w_n, z_n) = \lim_{n\to\infty} \frac{f(z_n)-f(w_n)}{z_n - w_n} = \frac{f(z)-f(w)}{z-w} = \varphi(w,z).$$

And in the case where $z = w$, then we can find $\delta > 0$ and $n_0 \in \mathbb{N}$ so that $D(w, 2\delta) \subset \Omega$ and $z_n, w_n \in D(w,\delta)$ for all $n \geq n_0$. For those $n \geq n_0$ such that $w_n = z_n$, one has, by the continuity of f' (because f is holomorphic)

$$\varphi(w_n, z_n) = f'(w_n) \longrightarrow f'(w) = \varphi(w,w), \quad \text{as } n \to \infty.$$

And if $w_n \neq z_n$, we apply the Fundamental Theorem of Calculus for the path-integral (see [5, Corollary 4.16]) on the line segment $[w_n, z_n]$, and we get that

$$\begin{aligned} |\varphi(w_n,z_n) - \varphi(w,w)| &= \left| \frac{f(z_n)-f(w_n)}{z_n-w_n} - f'(w) \right| = \left| \frac{f(z_n)-f(w_n)-f'(w)(z_n-w_n)}{z_n-w_n} \right| \\ &= \left| \frac{1}{z_n-w_n} \int_{[w_n,z_n]} \big(f'(\xi)-f'(w)\big)\,\mathrm{d}\xi \right| \leq \sup_{\xi\in[w_n,z_n]} |f'(\xi)-f'(w)|. \end{aligned}$$

Since f is holomorphic in Ω, we know that f' is continuous, and so the last term tends to 0 as n goes to infinity. This shows the continuity of φ in $\Omega \times \Omega$. We next claim that, for each $w \in \Omega$, the function $\Omega \ni z \mapsto \varphi(w,z)$ is holomorphic in Ω with

$$\frac{\partial\varphi}{\partial z}(w,z) = \begin{cases} \dfrac{f(w)-f(z)-f'(z)(w-z)}{(w-z)^2} & \text{if } z \neq w \\ \frac{1}{2} f''(w) & \text{if } z = w. \end{cases} \quad \text{for all} \quad (w,z) \in \Omega \times \Omega. \tag{1.2.6}$$

Indeed, if we fix $w \in \Omega$, for those $z \in \Omega$ with $z \neq w$, a simple computation from the definition of φ (1.2.5) shows the formula for $\frac{\partial\varphi}{\partial z}(w,z)$ in the first line of (1.2.6). And when $z = w$, looking at the definition of φ in (1.2.5) we have

$$\frac{\partial\varphi}{\partial z}(w,w) = \lim_{\xi\to w} \frac{\frac{f(\xi)-f(w)}{\xi-w} - f'(w)}{\xi - w} = \lim_{\xi \to w} \frac{f(\xi)-f(w)-f'(w)(\xi-w)}{(\xi-w)^2} = \frac{1}{2} f''(w);$$

where the last equality can be justified, for example, using that f is of class C^∞ (as a holomorphic function), and so Taylor's theorem applies for any order of smoothness in a disk around w. This shows (1.2.6). Repeating the arguments we used for the continuity of φ, we now show the continuity of $\frac{\partial\varphi}{\partial z}$ in $\Omega \times \Omega$. Indeed, let $(w,z) \in \Omega \times \Omega$ and a sequence $\{(w_n,z_n)\}_n \subset \Omega \times \Omega$ converging to (w,z). In the case where $w \neq z$, then for some $n_0 \in \mathbb{N}$ we have $w_n \neq z_n$ for all $n \geq n_0$, and it is clear from the continuity of f and f' that

$$\begin{aligned} \lim_{n\to\infty} \frac{\partial\varphi}{\partial z}(w_n,z_n) &= \lim_{n\to\infty} \frac{f(w_n)-f(z_n)-f'(z_n)(w_n-z_n)}{(w_n-z_n)^2} \\ &= \frac{f(w)-f(z)-f'(z)(w-z)}{(w-z)^2} = \frac{\partial\varphi}{\partial z}(w,z). \end{aligned}$$

In the case where $z = w$, then we can find $\delta > 0$ and $n_0 \in \mathbb{N}$ so that $D(w, 4\delta) \subset \Omega$ and $z_n, w_n \in D(w, \delta)$ for all $n \geq n_0$. For those $n \geq n_0$ such that $w_n = z_n$, one has, by the continuity of f'' (because f is holomorphic)

$$\frac{\partial \varphi}{\partial z}(w_n, z_n) = f''(w_n) \longrightarrow f''(w) = \frac{\partial \varphi}{\partial z}(w, w), \quad \text{as } n \to \infty.$$

And if $w_n \neq z_n$, note that $w_n \in D(z_n, 2\delta) \subset D(w, 3\delta)$ and we can use the analyticity of f as follows:

$$\begin{aligned}
&\left| \frac{f(w_n) - f(z_n) - f'(z_n)(w_n - z_n)}{(w_n - z_n)^2} - \frac{1}{2} f''(z_n) \right| = \left| \sum_{k=3}^{\infty} \frac{f^{(k)}(z_n)}{k!} (w_n - z_n)^{k-2} \right| \\
&\quad \leq |w_n - z_n| \sum_{k=3}^{\infty} \frac{|f^{(k)}(z_n)|}{k!} |w_n - z_n|^{k-3} \leq |w_n - z_n| \sum_{k=3}^{\infty} \frac{\sup\{|f(\xi)| \,:\, \xi \in D(w, 3\delta)\}}{(3\delta)^k} |w_n - z_n|^k \\
&\quad \leq |w_n - z_n| \sup\{|f(\xi)| \,:\, \xi \in D(w, 3\delta)\} \sum_{k=3}^{\infty} \frac{1}{(3\delta)^k} (2\delta)^{k-3} \\
&\quad = |w_n - z_n| \sup\{|f(\xi)| \,:\, \xi \in D(w, 3\delta)\} (2\delta)^{-3} \sum_{k=3}^{\infty} (2/3)^k;
\end{aligned}$$

and the last term converges to 0 as $n \to \infty$. This and the continuity of f'' imply

$$\left| \frac{\partial \varphi}{\partial z}(w, z) - \frac{\partial \varphi}{\partial z}(w, w) \right| \leq \left| \frac{f(w_n) - f(z_n) - f'(z_n)(w_n - z_n)}{(w_n - z_n)^2} - \tfrac{1}{2} f''(z_n) \right| + \left| \tfrac{1}{2} f''(z_n) - \tfrac{1}{2} f''(w) \right|$$

converges to 0, as $n \to \infty$.

To summarize, since Γ is a cycle in Ω, we have shown that φ satisfies the assumptions of Theorem 1.9 for the cycle Γ, and so the function $\Omega \ni z \mapsto \int_\Gamma \varphi(w, z) \, \mathrm{d}w$ is holomorphic. We introduce a new function $h : \mathbb{C} \to \mathbb{C}$ via the formula

$$h(z) = \begin{cases} \int_\Gamma \varphi(w, z) \, \mathrm{d}w & \text{if } z \in \Omega \\ \int_\Gamma \frac{f(w)}{w - z} \, \mathrm{d}w & \text{if } z \in U. \end{cases} \tag{1.2.7}$$

First of all, we need to verify that h is well defined. Let $z \in \Omega \cap U$. Then, $z \notin \Gamma^*$ and $W(\Gamma, z) = 0$, by the definition of U. In particular $w \neq z$ for all $w \in \Gamma^*$. Thus, looking at the definition of $\varphi(w, z)$ in (1.2.5), we see that

$$\int_\Gamma \varphi(w, z) \, \mathrm{d}w = \int_\Gamma \frac{f(w) - f(z)}{w - z} \, \mathrm{d}w = -2\pi i f(z) W(\Gamma, z) + \int_\Gamma \frac{f(w)}{w - z} \, \mathrm{d}w = \int_\Gamma \frac{f(w)}{w - z} \, \mathrm{d}w.$$

Thus the two branches of definition of h agree, and h is well-defined. Also, notice that h is defined in all of $\mathbb{C}$, by the assumption $\mathbb{C} \setminus \Omega \subset U$. As we conclude right before (1.2.7), the first branch of definition of h is a holomorphic function. Also, since f is holomorphic in Ω, by Theorem 1.9, we get that also h is holomorphic in U. Therefore, we have that h is holomorphic in $\mathbb{C}$. Let us now show that $\lim_{|z| \to \infty} |h(z)| = 0$. Indeed, by Remark 1.7, there exists $r > 0$ so that $\Gamma^* \subset D(0, r)$ and $W(\Gamma, z) = 0$ for all $|z| \geq r$. Thus, assuming $|z| \geq 2r$ and writing $\Gamma = \{\gamma_1, \ldots, \gamma_N\}$, we can estimate

$$|h(z)| = \left| \int_\gamma \frac{f(w)}{w - z} \, \mathrm{d}w \right| \leq \sum_{j=1}^{N} \int_{\gamma_j} \frac{|f(w)|}{|z - w|} |\mathrm{d}w| \leq \frac{\sup\{|f(w)| \,:\, w \in \Gamma^*\}}{|w| - r} \sum_{j=1}^{N} \mathrm{length}(\gamma_j).$$

Since the supremum is finite, letting $|w| \to \infty$ gives $\lim_{|w| \to \infty} |h(w)| = 0$. By the continuity of h, this implies that h is bounded in $\mathbb{C}$. Hence, Liouville's Theorem tells us that h is constant, and actually constantly equal to 0, due to $\lim\limits_{|w|\to\infty} |h(w)| = 0$. Therefore, for any $z \in \Omega \setminus \Gamma^*$, we have that

$$\begin{aligned}
0 = h(z) &= \int_\Gamma \varphi(w,z)\,\mathrm{d}w = \int_\Gamma \frac{f(z)-f(w)}{z-w}\,\mathrm{d}w \\
&= f(z)\int_\Gamma \frac{\mathrm{d}w}{z-w} + \int_\Gamma \frac{f(w)}{w-z}\,\mathrm{d}w = -2\pi i f(z) W(\Gamma,z) + \int_\Gamma \frac{f(w)}{w-z}\,\mathrm{d}w;
\end{aligned}$$

which yields formula (1.2.3).

Now we prove (1.2.4). Writing $\Gamma = \{\gamma_1,\dots,\gamma_N\}$, where each γ_j is a closed and piecewise C^1-path, integrating by parts immediately implies, for $g,h:\Omega\to\mathbb{C}$ holomorphic, that $\int_{\gamma_j} h(w)g'(w)\,\mathrm{d}w = -\int_{\gamma_j} g(w)h'(w)\,\mathrm{d}w$ and therefore

$$\int_\Gamma h(w)g'(w)\,\mathrm{d}w = -\int_\Gamma g(w)h'(w)\,\mathrm{d}w.$$

If $z \in \Omega\setminus\Gamma^*$, applying (1.2.3) for $f^{(n)}$ and then repeatedly the above formula, we get

$$\begin{aligned}
W(\Gamma,z)f^{(n)}(z) &= \frac{1}{2\pi i}\int_\Gamma \frac{f^{(n)}(w)}{w-z}\,\mathrm{d}w = \frac{1}{2\pi i}\int_\Gamma \frac{f^{(n-1)}(w)}{(w-z)^2}\,\mathrm{d}w = \frac{2}{2\pi i}\int_\Gamma \frac{f^{(n-2)}(w)}{(w-z)^3}\,\mathrm{d}w \\
&= \cdots = \frac{(n-1)!}{2\pi i}\int_\Gamma \frac{f'(w)}{(w-z)^n}\,\mathrm{d}w = \frac{n!}{2\pi i}\int_\Gamma \frac{f(w)}{(w-z)^{n+1}}\,\mathrm{d}w.
\end{aligned}$$

□

As a consequence of Theorem 1.13 is the Cauchy Homological Theorem.

Corollary 1.14 (Cauchy Homological Theorem)**.** *Let $\Omega \subset \mathbb{C}$ be open and Γ_1,Γ_2 two cycles in Ω. Then, the following statements are equivalent*

(i) $\Gamma_1 \simeq \Gamma_2$ *in* Ω.

(ii) For every $f:\Omega\to\mathbb{C}$ holomorphic in Ω, one has

$$\int_{\Gamma_1} f(w)\,\mathrm{d}w = \int_{\Gamma_2} f(w)\,\mathrm{d}w.$$

Also, if Γ is a cycle in Ω, then

$$\Gamma \simeq 0 \;\; \textit{in} \;\; \Omega \iff \int_\Gamma f(w)\,\mathrm{d}w = 0 \quad \textit{for every holomorphic function} \;\; f:\Omega\to\mathbb{C}.$$

Proof. By Remark 1.12, it suffices to show the part concerning null-homology. Assume first that $\Gamma \simeq 0$ in Ω. If $f:\Omega\to\mathbb{C}$ is holomorphic, we fix a point $z_0 \in \Omega\setminus\Gamma^*$, and define $g(z) = f(z)(z-z_0)$ for all $z\in\Omega$. Clearly $g \in \mathcal{H}(\Omega)$ and we can apply Theorem 1.13 to g at the point z_0 to obtain

$$0 = W(\Gamma,z_0)g(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{g(w)}{w-z_0}\,\mathrm{d}w = \int_\Gamma \frac{f(w)(w-z_0)}{w-z_0}\,\mathrm{d}w = \int_\Gamma f(w)\,\mathrm{d}w.$$

Conversely, assume $\int_\Gamma f(w)\,\mathrm{d}w = 0$ for every holomorphic function $f:\Omega\to\mathbb{C}$. Then, for each $z\in\mathbb{C}\setminus\Omega$, define the function $f(w) = \frac{1}{w-z}$ for all $w\in\Omega$. Clearly $f\in\mathcal{H}(\Omega)$, so, by the assumption

$$0 = \int_\Gamma \frac{\mathrm{d}w}{w-z} = 2\pi i W(\Gamma,z),$$

implying that $W(\Gamma,z)=0$. Since $z\in\mathbb{C}\setminus\Omega$ is arbitrary, this shows that $\Gamma\simeq 0$ in Ω. □

As we will see in Theorem 1.19, the validity of the Cauchy Homological Theorem is closely related to the existence of primitives in the pertinent domain. On the other hand, the following elementary example illustrates the equivalence in Corollary 1.14.

Example 1.15. Consider the open set $\Omega := \mathbb{C} \setminus \{0\}$, and the path $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$, which is the unit circle traveled once and in the positive direction. Then γ is a closed path which is null-homologous in Ω, that is, $\gamma \simeq 0$ in Ω. And the function $f(z) = 1/z$ is holomorphic in Ω, with

$$\int_\gamma f(w)\,\mathrm{d}w = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,\mathrm{d}t = 2\pi i \neq 0.$$

However, if we consider $g(z) = 1/z^2$, $z \in \Omega$, we see that g has a primitive in Ω (for example $\Omega \ni z \mapsto -1/z$), and the Fundamental Theorem of Calculus for the Complex Path-Integral implies that $\int_\Gamma g = 0$ for any cycle Γ in Ω.

1.2.3 Holomorphic Primitives, Logarithms, and Roots

We now derive some relevant consequences of the Cauchy Homological Theorem; Corollary 1.14. In order to construct certain primitives, we will use *polygonal lines* within a domain.

Definition 1.16 (Polygonally Connected Sets)**.** *If $A \subset \mathbb{C}$ is a set, a **polygonal line in** A is a piecewise C^1-path γ that is also **piecewise affine**. Therefore the trace γ^* of γ is a finite union of oriented line segments $L_1, \ldots, L_N$ such that the end point of L_j coincides with the initial point of L_{j+1} for each $j = 1, \ldots, N-1$.*

*We say that a set $A \subset \mathbb{C}$ is **polygonally connected** if for every two points $z, w \in A$ we can find a polygonal line $\gamma : [0,1] \to A$ in A so that $\gamma(0) = z$ and $\gamma(1) = w$.*

In the next proposition, we show that the notions of connectedness and polygonal connectedness are the same in the case of open sets.

Proposition 1.17. *Let $\Omega \subset \mathbb{C}$ be an open connected set. Then Ω is polygonally connected.*

Proof. Fix a point $z_0 \in \Omega$ and define

$$A = \{z \in \Omega \,:\, \text{there exists a polygonal line } \ell : [0,1] \to \Omega \text{ with } \varphi(0) = z_0,\ \varphi(1) = z\}.$$

Obviously $z_0 \in A$, and so $A \neq \emptyset$.

Let us check that A is open. If $z \in A$, we can find $\delta > 0$ with $D(z, \delta) \subset \Omega$, as Ω is open. Also, there exists a polygonal line $\ell_{z_0,z} : [0,1] \to \Omega$ joining z_0 to z. And given $w \in D(z,\delta)$, the line segment $[z, w]$ is contained in $D(z, \delta)$, and so in Ω. Thus, composing $\ell_{z_0,z}$ and $[z, w]$, we obtain a new polygonal line $\gamma_{z_0,w} : [0,1] \to \Omega$ with $\gamma_{z_0,w}(0) = z_0$ and $\gamma_{z_0,w}(1) = w$. This shows that $w \in A$, and since $w \in D(z, \delta)$ is arbitrary, that $D(z, \delta) \subset A$.

Let us now check that A is also closed relative to A. Let $\{z_n\}_n \subset A$ be a sequence convergent to $z \in \Omega$. Again there is $\delta > 0$ with $D(z, \delta) \subset \Omega$, and by the convergence, we can find n so that $z_n \in D(z, \delta)$. In particular the line segment $[z_n, z]$ is contained in Ω. Since $z_n \in A$, there is a polygonal line $\ell_{z_0,z_n} : [0,1] \to \Omega$ joining z_0 and z_n. Composing ℓ_{z_0,z_n} with $[z_n, z]$, we obtain a polygonal line $\gamma_{z_0,z} : [0,1] \to \Omega$ with $\gamma_{z_0,z}(0) = z_0$ and $\gamma_{z_0,z}(1) = z$. Therefore $z \in A$.

We have shown that A is a nonempty subset of Ω, with A both open and closed relative to Ω. Since Ω is connected, this implies that $A = \Omega$, as desired.

□

Abstract logarithms and square roots of functions are defined as inverses of the exponential and square functions, respectively.

Definition 1.18 (Holomorphic Logarithms and Square Roots). *Let $f : \Omega \to \mathbb{C}$ be a holomorphic function with $f(z) \neq 0$ for all $z \in \mathbb{C}$.*

*We say that $g : \Omega \to \mathbb{C}$ is a **holomorphic logarithm** of f if*

$$g \in \mathcal{H}(\Omega) \quad \text{and} \quad e^{g(z)} = f(z) \ \text{ for all } \ z \in \Omega.$$

*Also, if $n \in \mathbb{N}$, a function $h : \Omega \to \mathbb{C}$ is a **holomorphic n^{th} root** of f if*

$$h \in \mathcal{H}(\Omega) \quad \text{and} \quad (g(z))^n = f(z) \ \text{ for all } \ z \in \Omega.$$

*In the case $n = 2$, such function h is called a **holomorphic square root** of f.*

The following theorem is one of the most important results of the chapter. It provides different types of characterizations for the Cauchy Homological Theorem.

Theorem 1.19. *For an open set $\Omega \subset \mathbb{C}$, the following statements are equivalent.*

(i) $\Gamma \simeq 0$ *in* Ω *for all cycles* Γ *in* Ω.

(ii) $\int_\Gamma f(w)\,\mathrm{d}w = 0$ *for all functions* $f \in \mathcal{H}(\Omega)$, *and cycles* Γ *in* Ω.

(iii) Every $f \in \mathcal{H}(\Omega)$ *has a primitive in* Ω.

(iv) Every $f \in \mathcal{H}(\Omega)$ *with* $f(z) \neq 0$ *for all* $z \in \Omega$, *has a holomorphic logarithm in* Ω.

(v) Every $f \in \mathcal{H}(\Omega)$ *with* $f(z) \neq 0$ *for all* $z \in \Omega$, *has a holomorphic* n^{th} *root in* Ω, *for all* $n \in \mathbb{N}$.

(vi) Every $f \in \mathcal{H}(\Omega)$ *with* $f(z) \neq 0$ *for all* $z \in \Omega$, *has a holomorphic square root in* Ω.

Proof. The equivalence $(i) \iff (ii)$ was proven in Corollary 1.14. Let us prove the other implications.

$(ii) \implies (iii)$: Let $f : \Omega \to \mathbb{C}$ be holomorphic. We can write Ω as a disjoint union of its connected components Ω_j. Since Ω is open, each Ω_j is open, and so it suffices to find a primitive F_j of f on each Ω_j, and then just define $F = f_j$ on Ω_j, for all j, giving a primitive of f in all of Ω. Moreover, since (ii) holds for Ω, then it also holds for each of Ω_j. Therefore, we may and do assume that Ω is connected. Then Proposition 1.17 implies that Ω is polygonally connected. If we fix a point $z_0 \in \Omega$ and, for each $z \in \Omega$ we can define

$$F(z) = \int_{\gamma_{z_0,z}} f(w)\,\mathrm{d}w;$$

where $\gamma_{z_0,z}$ is a polygonal line in Ω joining z_0 and z. Let $z \in \Omega$ and let us show that F is complex-differentiable at z with $F'(z) = f(z)$. Given $\varepsilon > 0$, the continuity of f at z gives a $\delta > 0$ so that $D(z,\delta) \subset \Omega$ and

$$|\xi - z| < \delta \implies |f(\xi) - f(z)| \leq \varepsilon.$$

If $w \in D(z,\delta)$, and $\gamma_{z_0,w}$ is a polygonal line in Ω joining z_0 and w, then we can form a *closed* piecewise C^1-path Γ as the union of $\gamma_{z_0,w}$, the line segment $[w,z]$ (contained in $D(z,\delta)$ by convexity), and the reverse path $\gamma^-_{z_0,z}$. By our assumption,

$$0 = \int_\Gamma f(\xi)\,\mathrm{d}\xi = \int_{\gamma_{z_0,w}} f(\xi)\,\mathrm{d}\xi + \int_{[w,z]} f(\xi)\,\mathrm{d}\xi - \int_{\gamma_{z_0,z}} f(\xi)\,\mathrm{d}\xi = F(w) - F(z) - \int_{[z,w]} f(\xi)\,\mathrm{d}\xi.$$

This equality permits to write

$$\left|\frac{F(w) - F(z)}{w - z} - f(z)\right| = \left|\frac{\int_{[z,w]} f(\xi)\,\mathrm{d}\xi - (w-z)f(z)}{w - z}\right| = \left|\frac{\int_{[z,w]} f(\xi)\,\mathrm{d}\xi - \int_{[z,w]} f(z)\,\mathrm{d}\xi}{w - z}\right|$$

$$\leq \frac{1}{|w-z|}\left|\int_{[z,w]} (f(\xi) - f(z))\,\mathrm{d}\xi\right| \leq \sup_{\xi \in [z,w]} |f(\xi) - f(z)| \leq \varepsilon.$$

$(iii) \implies (iv)$: As we did in the proof of $(ii) \implies (iii)$, we may assume that Ω is connected, as, otherwise, we can construct holomorphic logarithms g_j of f on each connected component of Ω_j, and then define $g = g_j$ on each Ω_j, thus obtaining that g is a holomorphic logarithm of f in Ω.

If $f \in \mathcal{H}(\Omega)$ with $f(z) \neq 0$ for all $z \in \Omega$, the function f'/f is holomorphic in Ω, and by (iii) there exists $H \in \mathcal{H}(\Omega)$ with $H' = f'/f$ in Ω. But the function fe^{-H} is also holomorphic in Ω, and its derivative is

$$(fe^{-H})' = f'e^{-H} - fe^{-H}H' = f'e^{-H} - e^{-H}f' = 0.$$

Thus, since Ω is assumed to be connected, there exists a constant $w_0 \in \mathbb{C}\setminus\{0\}$ with $f(z) = w_0 e^{H(z)}$ for all $z \in \Omega$. Writing $w_0 = e^{\log|w_0| + i\operatorname{Arg}(w_0)}$, we get that

$$f(z) = e^{\log|w_0| + i\operatorname{Arg}(w_0) + H(z)}, \quad z \in \Omega.$$

Thus the function $g(z) := \log|w_0| + i\operatorname{Arg}(w_0) + H(z)$ is a logarithm of f in Ω.

$(iv) \implies (v)$: If $n \in \mathbb{N}$, and $f \in \mathcal{H}(\Omega)$ with $f(z) \neq 0$ for all $z \in \Omega$, by (iv) we can find $g \in \mathcal{H}(\Omega)$ with $f = e^g$ in Ω. Define

$$h(z) := e^{\frac{g(z)}{n}}, \quad z \in \Omega.$$

We have that $(h(z))^n = e^{g(z)} = f(z)$ for all $z \in \Omega$, thus f has a holomorphic n^{th} root in Ω.

$(v) \implies (vi)$: This implication is obvious.

$(vi) \implies (ii)$: Let Γ be a cycle in Ω and let $z \in \mathbb{C}\setminus\Omega$. The function $f(w) = w - z$, $w \in \Omega$ is never zero at Ω, and so (v) tells us that there is $g_1 \in \mathcal{H}(\Omega)$ with $g_1^2 = f$ in Ω. But g_1 is also never zero at Ω, and again by (v) we can find $g_2 \in \mathcal{H}(\Omega)$ with $g_2^2 = g_1$, whence $g_2^4 = f$, in Ω. Iterating, we get a sequence of holomorphic functions $\{g_n\}_n$ in Ω that never vanish in Ω and $g_n^{2^n} = f$ in Ω for every $n \in \mathbb{N}$. Differentiating the identity $g_n^{2^n} = f$, we get that

$$\frac{1}{w - z} = \frac{f'(w)}{f(w)} = 2^n \cdot \frac{g_n'(w)}{g_n(w)}, \quad w \in \Omega.$$

Using Exercise 1.4, we get that, for each $n \geq 2$, the number

$$\frac{2^{-n}}{2\pi i}\int_\Gamma \frac{\mathrm{d}w}{w - z} = \frac{1}{2\pi i}\int_\Gamma \frac{g_n'(w)}{g_n(w)}$$

is an integer, which we call $m_n \in \mathbb{Z}$. Thus for all $n \in \mathbb{N}$, we have that,

$$W(\Gamma, z) = \frac{1}{2\pi i}\int_\Gamma \frac{\mathrm{d}w}{w - z} = 2^n m_n$$

and since $m_n \in \mathbb{Z}$, we have that necessarily $m_n = 0$ for some n large enough. Thus $W(\Gamma, z) = 0$, which implies that $\Gamma \simeq 0$ in Ω because $z \in \mathbb{C}\setminus\Omega$ was arbitrary. □

We record also the following useful criteria for the existence of primitives of concrete functions, in arbitrary open sets.

Proposition 1.20. *Let $\Omega \subset \mathbb{C}$ be open, and $f : \Omega \to \mathbb{C}$ holomorphic. The following statements are equivalent.*

(i) There exists $F \in \mathcal{H}(\Omega)$ with $F' = f$ in Ω.

(ii) For every closed piecewise C^1-path γ in Ω, one has $\int_\gamma f = 0$.

Proof. The implication $(i) \implies (ii)$ is a consequence of the Fundamental Theorem of Calculus for the Complex Path-Integral: if $\gamma : [a, b] \to \Omega$ is closed and piecewise C^1, then

$$\int_\gamma f(w)\,\mathrm{d}w = \int_\gamma F'(w)\,\mathrm{d}w = F(\gamma(b)) - F(\gamma(a)) = 0.$$

The proof of the implication $(ii) \implies (i)$ can be copied verbatim from the proof of Theorem 1.19$(ii) \implies (iii)$. □

1.2.4 Integral Representation in Compact Sets

Closely related to Theorem 1.13 is an integral representation formula for holomorphic maps in compact sets, as an integral over finitely-many segments. This will be essential in the approximation theorems of Chapter 6. We examine first some easy examples.

Example 1.21. Given $\Omega \subset \mathbb{C}$ open and $K \subset \Omega$ compact, we try to write f in K, with f holomorphic in Ω, as a finite sum of integrals over oriented line segments contained in $\Omega \setminus K$.

(1) Let $\Omega = \mathbb{C}$ and $K \subset \mathbb{C}$ a compact set. Let Q be a square with $K \subset \operatorname{int}(Q)$ and give ∂Q the positive orientation. Then ∂Q can be seen as a cycle $\Gamma := \{L_1, L_2, L_3, L_4\}$, with 4 line segments. Note that $\Gamma^* \subset \Omega \setminus K$. For any $f : \Omega \to \mathbb{C}$, we may apply Theorem 1.13 (or just the version for convex domains; Theorem 1.3) for this Γ to obtain

$$f(z) = f(z)W(\Gamma, z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}\,\mathrm{d}w = \frac{1}{2\pi i}\sum_{j=1}^{4}\int_{L_j}\frac{f(w)}{w-z}\,\mathrm{d}w, \quad \text{for all} \quad z \in K.$$

(2) Let $\Omega = \mathbb{C} \setminus \overline{D}(0,1)$ and $K = S(0,3)$, the circle centered at the origin with radius 3. Let Q_1 be a closed square centered at 0 and surrounding K, and give to ∂Q_1 the positive orientation. Let Q_2 be a closed square centered at 0, and so that Q_2 is contained in the open disk $D(0,3)$; and give ∂Q_2 the negative orientation. Construct the cycle $\Gamma := \{\partial Q_1, \partial Q_2\}$, with the mentioned orientations, and observe that $\Gamma^* \subset \Omega \setminus K$ and $\Gamma \simeq 0$ in Ω. Denote by $\{L_j\}_{j=1}^8$, the 8 oriented line segments that are the edges of ∂Q_1 and ∂Q_2. Applying Theorem 1.13, for any $f \in \mathcal{H}(\Omega)$, we have

$$f(z) = f(z)W(\Gamma, z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}\,\mathrm{d}w = \frac{1}{2\pi i}\sum_{j=1}^{8}\int_{L_j}\frac{f(w)}{w-z}\,\mathrm{d}w, \quad \text{for all} \quad z \in K.$$

Example 1.21 shows that the integral representation depends very much on the shape of Ω and K. The following construction works for all cases.

Theorem 1.22 (Integral Representation in Compact Sets)**.** *Let $\Omega \subset \mathbb{C}$ be open, and $K \subset \Omega$ a nonempty compact set. There are oriented line segments $L_1, \dots, L_m \subset \Omega \setminus K$ such that for every $f : \Omega \to \mathbb{C}$ holomorphic, we have*

$$f(z) = \frac{1}{2\pi i}\sum_{j=1}^{m}\int_{L_j}\frac{f(w)}{w-z}\,\mathrm{d}w, \quad \textit{for all} \quad z \in K. \tag{1.2.8}$$

Proof. If $\Omega = \mathbb{C}$, we know how to prove the theorem by virtue of Example 1.21(1). Since K is nonempty, we have that $\delta = \frac{1}{2}\operatorname{dist}(K, \mathbb{C} \setminus \Omega) > 0$. Decompose $\mathbb{C}$ in a mesh of squares, with mutually disjoint interiors, and all of them with side length equal to δ. Since K is bounded, only finitely many of those squares intersect K, so let $\mathcal{F} := \{Q_1, \dots, Q_n\}$ those squares in the grid that intersect K. First we claim that each Q_j is contained in Ω. Indeed, otherwise there exist $z \in Q_j \setminus \Omega$ and $w \in K \cap Q_j$, and therefore

$$2\delta = \operatorname{dist}(K, \mathbb{C} \setminus \Omega) \le |w - z| \le \operatorname{diam}(Q_j) = \sqrt{2}\,\delta,$$

a contradiction. This shows that $Q_1, \dots, Q_n \subset \Omega$. We now give to each closed path ∂Q_j the positive orientation, and consider the collection of all the resulting line segments $\mathcal{S} := \{\ell_j^k : k = 1,2,3,4,\ j = 1, \dots, n\}$. We now select from $\mathcal{S}$ only those $\ell_j^k \in \mathcal{S}$ whose reverse segment $(\ell_j^k)^- \notin \mathcal{S}$, and we denote this new collection by $\{L_1, \dots, L_m\}$. In other words, we have removed from $\mathcal{S}$ those segments that are common edges to two adjacent squares from $\{Q_1, \dots, Q_n\}$. None of the segments

L_j, $j = 1, \dots, m$ intersect K, as otherwise K would intersect some L_k that is a common edge of two squares Q_j, $Q_l \in \mathcal{F}$, contradicting the choice of the segment L_k. We have thus shown that

$$L_1, \dots, L_m \subset \Omega \setminus K.$$

This proves the very first part of the theorem. Also, observe from the selection of the segments $L_1, \dots, L_m$, that for every continuous function $\varphi : \bigcup_{j=1}^m \partial Q_j \to \mathbb{C}$ one has

$$\sum_{j=1}^n \int_{\partial Q_j} \varphi(w)\,\mathrm{d}w = \sum_{j=1}^n \sum_{k=1}^4 \int_{\ell_j^k} \varphi(w)\,\mathrm{d}w = \sum_{j=1}^m \int_{L_j} \varphi(w)\,\mathrm{d}w, \tag{1.2.9}$$

because the segments we discarded from $\mathcal{S}$ to obtain $\{L_1, \dots, L_m\}$ are precisely couples of the form γ_j^k, $(\gamma_j^k)^-$, over which the integrals cancel out.

Now let $f : \Omega \to \mathbb{C}$ be holomorphic. Observe that if $z \in \bigcup_{j=1}^n \mathrm{int}(Q_j)$ then actually z belongs to a unique $\mathrm{int}(Q_k)$, and we can use Theorem 1.13 for each cycle $\Gamma := \partial Q_j$ to deduce that

$$\frac{1}{2\pi i}\int_{\partial Q_k} \frac{f(w)}{w-z}\,\mathrm{d}w = f(z)W(\partial Q_k, z) = f(z), \qquad \int_{\partial Q_j} \frac{f(w)}{w-z}\,\mathrm{d}w = f(z)W(\partial Q_j, z) = 0 \text{ for all } j \neq k.$$

In combination with (1.2.9), the above shows that

$$f(z) = \frac{1}{2\pi i}\sum_{j=1}^n \int_{L_j} \frac{f(w)}{w-z}\,\mathrm{d}w, \quad \text{for all} \quad z \in A := K \cap \bigcup_{j=1}^m \mathrm{int}(Q_j). \tag{1.2.10}$$

To extend the validity of (1.2.10) to all points of K, observe that the mapping

$$K \ni z \mapsto g(z) := \frac{1}{2\pi i}\sum_{j=1}^n \int_{L_j} \frac{f(w)}{w-z}\,\mathrm{d}w.$$

is continuous in K, as per Exercise 1.10. But also $K \ni z \mapsto f(z)$ is continuous in K, and, by (1.2.10), $f = g$ in the set $A \subset K$; which is clearly dense in K, as the interior of each closed square is dense in the square. Therefore, we may conclude that $f = g$ in all of K, which gives (1.2.8). $\square$

1.3 Homotopy of Paths. Simply Connected Domains

Our goal here is to introduce a special class of domains for which one (and so all) of the properties of Theorem 1.19 hold true. For this purpose, we need to define rigorously the idea of *continuously deforming one curve into another.* This can be achieved via the definition of homotopy.

Definition 1.23 (Homotopy of Paths). *Let $A \subset \mathbb{C}$ be a subset, and $\gamma_0, \gamma_1 : [a,b] \to \Omega$ two continuous and closed curves. We say that γ_0 and γ_1 **are homotopic in** A, express it as $\gamma_0 \sim \gamma_1$ in A, when there exists a continuous mapping $H : [a,b] \times [0,1] \to A$ such that*

- *$H(t,0) = \gamma_0(t)$ for all $t \in [0,1]$,*
- *$H(t,1) = \gamma_1(t)$ for all $t \in [0,1]$, and*
- *$H(a,s) = H(b,s)$ for all $s \in [0,1]$.*

*Such a function H is called an **homotopy between** γ_0 **and** γ_1 **in** A.*

*Also, if $\gamma : [a,b] \to A$ is homotopic to a constant path $\gamma_0 : [a,b] \to \Omega$, $\gamma_0(t) = w_0$ for all $t \in [a,b]$, then we say that A is **null-homotopic in** A, and express it as $\gamma \sim 0$ in A.*

Looking that Definition 1.23, note that, for each $s \in [0,1]$, the function $[a,b] \ni t \mapsto \gamma_s(t) := H(t,s)$ defines a continuous closed curve in A. Thus, this homotopy H provides us with a family $\{\gamma_s\}_{s\in[0,1]}$ of continuous closed curves in A, the *initial one* being γ_0, and the *final one* being γ_1. The curve γ_s is the *deformation of* γ_0 *to* γ_1 at the instant $s \in [0,1]$.

In addition, the following remark is pertinent.

Remark 1.24. If $A \subset \mathbb{C}$ is a subset, then the relation "$\sim$ in A" between paths defines an equivalence relation. Indeed, $\gamma \sim \gamma$ in A for each continuous curve $\gamma : [a,b] \to A$, because we can define the trivial homotopy $H(t,s) = \gamma(t)$ for all $(t,s) \in [a,b] \times [0,1]$. So the reflexivity holds. Also, if $\gamma_0 \sim \gamma_1$ in A, and H is an homotopy as in Definition 1.23, then we can define $\widetilde{H}(t,s) = H(t,1-s)$ for all $(t,s) \in [a,b] \times [0,1]$, and clearly $\widetilde{H}$ is an homotopy between γ_1 and γ_0 in A, therefore $\gamma_1 \sim \gamma_0$ in A, showing the property of symmetry. To prove the transitivity, let $\gamma_0 \sim \gamma_1$ in A via some homotopy H_0 and $\gamma_1 \sim \gamma_2$ in A via some homotopy H_1. If we define $H : [a,b] \times [0,1] \to A$ by formula

$$H(t,s) := \begin{cases} H_0(t,2s) & \text{if } (t,s) \in [a,b] \times [0,1/2] \\ H_0(t,2s-1) & \text{if } (t,s) \in [a,b] \times [1/2,1]; \end{cases}$$

and clearly H defines an homotopy between γ_0 and γ_2 in A, that is $\gamma_0 \sim \gamma_2$ in A.

Moreover, by considering re-parametrizations, we may assume that all the curves involved are defined in $[0,1]$, instead of a general closed interval $[a,b]$.

The *simply connected domains* are those on which every closed curve can *be continuously shrunk to a point,* where the *deformation* is always within the domain.

Definition 1.25 (Simply Connected Domains). *Let $\Omega \subset \mathbb{C}$ be open and connected. We say that Ω is* ***simply connected*** *if every continuous closed curve $\gamma : [a,b] \to \Omega$ is null-homotopic in Ω.*

An interesting particular example is as follows.

Example 1.26. A set $A \subset \mathbb{C}$ is called **star-shaped** if there exists $z_0 \in A$ so that all segments $[z_0, z]$, $z \in A$, are entirely contained in A. An open star-shaped open set Ω is simply connected. Indeed, if $\gamma : [a,b] \to \Omega$ is continuous and closed, then we can define $H : [a,b] \times [0,1] \to \mathbb{C}$ by formula

$$H(t,s) = sz_0 + (1-s)\gamma(t), \quad (t,s) \in [a,b] \times [0,1].$$

Since for each $t \in [a,b]$, the line segment $[\gamma(t), z_0]$ is contained in Ω, we see that H takes values only in Ω. It is also immediate to verify the properties of homotopies for H; see Definition 1.23. We have shown that $\gamma \sim 0$ in Ω, and thus Ω is simply-connected.

Note that every **convex** set is star-shaped, and so simply connected as well.

Our next goal is to prove that two homotopic paths are always homologous. The following lemma is very helpful for that purpose.

Lemma 1.27. *Let $\gamma_0, \gamma_1 : [a,b] \to \mathbb{C}$ two closed and piecewise C^1 paths. Assume there exists $z \in \mathbb{C}$ so that*

$$|\gamma_1(t) - \gamma_0(t)| < |z - \gamma_0(t)|, \quad \textit{for all } t \in [a,b]. \tag{1.3.1}$$

Then $W(\gamma_0, z) = W(\gamma_1, z)$.

Proof. First note that the the condition (1.3.1) implies that $z \notin \gamma_0^* \cup \gamma_1^*$. Thus we can define a new closed and piecewise C^1-path $\gamma : [a,b] \to \mathbb{C}$ by

$$\gamma(t) = \frac{\gamma_1(t) - z}{\gamma_0(t) - z}, \quad t \in [a,b].$$

Then precisely by (1.3.1), we see that γ^* is contained in the open disk $D(1,1)$, and so the origin 0 belongs to the unbounded connected component of $\mathbb{C}\setminus\gamma^*$. Thus $W(\gamma,0)=0$. But calculating this winding number, we see that

$$W(\gamma,0)=\frac{1}{2\pi i}\int_a^b\frac{\gamma'(t)}{\gamma(t)}\,\mathrm{d}t=\frac{1}{2\pi i}\int_a^b\frac{\gamma_1'(t)}{\gamma_1(t)-z}\,\mathrm{d}t-\frac{1}{2\pi i}\int_a^b\frac{\gamma_2'(t)}{\gamma_2(t)-z}\,\mathrm{d}t=W(\gamma_1,z)-W(\gamma_2,z),$$

thus obtaining $W(\gamma_0,z)=W(\gamma_1,z)$. □

As we mentioned above, the notion of homotopy equivalence is stronger than homology equivalence.

Theorem 1.28. *Let $\Omega\subset\mathbb{C}$ be open, and $\gamma_0,\gamma_1:[a,b]\to\Omega$ two closed piecewise C^1-paths in Ω that are homotopic in Ω. Then $\gamma_0\simeq\gamma_1$ in Ω, that is,*

$$W(\gamma_0,z)=W(\gamma_1,z)\quad\text{for all }\; z\in\mathbb{C}\setminus\Omega.$$

In particular, if γ is a closed and piecewise C^1-path that is null-homologous in Ω, then γ is null-homotopic in Ω.

Proof. By Remark 1.24, and because winding numbers are stable under reparametrizations, we may assume that $[a,b]=[0,1]$. Let $H:[0,1]\times[0,1]\to\Omega$ an homotopy between γ_0 and γ_1 in Ω as in Definition 1.23. Fix a point $z\in\mathbb{C}\setminus\Omega$. By the continuity of H, the set $H([0,1]\times[0,1])$ is compact subset of Ω, and so there exists $\varepsilon>0$ so that

$$|z-H(t,s)|>2\varepsilon,\quad\text{for all }\;(t,s)\in[0,1]\times[0,1]. \tag{1.3.2}$$

And again by the continuity of H, we can find $n\in\mathbb{N}$ so that

$$|H(t,s)-H(t',s')|\le\varepsilon,\quad\text{whenever }\;|t-t'|+|s-s'|\le\frac{1}{n},\quad(t,s)\in[0,1]\times[0,1]. \tag{1.3.3}$$

Now, for every $k=0,\dots,n$ consider the polygonal line $\gamma_k:[0,1]\to\mathbb{C}$ given by

$$\sigma_k(t)=H\left(\tfrac{j-1}{n},\tfrac{k}{n}\right)(j-nt)+H\left(\tfrac{j-1}{n},\tfrac{k}{n}\right)(nt-(j-1))\;\text{ on each }\;t\in\left[\tfrac{j-1}{n},\tfrac{j}{n}\right],\;j=1,\dots,n. \tag{1.3.4}$$

Notice that σ_k is continuous in $[0,1]$. Also,

$$\sigma_k(0)=H(0,\tfrac{k}{n})=H(1,\tfrac{k}{n})=\sigma_k(1),$$

and σ_k is a closed piecewise C^1-path. Now observe that the definition (1.3.4) and the estimate (1.3.3) imply

$$\begin{aligned}|\sigma_{k-1}(t)-\sigma_k(t)|&\le\left|H\left(\tfrac{j-1}{n},\tfrac{k-1}{n}\right)-H\left(\tfrac{j-1}{n},\tfrac{k}{n}\right)\right|(j-nt)+\left|H\left(\tfrac{j}{n},\tfrac{k-1}{n}\right)-H\left(\tfrac{j}{n},\tfrac{k}{n}\right)\right|(nt-(j-1))\\&\le\varepsilon(j-nt)+\varepsilon(nt-(j-1))=\varepsilon,\quad\text{whenever}\quad t\in\left[\tfrac{j-1}{n},\tfrac{j}{n}\right],\;k=1,\dots,n;\end{aligned} \tag{1.3.5}$$

and similarly

$$\begin{aligned}\left|\sigma_k(t)-H\left(t,\tfrac{k}{n}\right)\right|&\le\left|H\left(\tfrac{j-1}{n},\tfrac{k}{n}\right)-H\left(t,\tfrac{k}{n}\right)\right|(j-nt)+\left|H\left(\tfrac{j}{n},\tfrac{k}{n}\right)-H\left(t,\tfrac{k}{n}\right)\right|(nt-(j-1))\\&\le\varepsilon(j-nt)+\varepsilon(nt-(j-1))=\varepsilon,\quad\text{whenever}\quad t\in\left[\tfrac{j-1}{n},\tfrac{j}{n}\right],\;k=1,\dots,n.\end{aligned} \tag{1.3.6}$$

In the particular cases $k = 0$ and $k = n$, (1.3.6) tells us that

$$|\sigma_0(t) - \gamma_0(t)| \le \varepsilon \quad \text{and} \quad |\sigma_n(t) - \gamma_1(t)| \le \varepsilon, \quad t \in [0,1]. \tag{1.3.7}$$

But then, by (1.3.2) and (1.3.6), we deduce

$$|z - \sigma_k(t)| \ge \left|z - H\left(t, \tfrac{k}{n}\right)\right| - \left|H\left(t, \tfrac{k}{n}\right) - \sigma_k(t)\right| > 2\varepsilon - \varepsilon = \varepsilon, \quad t \in [0,1],\, k = 0, \dots, n. \tag{1.3.8}$$

The lower bound (1.3.8) in combination with (1.3.7) and (1.3.5) gives the inequalities

$$|\gamma_0(t) - \sigma_0(t)| < |z - \sigma_0(t)|, \quad |\sigma_{k-1}(t) - \sigma_k(t)| < |z - \sigma_k(t)|, \quad |\gamma_1(t) - \sigma_n(t)| < |z - \sigma_n(t)|,$$

for all $t \in [0,1]$ and $k \in \{1, \dots, n\}$. Applying Lemma 1.27 for the paths $\gamma_0, \sigma_0, \dots, \sigma_n, \gamma_1$, we conclude

$$W(\gamma_0, z) = W(\sigma_0, z) = W(\sigma_1, z) = \cdots = W(\sigma_{n-1}, z) = W(\sigma_n, z) = W(\gamma_1, z).$$

□

Theorem 1.28 can be now combined with Theorems 1.13 and 1.19 to derive the following properties for holomorphic maps in simply-connected domains. These properties actually characterize the simple connectedness, as we will see in Chapter 4, Corollary 4.44.

Corollary 1.29. *Let $\Omega \subset \mathbb{C}$ be open and simply-connected. The following statements hold.*

(i) $\Gamma \simeq 0$ *in* Ω *for all cycles* Γ *in* Ω.

(ii) For every $f \in \mathcal{H}(\Omega)$ *and every cycle* Γ *in* Ω, *one has*

$$W(\Gamma, z) f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(w)}{(w-z)^{n+1}}\, \mathrm{d}w, \quad z \in \Omega \setminus \Gamma^*,\, n \in \mathbb{N} \cup \{0\}.$$

(iii) $\int_\Gamma f(w)\, \mathrm{d}w = 0$ *for all functions* $f \in \mathcal{H}(\Omega)$, *and cycles* Γ *in* Ω.

(iv) Every $f \in \mathcal{H}(\Omega)$ *has a primitive in* Ω.

(v) Every $f \in \mathcal{H}(\Omega)$ *with* $f(z) \neq 0$ *for all* $z \in \Omega$, *has a holomorphic logarithm in* Ω.

(vi) Every $f \in \mathcal{H}(\Omega)$ *with* $f(z) \neq 0$ *for all* $z \in \Omega$, *has a holomorphic* n^{th} *root in* Ω, *for all* $n \in \mathbb{N}$.

Proof. By definition of simple connectedness, we have that $\gamma \sim 0$ in Ω for all continuous closed curves in Ω. By Theorem 1.28 we have that all closed piecewise C^1-paths are null-homologous, and so $\Gamma \simeq 0$ in Ω for all cycles in Ω. This proves (i), then Theorem 1.13 gives (ii), and Theorem 1.19 implies the rest (iii)–(vi). □

1.4 Exercises

Exercise 1.1. *Let $D(z_0, r_0) \subset \mathbb{C}$ be an open disk and $f : D(z_0, r_0) \to \mathbb{C}$ be continuous. Prove that*

$$f(z_0) = \lim_{r \to 0^+} \frac{1}{2\pi i} \int_{\partial D(z_0, r)} \frac{f(z)}{z - z_0}\, \mathrm{d}z,$$

where the circles $\partial D(z_0, r)$ are traveled once and with the positive orientation.

Exercise 1.2. *Let Ω be open and convex, and $f : \Omega \to \mathbb{C}$ holomorphic in Ω with $\mathrm{Re}(f'(z)) > 0$ for all $z \in \Omega$. Show that f is injective in Ω.*

Exercise 1.3. *Let $\Omega \subset \mathbb{C}$ open, $f : \Omega \to \mathbb{C}$ holomorphic and $\gamma : [a,b] \to \Omega$ a closed piecewise C^1-path. Prove that if $n \in \mathbb{N} \cup \{0\}$ and $z_0 \notin \gamma^*$, then*

$$\int_\gamma \frac{f'(z)}{(z-z_0)^n}\,\mathrm{d}z = n \int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z.$$

Exercise 1.4. *Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \to \mathbb{C}$ holomorphic and $\gamma : [a,b] \to \Omega$ a piecewise C^1-path with $f(z) \neq 0$ for all $z \notin \gamma^*$ and $f(\gamma(b)) = f(\gamma(a))$. Show that*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(w)}{f(w)}\,\mathrm{d}w \in \mathbb{Z}.$$

Exercise 1.5. *Let γ be the ellipse $\{z \in \mathbb{C} : |z-2|+|z+2| = 10\}$ traveled once and counterclockwise. Use the Cauchy Integral Formulae to find*

$$\int_\gamma \frac{e^{z^2}\sin z}{(z-\pi i)^2}\,\mathrm{d}z.$$

Exercise 1.6. *Define, for each $r > 0$, the path $\gamma_r : [0, \pi/4] \to \mathbb{C}$ by $\gamma_r(t) = re^{it}$. Prove that*

$$\lim_{r\to+\infty} \int_{\gamma_r} e^{-z^2}\,\mathrm{d}z = 0.$$

Then, integrate the function e^{-z^2} over the paths $\Gamma_r := [0,r] \star \gamma_r \star [re^{\frac{i\pi}{4}}, 0]$, $r > 0$, to show that

$$\int_0^\infty \sin(x^2)\,\mathrm{d}x = \int_0^\infty \cos(x^2)\,\mathrm{d}x = \frac{\sqrt{2\pi}}{4}.$$

Suggestion: For the limit part, take into account the inequality $\cos(2t) \geq 1 - \frac{4}{\pi}t$ for all $t \in [0, \frac{\pi}{4}]$.

Exercise 1.7. *Given $n \in \mathbb{N}$, with $n \geq 3$, consider the closed path γ given by the n-polygon whose vertices are the nth-roots of unity, and traveled counterclockwise. Show that $W(\gamma, 0) = 1$.*

Exercise 1.8. *Use the Cauchy Homological Theorem (Corollary 1.14) to prove that*

$$\int_{\partial D(0,1)} \frac{(z+1)e^{1/z}}{z^2}\,\mathrm{d}z = 2\pi i.$$

Suggestion: Compare with the same integral over circles of radius $r > 1$, and let $r \to \infty$.

Exercise 1.9. *Let $\Omega \subset \mathbb{C}$ be open with the property that $\Gamma \simeq 0$ in Ω for all cycles Γ in Ω. Let $f, g : \Omega \to \mathbb{C}$ be two holomorphic functions such that $f^2 + g^2 = 1$ at all points of Ω. Prove that there is a holomorphic function $\varphi : \Omega \to \mathbb{C}$ in Ω with*

$$f = \cos(\varphi), \quad g = \sin(\varphi), \quad \text{in } \Omega.$$

Suggestion: First find a holomorphic logarithm for the function $f + ig$.

Exercise 1.10. *Let $A \subset \mathbb{C}$ be a set, $\gamma : [a,b] \to \mathbb{C}$ a piecewise C^1-path, and $\varphi : \gamma^* \times A \to \mathbb{C}$ a continuous mapping. Then the function $f : A \to \mathbb{C}$ given by*

$$f(z) = \int_\gamma \varphi(w,z)\,\mathrm{d}w, \quad z \in A,$$

is continuous in A.

Exercise 1.11. *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{C}$ open and connected, and $f \in \mathcal{H}(\Omega)$. Let $g : \Omega \to \mathbb{C}$ be a holomorphic function with $g^n = f$ in Ω. Show that if f is not identically zero in Ω, then the functions $\{g, \xi g, \ldots, \xi^{n-1} g\}$ are all the holomorphic n^{th}-roots of f; where $\xi = e^{\frac{2\pi i}{n}}$.*

Exercise 1.12. *Consider the function $f : \mathbb{C} \setminus \{-1, 1\} \to \mathbb{C}$ given by*

$$f(z) = \frac{1}{z^2 - 1}, \quad z \in \mathbb{C} \setminus \{-1, 1\}.$$

*First verify that $\Omega := \mathbb{C} \setminus \{-1, 1\}$ does **not** satisfy the property "$\Gamma \simeq 0$ in Ω for all cycles in Ω". Then show that the f does **not** have a primitive in $\mathbb{C} \setminus \{-1, 1\}$. Finally prove that, however, f **has** a primitive in the smaller domain $\mathbb{C} \setminus [-1, 1]$.*

Suggestion: For the last part, show first that $\int_\gamma f = 0$ for all closed paths γ in $\mathbb{C} \setminus [-1, 1]$. Then proceed as in Theorem 1.19 (ii) $\implies$ (iii).

Chapter 2

Meromorphic Functions

A complex function f has an isolated singularity at a point z_0, when f is holomorphic in some *punctured disk* around z_0. We classify the type of isolated singularities (removable, pole, essential) by looking at the terms of the Laurent Series of f at z_0. Characterizations of these singularities are also provided by Riemann's Criterion and by Casorati-Weierstrass Theorem. The residue of f at z_0 is the coefficient of term $(z-z_0)^{-1}$ in the Laurent Series, which can be recovered with a Cauchy-type integral formula in a disk. In combination with the Cauchy Homological Theorem, we will use this formula to deduce the homological version of the Cauchy Residues Theorem. Then we define the *meromorphic functions* as those functions that are holomorphic except for isolated singularities, all of which are poles. The *Argument Principle* for meromorphic functions f and a closed path γ states that the number of times that $f \circ \gamma$ travels around the origin coincides with the number of zeros of f minus the number of poles of f within the inside of γ, and counted with multiplicity. Rouché's Theorem is an useful criteria for localizing the zeros of a holomorphic function in a domain. Finally Hurwitz's theorem states that the number of zeros of a locally uniformly convergent sequence of holomorphic functions is essentially stationary.

2.1 Isolated Singularities

Definition 2.1 (Isolated Singularity)**.** *Given $z_0 \in \mathbb{C}$, we say that a complex function f has an* ***isolated singularity at*** z_0 *if there exists $r > 0$ so that $f : D(z_0, r) \setminus \{z_0\} \to \mathbb{C}$ is holomorphic in the set $D(z_0, r) \setminus \{z_0\}$.*

Let us examine the Laurent Series expansion at isolated singularities. We refer the reader to [5, Chapter 5] for explanations on the Laurent Series expansions on annuli.

Remark 2.2. Let $z_0 \in \mathbb{C}$ and let f be holomorphic in $D(z_0, r) \setminus \{z_0\} \to \mathbb{C}$. Using [5, Theorem 5.5] we find a sequence $\{a_n\}_{n\in\mathbb{Z}}$ so that

$$f(z) = \sum_{n\in\mathbb{Z}} a_n (z - z_0)^n, \quad 0 < |z - z_0| < r; \tag{2.1.1}$$

with absolute-uniform convergence in annuli $\{z \in \mathbb{C} : t \leq |z - z_0| \leq s\}$, with $0 < t < s < r$. More precisely, defining $b_n := a_{-n}$ for all $n \in \mathbb{N}$, we can write

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, \tag{2.1.2}$$

where $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges absolutely in $D(z_0, r)$, and absolutely–uniformly in $\overline{D}(z_0, s)$ for all $0 < s < r$, and the **principal part of the Laurent series** $\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ converge absolutely in $\mathbb{C} \setminus \{z_0\}$ and absolutely–uniformly on $\mathbb{C} \setminus D(z_0, \varepsilon)$ for all $\varepsilon > 0$.

On the other hand, the **residue of f at** z_0 is the coefficient $b_1 = a_{-1}$ in the series expansions (2.1.1)–(2.1.2). The residue is also given by an integral formula, see [5, Theorem 5.5] for a proof:

$$\operatorname{Res}(f, z_0) = \frac{1}{2\pi i} \int_{\partial D(0,s)} f(w)\, \mathrm{d}w, \quad \text{for all} \quad 0 < s < r.$$

We now classify the isolated singularities.

Definition 2.3 (Types of Isolated Singularity)**.** *Let $f : D(z_0, r) \setminus \{z_0\} \to \mathbb{C}$ be holomorphic in $D(z_0, r) \setminus \{z_0\}$, that is, with an isolated singularity at z_0. Let $\{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$ as in the Laurent expansion* (2.1.1) *of f at z_0, in the punctured disk $D(z_0, r) \setminus \{z_0\}$. We say that*

- *f has **a removable singularity at** z_0 if $a_n = 0$ for all $n < 0$. Note that then* $\operatorname{Res}(f, z_0) = 0$.
- *f has **a pole at** z_0 if there exists $N \in \mathbb{N}$ with $a_{-N} \neq 0$ and $a_n = 0$ for all $n < -N$. More precisely, in this case we say that f has **a pole of order** N **at** z_0. Sometimes, poles of order* 1 *are called **simple poles**.*
- *f has **an essential singulartiy at** z_0 if $a_n \neq 0$ for infinitely many $n < 0$.*

Let us now state useful criteria for the type of singularity. For removable singularities, we use the following Riemann's theorem.

Theorem 2.4 (Riemann's Criterion)**.** *If $f : D(z_0, r) \setminus \{z_0\} \to \mathbb{C}$ is holomorphic, then f admits an extension $F : D(z_0, r) \to \mathbb{C}$ holomorphic in all of $D(z_0, r)$ if and only if f is bounded in $D(z_0, r) \setminus \{z_0\}$. In other words, f has a removable singularity at z_0 if and only if f is bounded in the punctured disk $D(z_0, r) \setminus \{z_0\}$.*

Proof. See [5, Theorem 5.12] □

We can characterize poles with or without specifying the order of the pole.

Proposition 2.5. *Let $f : D(z_0, r) \setminus \{z_0\} \to \mathbb{C}$ be holomorphic, and $N \in \mathbb{N}$. The following statements are equivalent.*

(i) f has a pole of order N at z_0.

(ii) There exists $g \in \mathcal{H}(D(z_0, r))$ with $g(z_0) \neq 0$ and so that

$$g(z) = (z - z_0)^N f(z), \quad z \in D(z_0, r) \setminus \{z_0\}. \tag{2.1.3}$$

(iii) The function $1/f$ admits a holomorphic extension $\varphi : D(z_0, s)\setminus\{z_0\} \to \mathbb{C}$ in a disk $D(z_0, s)$, so that φ has a zero of order N at z_0. This means that φ can be written as $\varphi(z) = (z - z_0)^N \psi(z)$, where ψ is a holomorphic function with $\psi(z) \neq 0$ for all $z \in D(z_0, s)$.

And, in general, without specifying the order of the pole, the following statements are also equivalent.

(i) f has a pole at z_0.

(ii) $\lim\limits_{z\to z_0} |f(z)| = \infty$.

Proof. See [5, Propositions 5.13 and 5.14]. □

Concerning essential singularities, the criterion is given by Casorati-Weierstrass theorem.

Theorem 2.6 (Casorati-Weierstrass)**.** *Let $f : D(z_0, r) \setminus \{z_0\} \to \mathbb{C}$ be holomorphic. Then, the following statements are equivalent.*

(i) f has an essential singularity at z_0.

(ii) $\overline{f\left(D(z_0, s) \setminus \{z_0\}\right)} = \mathbb{C}$ for every $0 < s \leq r$. That is, for every $w \in \mathbb{C}$ there exists a sequence $\{z_n\}_n$ converging to z_0, with $z_n \neq z_0$ for all $n \in \mathbb{N}$, so that $\{f(z_n)\}_n$ converges to w.

Proof. See [5, Theorem 5.15]. □

2.1.1 Holomorphicity and Singularities at Infinity

Definition 2.7 (Holomorphicity at Infinity)**.** *We say that a function f is* ***holomorphic at*** ∞ *if there exists $r > 0$ so that the function $g(z) = f(1/z)$ is holomorphic and well-defined in $D(0,r)\setminus\{0\}$ and g admits a holomorphic extension to all of $D(0,r)$.*

Also, if $n \in m \cup \{0\}$ and g has a zero of order m at 0, then we say that f ***has a zero of order*** *m* ***at*** *∞.*

For example the function $f(z) = \frac{1}{z^m}$, $m \in \mathbb{N}$, has an isolated singularity at ∞, as $g(w) = f(1/w) = w^m$ is holomorphic in a disk around 0.

Actually, this definition can be seen as a particular case of the following singularities at infinity, in the case where they are removable.

Definition 2.8 (Singularity at Infinity)**.** *A function f has* ***an isolated singularity at*** ∞ *if the function $\mapsto g(w) := f(1/w)$ has an isolated singularity at* 0.

Moreover, if f has an isolated singularity at ∞, we say that f has ***a removable singularity, a pole of order*** *$N \in \mathbb{N}$****, or an essential singularity at*** *∞ if the function above g has respectively a removable singularity, a pole of order N, or an essential singularity at* 0.

We have the following interpretation from the topological point of view. The **extended complex plane** $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ is equipped with the collection of sets

$$\mathcal{T}_{\mathbb{C}_\infty} := \{U \subset \mathbb{C} \,:\, U \text{ is open}\} \bigcup \{V \cup \{\infty\} \,:\, V = \mathbb{C} \setminus K, \text{ with } K \text{ compact in } \mathbb{C}\}.$$

This family defines a *topology* in $\mathbb{C}_\infty$, and the space $(\mathbb{C}_\infty, \mathcal{T}_{\mathbb{C}_\infty})$ is a **compact** topological space, called the *Alexandroff-compactification of* $\mathbb{C}$. Moreover, this space is *metrizable*, as can be shown using the **spherical distance** in $\mathbb{C}_\infty$, which is the Euclidean distance in $\mathbb{S}^2$ composed with the inverse of the stereographic projection; see [5, Section 1.7]. Therefore, sets of the form $\Omega_\infty := \mathbb{C}_\infty \setminus \overline{D}(0,R)$ are open neighbourhoods of ∞ with this topology, and if f has an isolated singularity at ∞, we can say that f is holomorphic in some neighbourhood Ω_∞ of ∞, except at ∞, that is, $f \in \mathcal{H}\left(\Omega_\infty \setminus \{\infty\}\right)$.

Let us see some easy examples.

Example 2.9. If $f(z) = a_0 + a_1 z + \cdots + a_m z^m$ is a polynomial of degree $m \in \mathbb{N}$, then f has a pole of order m at infinity, as the function $g(w) = f(1/w)$ is

$$g(w) = a_0 + \frac{a_1}{w} + \cdots + \frac{a_m}{w^m},$$

which clearly has a pole of order m at 0.

Also, if $f(z)$ is any the functions $\sin z$, $\cos z$, e^z, then f has an essential singularity at ∞, because $g(w) = f(1/w)$ is (respectively) $\sin(1/w)$, $\cos(1/w)$, $e^{1/w}$, with isolated singularities at 0.

Alos, let us look at the *Laurent Series of a function at infinity.*

Remark 2.10. If f has an isolated singularity at ∞, then for some $r > 0$ we have the Laurent Series expansion of $g(w) = f(1/w)$:

$$g(w) = \sum_{n=0}^{\infty} a_n w^n + \sum_{n=1}^{\infty} \frac{b_n}{w^n}, \quad \text{for} \quad 0 < |w| < 1/r.$$

Therefore,

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n} + \sum_{n=1}^{\infty} b_n z^n, \quad \text{for} \quad |z| > r,$$

and according to Definition 2.8, we have the following.

(i) If f has a removable singularity at ∞, then $b_n = 0$ for all $n \in \mathbb{N}$, and

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}, \quad \text{for} \quad |z| > r.$$

(ii) If f has a pole of order $N \in \mathbb{N}$ at infinity, then $b_N \neq 0$ and $b_n = 0$ for all $n > N$, and

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n} + \sum_{n=1}^{N} b_n z^n, \quad \text{for} \quad |z| > r.$$

(iii) If f has an essential singularity at infinity, then $b_N \neq 0$ for infinitely many $n \in \mathbb{N}$.

Proposition 2.11. *If $f : \mathbb{C} \to \mathbb{C}$ be holomorphic, the following hold.*

(i) If f has a removable singularity at ∞, then f is constant.

(ii) If f has a pole of order $m \in \mathbb{N}$ at ∞, then f is a polynomial of degree m.

Proof. Since $f : \mathbb{C} \to \mathbb{C}$ is holomorphic, there are unique numbers $\{c_n\}_{n \geq 0} \subset \mathbb{C}$ so that

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \quad z \in \mathbb{C},$$

and so the function $g(w) = f(1/w)$ can be written as

$$g(w) = c_0 + \sum_{n=1}^{\infty} \frac{c_n}{w^n}, \quad w \in \mathbb{C} \setminus \{0\}.$$

Looking at the Laurent Series of g in Remark 2.10, and bearing in mind that the Laurent Expansions are unique, we see that if g has a removable singularity at 0, then $c_n = 0$ for all $n \in \mathbb{N}$, and f is constant.

And by the same argument, if f has a pole of order m at infinity, then g has a pole of order m at 0, implying that $c_m \neq 0$ and $c_n = 0$ for all $n > m$.

□

2.1.2 The Cauchy Residues Theorem

In this subsection we will state results for an open set Ω, and a function f *holomorphic in Ω except in a set $S \subset \Omega$, where f has isolated singularities.* The precise meaning of this sentence is the following dichotomy:

(1) If $z \in \Omega \setminus S$, then there exists $\varepsilon > 0$ with $D(z, \varepsilon) \subset \Omega$ and f is holomorphic in $D(z, \varepsilon)$.

(2) If $z \in S$, then there exists $\varepsilon > 0$ with $D(z, \varepsilon) \subset \Omega$, f is holomorphic in the punctured disk $D(z, \varepsilon) \setminus \{z\}$, and $D(z, \varepsilon) \cap S = \{z\}$.

Before the Cauchy Residues Theorem, let us make some topological observations. In this section, for a subset $A \subset \mathbb{C}$, we will denote by A' **the set of accumulation points of** A. Recall that $z \in A'$ if and only if there exists $\{z_k\}_k \subset A$ with $z_k \neq z$ for all $k \in \mathbb{N}$, and $\lim_{k \to \infty} z_k = z$.

Proposition 2.12. *Let $\Omega \subset \mathbb{C}$ be open, and $S \subset \Omega$ a subset with $S' \cap \Omega = \emptyset$. The following hold.*

(i) S is countable.

(ii) $\Omega \setminus S = \Omega \setminus \overline{S}$ is an open set.

(iii) *If Γ is a cycle in Ω with $\Gamma \simeq 0$ in Ω, and $\Gamma^* \cap S = \emptyset$, then the set*

$$S_0 := \{z \in S \,:\, W(\Gamma, z) \neq 0\}$$

is finite.

(iv) *If, in addition, Ω is connected, then $\Omega \setminus S$ is connected too.*

Proof.

(i) For every $z \in \Omega$, there exists $r_z > 0$ so that $\overline{D}(z, r_z) \subset \Omega$. The intersection $S \cap \overline{D}(z, r_z)$ must be finite, as otherwise, by Bolzano-Weierstrass, this set would have an accumulation point $z \in S' \cap \overline{D}(z_n, r_n) \subset S' \cap \Omega$, a contradiction. The disks $\{D(z, r_z)\}_{z\in\Omega}$ form an open covering of Ω, and so there exists a countable subcovering $\{D(z_n, r_n)\}_{n\in\mathbb{N}}$. Therefore

$$S = S \cap \Omega = S \cap \left(\bigcup_{n\in\mathbb{N}} D(z_n, r_n) \right) = \bigcup_{n\in\mathbb{N}} S \cap D(z_n, r_n),$$

which is a countable union of finite sets, and thus S is countable.

(ii) We can write $\overline{S} = S \cup S'$, from which

$$\Omega \setminus \overline{S} = \Omega \setminus \big(S \cup S'\big) = (\Omega \setminus S) \cap \big(\Omega \setminus S'\big) = \Omega \setminus S.$$

This proves the identity between sets. Moreover, since $\Omega \setminus S = \Omega \cap \big(\mathbb{C} \setminus \overline{S}\big)$ is intersection of two open sets, we get that $\Omega \setminus S$.

(iii) By Remark 1.7, there exists $r > 0$ such that $\Gamma^* \subset D(0, r)$ and $W(\gamma, z) = 0$ for all $z \in \mathbb{C} \setminus D(0, r)$. This clearly implies that $S_0 \subset D(0, r)$, whence S_0 is bounded. Suppose, for the sake of contradiction, that S_0 is infinite. Since S_0 is also bounded, there exists $z \in S_0' \subset \mathbb{C}$, that is, there is a sequence $\{z_k\}_k \subset S_0$ convergent to $z \in \mathbb{C}$, with $z_k \neq z$ for all $k \in \mathbb{N}$. This point z does not belong to Ω, because otherwise we would have $z \in S' \cap \Omega$, contradicting that $S' \cap \Omega = \emptyset$. Thus $z \in \mathbb{C} \setminus \Omega$, and therefore $W(\Gamma, z) = 0$. Moreover, since Γ^* is compact and contained in Ω, and $z \notin \Omega$, we can find $\varepsilon > 0$ such that $D(z, \varepsilon) \cap \Gamma^* = \emptyset$. But $D(z, \varepsilon)$ is a connected subset of $\mathbb{C} \setminus \Gamma^*$, and we know that $W(\Gamma, z) = 0$, implying that $W(\Gamma, w) = 0$ for all $w \in D(z, \varepsilon)$. But this is a contradiction because there is z_k (actually infinitely many z_k's) contained in that disk, for which we had that $W(\Gamma, z_k) \neq 0$.

(iv) By Proposition 1.17, given any two points $z, w \in \Omega \setminus S$, let γ be a polygonal line contained in Ω that joins z and w. The trace γ^* is a compact subset of Ω, and so $\varepsilon := \mathrm{dist}(\gamma^*, \mathbb{C} \setminus \Omega) > 0$. Letting $\delta = \varepsilon/100$, we have that the compact set $K := \{z \in \Omega \,:\, \mathrm{dist}(z, \gamma^*) \leq \delta\}$ contains γ^* and is contained in Ω. Since $S' \cap \Omega = \emptyset$, the set K contains at most finitely many points of S. Therefore, since the endpoints of γ are not in S, it is clear that the segment lines of γ can modified to obtain a new path $\widetilde{\gamma}$ contained in K, and joining z and w. Thus, $\Omega \setminus S$ is connected.

□

Proposition 2.13. *Let $\Omega \subset \mathbb{C}$ be open, and f be holomorphic in Ω except in the set $S \subset \Omega$ consisting only of isolated singularities that are not removable. Then S has no accumulation points in S, that is, $S' \cap \Omega = \emptyset$.*

Proof. Assume, towards a contradiction, that there is $z_0 \in S' \cap \Omega$. Let $\{z_k\}_k \subset S$ be a sequence convergent to z_0 with $z_k \neq z_0$ for all $k \in \mathbb{N}$. There exists $\delta > 0$ so that $D(z, \delta) \subset \Omega$. Given $0 < \varepsilon < \delta$, there exists $z_k \in D(z_0, \varepsilon) \setminus \{z_0\}$, and so f cannot be holomorphic in $D(z_0, \varepsilon) \setminus \{z_0\}$, as the singularities $\{z_k\}$ for f are not removable. This means that f is not holomorphic in any of the punctured disks $D(z_0, \varepsilon) \setminus \{z_0\}$, with $0 < \varepsilon < \delta$. In other words, f has a singularity at $z \in \Omega$ that is not isolated, contradicting that S is the set of all singularities of f. □

Theorem 2.14 (Cauchy Residues Theorem)**.** *Let $\Omega \subset \mathbb{C}$ be open, and $f : \Omega \setminus S \to \mathbb{C}$ be holomorphic, where $S \subset \Omega$ is the set of isolated singularities of f. If Γ is a cycle with $\Gamma^* \subset \Omega \setminus S$, and $\Gamma \simeq 0$ in Ω, then $W(\Gamma, z) \neq 0$ for at most finitely-many $z \in S$ and*

$$\int_\Gamma f(w)\,\mathrm{d}w = 2\pi i \sum_{z \in S} W(\Gamma, z)\,\mathrm{Res}(f, z).$$

Proof. We may and do assume that all the points of S are poles or essential singularities (not removable), as $\mathrm{Res}(f, z) = 0$ when the singularity z is removable. By Propositions 2.13 and 2.12, we know that $S' \cap \Omega = \emptyset$, that $\Omega \setminus S$ is open, and that the set

$$S_0 = \{z \in Z \,:\, W(\Gamma, z) \neq 0\}$$

is finite. So, let $S_0 = \{z_1, \dots, z_n\}$ and $\delta > 0$ be such that the closed disks $\{\overline{D}(z_k, \delta)\}_{k=1}^n$ are contained in Ω, and are mutually disjoint. Denoting by γ_k the circular path $\partial D(z_k, \delta/2)$ traveled once and counterclockwise, for each $k \in \{1, \dots, n\}$, we define a new cycle Γ_0 by setting

$$\Gamma_0 := \sum_{k=1}^n W(\Gamma, z_k)\gamma_k := \{W(\Gamma, z_k)\gamma_k\}_{k=1}^n.$$

The integer $W(\Gamma, z_j) \in \mathbb{Z}$ in front of γ_k simply means that γ_k is repeated $W(\gamma, z_j)$ times in Γ_0, with positive or negative orientation depending on the sign of $W(\Gamma, z_j)$.

Our next claim is that $\Gamma \simeq \Gamma_0$ in $\Omega \setminus S$. Indeed, if $z \notin \Omega \setminus S$, then either $z \notin \Omega$, or $z \in S \setminus S_0$, or $z \in S_0$. In the first two cases, we use either the assumption $\Gamma \simeq 0$ in Ω or the definitions of S_0 and the γ_k's to write

$$W(\Gamma, z) = 0, \quad W(\Gamma_0, z) = \sum_{k=1}^n W(\Gamma, z_k) W(\gamma_k, z) = 0,$$

thus $W(\Gamma, z) = W(\Gamma_0, z)$. And in the case where $z \in S_0$, we have that $z = z_j$ for some $j \in \{1, \dots, n\}$. Since the disks $\{\overline{D}(z_k, \delta)\}_{k=1}^n$ are disjoint, we see that

$$W(\gamma_k, z) = \begin{cases} 1 & \text{if} \;\; k = j \\ 0 & \text{if} \;\; k \neq j. \end{cases}$$

Therefore, the winding numbers of Γ and Γ' satisfy

$$W(\Gamma_0, z) = W(\Gamma_0, z_j) = \sum_{k=1}^n W(\Gamma, z_k) W(\gamma_k, z_j) = W(\Gamma, z_j) = W(\Gamma, z).$$

We have shown that $\Gamma \simeq \Gamma_0$ in $\Omega \setminus S$, over which we can apply the Cauchy Homological Theorem (Corollary 1.14), for the holomorphic function f in $\Omega \setminus S$:

$$\int_\Gamma f(w)\,\mathrm{d}w = \int_{\Gamma_0} f(w)\,\mathrm{d}w = \int_{\sum_{k=1}^n W(\Gamma, z_k)\gamma_k} f(w)\,\mathrm{d}w = \sum_{k=1}^n W(\Gamma, z_k) \int_{\partial D(z_k, \delta/2)} f(w)\,\mathrm{d}w.$$

By Remark 2.2, the last term coincides with

$$2\pi i \sum_{k=1}^n W(\Gamma, z_k)\,\mathrm{Res}(f, z_k) = 2\pi i \sum_{z \in S} W(\Gamma, z)\,\mathrm{Res}(f, z),$$

and the last equality is due to the definition of S_0. $\square$

2.2 Meromorphic Functions

Definition 2.15 (Meromorphic Function)**.** *Let $\Omega \subset \mathbb{C}$ be open. We say that a function f is* ***meromorphic on*** *Ω if f is holomorphic in Ω except (possibly) for isolated singularities in Ω, all of which are* ***poles****. We denote by $\mathcal{M}(\Omega)$ the set of all holomorphic functions on Ω.*

Also, we denote the poles of f in Ω by $\mathcal{P}_\Omega(f)$, or sometimes simply by $\mathcal{P}(f)$, when we are considering only one open set Ω.

A meromorphic function may have finitely or infinitely many poles, and all holomorphic functions are meromorphic.

Example 2.16. (1) If $f : \mathbb{C} \to \mathbb{C}$ is a polynomial, then f is meromorphic in $\mathbb{C}$. Indeed, writing $f = P/Q$ with $P, Q : \mathbb{C} \to \mathbb{C}$ polynomials, the singularities of f (some of them possibly removable) are the roots of Q. This is a finite set, and in particular, all the singularities of f are isolated. Moreover, if z_0 is a roots of order $m \in \mathbb{N}$ of Q, then z_0 is either removable (when z_0 is a root of P of order at least m), or a pole (then the order of z_0 as a zero of P is smaller than m), as

$$\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} \frac{|P(z)|}{|Q(z)|} \in \mathbb{R} \cup \{\infty\}.$$

(2) The function $f(z) = \frac{1}{\sin z}$ is meromorphic in $\mathbb{C}$. Indeed, the singularities of f is the set $\{n\pi : n \in \mathbb{Z}\}$. This singularities are all isolated, and they are actually poles, as

$$\lim_{z \to n\pi} \left| \frac{1}{\sin z} \right| = \infty.$$

2.2.1 Operations and Properties

Proposition 2.17 (Properties of Meromorphic functions)**.** *If $\Omega \subset \mathbb{C}$ is a nonempty open set, the following properties hold.*

(i) If $f \in \mathcal{M}(\Omega)$, then $\mathcal{P}(f)$ is countable, $\mathcal{P}(f)' \cap \Omega = \emptyset$, the set $\Omega \setminus \mathcal{P}(f)$ is open, and $f \in \mathcal{H}(\Omega \setminus \mathcal{P}(f))$.

(ii) If $f, g \in \mathcal{M}(\Omega)$, and $\lambda \in \mathbb{C}$, then $f + \lambda g \in \mathcal{M}(\Omega)$.

(iii) If $f, g \in \mathcal{M}(\Omega)$, then $fg \in \mathcal{M}(\Omega)$ and $\mathcal{P}(fg) \subset \mathcal{P}(f) \cup \mathcal{P}(g)$.

(iv) If $f \in \mathcal{M}(\Omega)$ and f is not identically zero in any connected component of Ω, then $1/f \in \mathcal{M}(\Omega)$, and $\mathcal{P}(1/f) = \mathcal{Z}(f)$.

(v) If $f \in \mathcal{M}(\Omega)$, then $f' \in \mathcal{M}(\Omega)$ and $\mathcal{P}(f) = \mathcal{P}(f')$.

(vi) $f \in \mathcal{M}(\Omega)$ and f is not identically zero in any connected component of Ω, then

(vi)(a) $f'/f \in \mathcal{M}(\Omega)$, with $f'/f \in \mathcal{H}(\Omega \setminus \mathcal{Z}(f) \cup \mathcal{P}(f))$ and $\mathcal{P}(f'/f) = \mathcal{Z}(f) \cup \mathcal{P}(f)$.

(vi)(b) If $z \in \mathcal{Z}(f)$, with order $m_0(z) \in \mathbb{N} \cup \{0\}$, then f'/f has a pole of order 1 at z, with

$$\operatorname{Res}\left(f'/f, z\right) = m_0(z).$$

(vi)(c) If $z \in \mathcal{P}(f)$, with order $m_\infty(z) \in \mathbb{N} \cup \{0\}$, then f'/f has a pole of order 1 at z, with

$$\operatorname{Res}\left(f'/f, z\right) = -m_\infty(z).$$

Proof. (i) is a consequence of Propositions 2.13 and 2.12, and (ii) is immediate.

(*iii*) Assume the non-trivial case where none of f, g are identically zero in any connected component of Ω. If $z_0 \in \mathcal{P}(f) \cap \mathcal{P}(g)$, then clearly $z_0 \in \mathcal{P}(fg)$, as

$$\lim_{z \to z_0} |f(z)g(z)| = \infty;$$

see Proposition 2.5. With the same argument, we see that if $z_0 \in \mathcal{P}(f) \setminus \mathcal{P}(g)$ and $g(z_0) \neq 0$, then also $z_0 \in \mathcal{P}(fg)$. And if $z_0 \in \mathcal{P}(f) \setminus \mathcal{P}(g)$ and $g(z_0) = 0$, then there are $n, m \in \mathbb{N}$, and functions φ, ψ holomorphic in $D(z_0, \varepsilon)$ with $\varphi(z_0), \psi(z_0) \neq 0$, and

$$f(z)g(z) = \frac{\varphi(z)}{(z - z_0)^n} \cdot (z - z_0)^m \psi(z) = (z - z_0)^{m-n} \varphi(z)\psi(z), \quad z \in D(z_0, \varepsilon) \setminus \{z_0\}.$$

Therefore, z_0 is either a removable singularity or a pole of fg.

(*iv*) The possible singularities of $1/f$ in Ω is the set of points $S := \mathcal{Z}(f) \cup \mathcal{P}(f)$. The set $\mathcal{Z}(f)$ has no accumulation points in Ω, by the assumption on f. Therefore $S' \cap \Omega = \emptyset$, whence $1/f$ has only isolated singularities in Ω. To verify that these singularities are removable or poles, note that

$$\lim_{z \to z_0} \left| \frac{1}{f(z)} \right| = \begin{cases} 0 & \text{if } z_0 \in \mathcal{P}(f), \\ \infty & \text{if } z_0 \in \mathcal{Z}(f), \end{cases}$$

thanks to Proposition 2.5. Combining Theorem 2.4 and Proposition 2.5, we see that each $z_0 \in S$ is either a removable singularity or a pole for $1/f$, and, more precisely $\mathcal{Z}(1/f) = \mathcal{P}(f)$ and $\mathcal{P}(1/f) = \mathcal{Z}(f)$.

(*v*) Since holomorphic functions are infinitely-many times differentiable, if f is holomorphic in a disk around a point z_0, then the same occurs for f'. Thus, the possible singularities of f' is the set $\mathcal{P}(f)$, which has no accumulation points in Ω, and so the singularities of f' are all isolated. Now, let $z_0 \in \mathcal{P}(f)$ a pole of order m of f, and write the Laurent expansion of f in some disk $D(z_0, r) \setminus \{z_0\}$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m}, \quad 0 < |z - z_0| < r, \, b_m \neq 0.$$

The series converges uniformly on each sub-annulus $\{z \in \mathbb{C} : \varepsilon \leq |z - z_0| \leq \delta\}$, with $0 < \varepsilon < \delta \leq r$, and so we can differentiate termwise to obtain that

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1} + \frac{-b_1}{(z - z_0)^2} + \cdots + \frac{-m b_m}{(z - z_0)^{m+1}}, \quad 0 < |z - z_0| < r, \, b_m \neq 0.$$

Since Laurent Expansions are unique, the above shows that f' has a pole of order $m + 1$ at z_0. We conclude that $\mathcal{P}(f) = \mathcal{P}(f')$.

(*vi*) By parts $(iii), (iv), (v)$, we have that $f'/f \in \mathcal{M}(\Omega)$, and

$$\mathcal{P}(f'/f) \subset \mathcal{P}(f') \cup \mathcal{P}(f) \cup \mathcal{Z}(f) = \mathcal{P}(f) \cup \mathcal{Z}(f).$$

The reverse inclusion and parts $(vi)(b), (c)$ will be checked at the same time. If $z_0 \in \mathcal{Z}(f)$ of order $m \in \mathbb{N}$, we can write $f(z) = \varphi(z)(z - z_0)^m$ for all $z \in D(z_0, r)$ for some $r > 0$ and $\varphi \in \mathcal{H}(D(z_0, r))$ and $\varphi(z) \neq 0$ for all $z \in D(z_0, r)$. Differentiating f, and then dividing by f, we get

$$\frac{f'(z)}{f(z)} = \frac{\varphi'(z)(z - z_0)^m + m(z - z_0)^{m-1}\varphi(z)}{\varphi(z)(z - z_0)^m} = \frac{\varphi'(z)}{\varphi(z)} + \frac{m}{z - z_0}, \quad z \in D(z_0, r) \setminus \{z_0\}.$$

Since φ does not vanish in $D(z_0, r)$, the expression φ'/φ defines a holomorphic function in $D(z_0, r)$, and therefore the above shows that $z_0 \in \mathcal{P}(f'/f)$, with a pole of order 1, and $\mathrm{Res}(f'/f, z_0) = m$. This shows part (b) and that $\mathcal{Z}(f) \subset \mathcal{P}(f'/f)$.

Now, if $z_0 \in \mathcal{P}(f)$ is a pole of order $m \in \mathbb{N}$, then we can write $f(z) = \psi(z)(z - z_0)^{-m}$ for all $z \in D(z_0, r)$ for some $r > 0$ and $\psi \in \mathcal{H}(D(z_0, r))$ and $\psi(z) \neq 0$ for all $z \in D(z_0, r)$. Differentiating f, and then dividing by f, we get

$$\frac{f'(z)}{f(z)} = \frac{\psi'(z)(z-z_0)^{-m} - m(z-z_0)^{-m-1}\psi(z)}{\psi(z)(z-z_0)^{-m}} = \frac{\psi'(z)}{\psi(z)} - \frac{m}{z-z_0}, \quad z \in D(z_0, r) \setminus \{z_0\}.$$

Again, ψ'/ψ is holomorphic in $D(z_0, r)$, and thus the above tells us that f'/f has a pole of order 1 at z_0, with $\mathrm{Res}(f'/f, z_0) = -m$. This shows part (c), that $\mathcal{P}(f) \subset \mathcal{P}(f'/f)$, and consequently $\mathcal{P}(f') = \mathcal{P}(f) \cup \mathcal{Z}(f)$.

□

2.2.2 Meromorphic Functions in the Extended Complex Plane

Definition 2.18 (Meromorphic Functions at Infinity). *Let $\Omega_\infty \subset \mathbb{C}_\infty$ be an open set of $\mathbb{C}_\infty$, that is, either $\Omega_\infty \subset \mathbb{C}$ is open, or $\Omega_\infty = \{\infty\} \cup \mathbb{C} \setminus K$, with K compact. A function f* ***is meromorphic in*** *Ω_∞ if f is holomorphic in Ω_∞ except in a set of isolated singularities, all of which are either removable or poles.*

If f and Ω_∞ are as in Definition 2.18, and $\infty \notin \Omega_\infty$, then $\Omega = \Omega_\infty$ is an open subset of $\mathbb{C}$, and f is meromorphic in Ω in the regular sense (Definition 2.15). However if $\infty \in \Omega_\infty$, then $\Omega_\infty = \{\infty\} \cup \mathbb{C} \setminus K$, with $K \subset \mathbb{C}$ compact, and we have two possibilities.

- f is holomorphic at ∞ (see Definition 2.7), then f is holomorphic in some annulus $\mathbb{C} \setminus \overline{D}(0, R)$. The rest of the poles of f form a subset of $\mathbb{C} \setminus K$ with no accumulation points in $\mathbb{C} \setminus K$, and all of them contained in $\overline{D}(0, 2R)$. Therefore, f has only finitely-many poles in $\Omega \subset \mathbb{C}$.

- f has a pole at ∞. The function $g(w) = f(1/w)$ is holomorphic in $D(0, \varepsilon) \setminus \{0\}$. But then f is holomorphic $\mathbb{C} \setminus \overline{D}(0, 1/\varepsilon)$, implying that the rest of the poles of f are contained in $\overline{D}(0, 2/\varepsilon) \setminus K \subset \Omega_\infty$. Since these poles have no accumulation points in $\mathbb{C} \setminus K$, we conclude that the poles of f in Ω_∞ are ∞ and finitely-many points of $\mathbb{C}$.

Consequently, if f is meromorphic in Ω_∞ with $\infty \in \Omega_\infty$, then f has (at most) finitely-many poles.

Theorem 2.19 (Rational Functions). *A function f is meromorphic in $\mathbb{C}_\infty$ if and only if f is rational in $\mathbb{C}$.*

Proof. Assume first that $f = P/Q$, where $P(z) = \sum_{k=0}^n a_k z^k$, $Q(z) = \sum_{k=0}^m b_k z^k$, and $b_n, a_m \neq 0$. The possible poles of f are the roots of Q, and so they are finitely many. We know already that f is meromorphic in $\mathbb{C}$. To see that it is also meromorphic in $\mathbb{C}_\infty$, we look at the function $g(w) = f(1/w)$:

$$g(w) = \frac{w^m \left(a_0 w^n + a_1 w^{n-1} + \cdots + a_n\right)}{w^n \left(b_0 w^m + b_1 w^{m-1} + \cdots + b_m\right)},$$

and notice that the singularities of g at 0 are either removable (when $n \leq m$) or a pole (when $n > m$). Therefore f is meromorphic in $\mathbb{C}_\infty$.

To prove the converse, let f be meromorphic in $\mathbb{C}_\infty$. By the observation right after Definition 2.18, f has finitely many poles $z_1, \ldots, z_n \in \mathbb{C}$ and (possibly) a pole at ∞. In particular $f \in \mathcal{H}(\mathbb{C} \setminus \{z_1, \ldots, z_N\})$, and there exists $r > 0$ so that the disks $\{\overline{D}(z_j, 2r)\}_{j=1}^N$ are mutually disjoint. There is a pole of order $m_k \in \mathbb{N}$ at z_k, and so we have the decomposition

$$f(z) = \psi_k(z) + \varphi_k(z),\ z \in D(z_k, r) \setminus \{z_k\}; \ \text{ where } \ \varphi_k(z) = \sum_{n=1}^{m_k} \frac{c_n^k}{(z - z_k)^n},\ z \in \mathbb{C} \setminus \{z_k\}, \quad (2.2.1)$$

and $\psi_k \in \mathcal{H}(D(z_k, r))$ for all $k = 1, \ldots, N$. Also, f has a pole or order $m \in \mathbb{N} \cup \{0\}$ (let us call pole of order 0 a removable singularity this time) at ∞, meaning that $f(1/w)$ has a pole of order m at

0. Thus we can find $\varepsilon > 0$, a function $\psi_\infty \in \mathcal{H}(D(0,\varepsilon))$, and a polynomial $P : \mathbb{C} \to \mathbb{C}$ of degree N so that

$$f(1/w) = \psi_\infty(w) + P(1/w), \quad \text{for all} \quad 0 < |w| < \varepsilon. \tag{2.2.2}$$

We define a function $h : \mathbb{C} \to \mathbb{C}$ in the following manner

$$h(z) = \begin{cases} \psi_k(z) - \sum_{j=1,\, j\neq k}^N \varphi_j(z) - P(z) & \text{if } z \in D(z_k, r),\ k \in \{1, \ldots, N\} \\ f(z) - \sum_{j=1}^N \varphi_j(z) - P(z) & \text{if } z \in \mathbb{C} \setminus \{z_1, \ldots, z_N\}. \end{cases} \tag{2.2.3}$$

Since the disks $\{\overline{D}(z_j, 2r)\}_{j=1}^N$ are disjoint, the first branch of definition of h is consistent. Also, if $z \in \mathbb{C} \setminus \{z_1, \ldots, z_N\}$ and at the same time z belongs to $D(z_k, r)$ for some (unique) $k \in \{1, \ldots, N\}$, then (2.2.1) implies that

$$f(z) - \sum_{j=1}^N \varphi_j(z) - P(z) = \psi_k(z) + \varphi_k(z) - \sum_{j=1}^N \varphi_j(z) - P(z) = \psi_k(z) - \sum_{j=1,\, j\neq k}^N \varphi_j(z) - P(z);$$

showing that h is well defined in $\mathbb{C}$. Moreover, since $\varphi_j \in \mathcal{H}(\mathbb{C} \setminus \{z_j\})$ and $\psi_k \in \mathcal{H}(D(z_k, r))$ for all j, k, it is clear that h is holomorphic in $\mathbb{C}$. Now, note that (2.2.1) and (2.2.2) yield

$$\lim_{|z|\to\infty} |h(z)| = \lim_{|z|\to\infty} \left| f(z) - \sum_{j=1}^N \varphi_j(z) - P(z) \right| = \lim_{|z|\to\infty} |f(z) - P(z)| = \lim_{w\to 0} |\psi_\infty(w)| = |\psi_\infty(0)|,$$

by the continuity of ψ_∞ at 0. This proves that h is bounded in $\mathbb{C}$, and therefore constant, by virtue of Liouville's Theorem. We call this constant $w_0 \in \mathbb{C}$, and we obtain the decomposition

$$f(z) = w_0 + P(z) + \sum_{j=1}^N \varphi_j(z) = w_0 + P(z) + \sum_{j=1}^N \sum_{n=1}^{m_j} \frac{c_n^j}{(z - z_j)^n}, \quad z \in \mathbb{C} \setminus \{z_1, \ldots, z_N\}.$$

Thus f is a rational function. □

As a corollary of Theorem 2.19 (more precisely, of its proof), we deduce the following.

Corollary 2.20 (Partial Fraction Decomposition). *Every rational function f in $\mathbb{C}$ admits a decomposition of the form*

$$f(z) = P(z) + \sum_{j=1}^N \sum_{n=1}^{m_j} \frac{c_n^j}{(z - z_j)^n},$$

for a polynomial P, and constants $\{c_n^j : n = 0, \ldots, m_j,\ j = 1, \ldots, N\} \subset \mathbb{C}$, where f has a pole of order m_j at z_j, $j = 1, \ldots, N$.

Proof. It is the conclusion of the proof of Theorem 2.19. □

2.3 The Argument Principle

Theorem 2.21 (The Argument Principle). *Let $\Omega \subset \mathbb{C}$ be open, and $f \in \mathcal{M}(\Omega)$, such that f is not identically zero in any connected component of Ω. Let Γ be a cycle in Ω with $\Gamma \simeq 0$ in Ω, and such that $\Gamma^* \cap (\mathcal{Z}_\Omega(f) \cup \mathcal{P}_\Omega(f)) = \emptyset$. Then $W(\Gamma, z) \neq 0$ for at most finitely-many points $z \in \mathcal{Z}_\Omega(f) \cup \mathcal{P}_\Omega(f)$, and*

$$\frac{1}{2\pi i} \int_\Gamma \frac{f'(w)}{f(w)}\, \mathrm{d}w = \sum_{z \in \mathcal{Z}_\Omega(f)} m_0(z) W(\Gamma, z) - \sum_{z \in \mathcal{P}_\Omega(f)} m_\infty(z) W(\Gamma, z); \tag{2.3.1}$$

where $m_0(z)$ and $m_\infty(z)$ are respectively the order of z as a zero of f and as a pole of f.

Proof. We abbreviate $\mathcal{Z} = \mathcal{Z}_\Omega$ and $\mathcal{P} = \mathcal{P}_\Omega$. Since f is not null on any connected component of Ω, by the Identity Theorems for Holomorphic Functions, we have that $\mathcal{Z}(f)' \cap \Omega = \emptyset$. Also $\mathcal{P}(f)' \cap \Omega = \emptyset$, and by Proposition 2.12, we deduce that $W(\Gamma, z) \neq 0$ for at most finitely-many $z \in \mathcal{Z}(f) \cup \mathcal{P}(f)$. In particular, the two sums in the right-hand side of (2.3.1) are both finite.

By Proposition 2.17$(vi)(a)$, $f'/f \in \mathcal{H}\left(\Omega \setminus \mathcal{Z}(f) \cup \mathcal{P}(f)\right)$, where $\mathcal{Z}(f) \cup \mathcal{P}(f)$ is precisely the set of isolated singularities (all of them poles) of f'/f in Ω. By the assumption $\Gamma^* \cap (\mathcal{Z}(f) \cup \mathcal{P}(f)) = \emptyset$, and so we can apply the Residues Theorem 2.14 to f'/f in order to write

$$\begin{aligned}
\frac{1}{2\pi i} \int_\Gamma \frac{f'(w)}{f(w)} \,\mathrm{d}w &= \sum_{z \in \mathcal{Z}(f) \cup \mathcal{P}(f)} \operatorname{Res}\left(f'/f, z\right) W(\Gamma, z) \\
&= \sum_{z \in \mathcal{Z}(f)} m_0(z) W(\Gamma, z) - \sum_{z \in \mathcal{P}(f)} m_\infty(z) W(\Gamma, z),
\end{aligned}$$

after applying Proposition 2.17$(vi)(b),(c)$. □

We can multiply by a holomorphic function in (2.3.1) and still get a similar formula.

Theorem 2.22. *Let $\Omega \subset \mathbb{C}$ be open, $g \in \mathcal{H}(\Omega)$, and $f \in \mathcal{M}(\Omega)$, such that f is not identically zero in any connected component of Ω. Let Γ be a cycle in Ω with $\Gamma \simeq 0$ in Ω, and such that $\Gamma^* \cap (\mathcal{Z}_\Omega(f) \cup \mathcal{P}_\Omega(f)) = \emptyset$. Then,*

$$\frac{1}{2\pi i} \int_\Gamma g(w) \frac{f'(w)}{f(w)} \,\mathrm{d}w = \sum_{z \in \mathcal{Z}_\Omega(f)} m_0(z) W(\Gamma, z) g(z) - \sum_{z \in \mathcal{P}_\Omega(f)} m_\infty(z) W(\Gamma, z) g(z); \tag{2.3.2}$$

where $m_0(z)$ and $m_\infty(z)$ are respectively the order of z as a zero of f and as a pole of f.

Proof. Consider the function $h := g\frac{f'}{f}$. Since $g \in \mathcal{H}(\Omega)$, and $f \in \mathcal{M}(\Omega)$, by Proposition 2.17$(vi)$ we get that $h \in \mathcal{H}(\Omega)$, with $\mathcal{P}(h) \subset \mathcal{P}(f'/f) = \mathcal{Z}(f) \cup \mathcal{P}(f)$. By the same proposition, the poles of f'/f are all of order 1. So, $z \in \mathcal{P}(f'/f)$ will be a pole of h if and only if $g(z) \neq 0$. In other words, $\mathcal{P}(h) = \mathcal{P}(f'/f) \setminus \mathcal{Z}(g)$, and when $z \in \mathcal{Z}(g)$, the function h has a removable singularity at z, and so $\operatorname{Res}(h, z) = 0$. On the other hand, if $z \in \mathcal{P}(f'/f)$ and $g(z) \neq 0$, then h has a pole of order 1 at z. These observations and Proposition 2.17(vi) tell us that, regardless of the value of $g(z)$, we have that

$$z \in \mathcal{Z}(f) \cup \mathcal{P}(f) \implies \operatorname{Res}(h, z) = g(z) \operatorname{Res}(f'/f, z) = \begin{cases} g(z) m_0(z) & \text{if } z \in \mathcal{Z}(f) \\ -g(z) m_\infty(z) & \text{if } z \in \mathcal{P}(f). \end{cases}$$

Since $\Gamma^* \cap \mathcal{P}(h) = \emptyset$, and $\mathcal{P}(h) \subset \mathcal{Z}(f) \cup \mathcal{P}(f)$ (as we saw in the proof of Theorem 2.21) has no accumulation points in Ω, we can apply the Residues Theorem 2.14 to h:

$$\begin{aligned}
\frac{1}{2\pi i} \int_\Gamma g(w) \frac{f'(w)}{f(w)} \,\mathrm{d}w &= \sum_{z \in \mathcal{P}(h)} \operatorname{Res}(h, z) W(\Gamma, z) = \sum_{z \in \mathcal{Z}(f) \cup \mathcal{P}(f)} \operatorname{Res}(h, z) W(\Gamma, z) \\
&= \sum_{z \in \mathcal{Z}(f)} m_0(z) W(\Gamma, z) g(z) - \sum_{z \in \mathcal{P}(f)} m_\infty(z) W(\Gamma, z) g(z).
\end{aligned}$$

□

2.4 Rouché's Theorem

Theorem 2.23 (Rouché's Theorem)**.** *Let $\Omega \subset \mathbb{C}$ be open, and $f, g \in \mathcal{M}(\Omega)$ functions that are not identically zero in any connected component of Ω. Let Γ be a cycle in Ω with $\Gamma \simeq 0$ in Ω, and such that $\Gamma^* \cap \mathcal{P}_\Omega(f) = \emptyset$. Assume further that*

$$|f(z) - g(z)| < |f(z)|, \quad \textit{for all} \quad z \in \Gamma^*. \tag{2.4.1}$$

Then, we have that

$$\sum_{z\in\mathcal{Z}_\Omega(f)} m_0(f,z)W(\Gamma,z) - \sum_{z\in\mathcal{P}_\Omega(f)} m_\infty(f,z)W(\Gamma,z)$$

$$= \sum_{z\in\mathcal{Z}_\Omega(g)} m_0(g,z)W(\Gamma,z) - \sum_{z\in\mathcal{P}_\Omega(g)} m_\infty(g,z)W(\Gamma,z); \tag{2.4.2}$$

where $m_0(f,z)$, $m_0(g,z)$, *and* $m_\infty(f,z)$, $m_\infty(g,z)$ *are respectively the orders of* z *as zeros of* f, g *and as a poles of* f, g.

Proof. Let us abbreviate $\mathcal{Z} = \mathcal{Z}_\Omega$ and $\mathcal{P} = \mathcal{P}_\Omega$. Observe that (2.4.1) implies $(\mathcal{Z}(g)\cup\mathcal{Z}(f))\cap\Gamma^* = \emptyset$. Similarly $\mathcal{P}(g)\cap\Gamma^* = \emptyset$, as the condition (2.4.1) and $\mathcal{P}(f)\cap\Gamma^* = \emptyset$ tell us that $\lim_{|z|\to\infty} |g(z)| \neq \infty$. Thus can apply Theorem 2.21 to both f and g. Before doing so, consider the function $h = g/f$, which is meromorphic in Ω by virtue of Proposition 2.17$(iii), (iv)$. The possible zeros or poles of h in Ω are contained in the set

$$A := \mathcal{Z}(f)\cup\mathcal{P}(f)\cup\mathcal{Z}(g)\cup\mathcal{P}(g).$$

But $A \subset \Omega$ has no accumulation points within Ω, and A does not intersect the compact set $\Gamma^* \subset \Omega$. Thus there exists $\varepsilon > 0$ so that the open set

$$U_\varepsilon = \{z \in \mathbb{C} : \operatorname{dist}(z, \Gamma^*) < \varepsilon\}$$

contains Γ^*, is contained in Ω, and still $A \cap U_\varepsilon = \emptyset$. Therefore $h \in \mathcal{H}(U_\varepsilon)$, and if $\Gamma = \{\gamma_j\}_{j=1}^N$, where each $\gamma_j : [a, b] \to \Omega$ is a closed and piecewise C^1-path, we can define new closed and piecewise C^1-paths by setting

$$\sigma_j : [a_j, b_j] \to \mathbb{C}, \quad \sigma_j(t) := h(\gamma_j(t)), \quad t \in [a_j, b_j], \quad j = 1, \dots, N.$$

By the assumption (2.4.1), each σ_j satisfies that

$$|\sigma_j(t) - 1| = \left|\frac{g(\gamma_j(t))}{f(\gamma_j(t))} - 1\right| < 1, \quad t \in [a_j, b_j].$$

This shows that $\sigma_j^* \subset D(1, 1)$. Therefore $\mathbb{C} \setminus D(1, 1)$ is contained in the unbounded connected component of σ_j, where $W(\sigma_j, \cdot)$ is identically zero. Therefore $W(\sigma_j, 0) = 0$ for all $j = 1, \dots, N$. But this yields

$$0 = \sum_{j=1}^N W(\sigma_j, 0) = \frac{1}{2\pi i}\sum_{j=1}^N \int_{\sigma_j} \frac{\mathrm{d}w}{w} = \frac{1}{2\pi i}\sum_{j=1}^N \int_{a_j}^{b_j} \frac{\sigma_j'(t)}{\sigma_j(t)}\,\mathrm{d}t = \frac{1}{2\pi i}\sum_{j=1}^N \int_{a_j}^{b_j} \frac{h'(\gamma_j(t))\gamma_j'(t)}{h(\gamma_j(t))}\,\mathrm{d}t$$

$$= \frac{1}{2\pi i}\sum_{j=1}^N \int_{\gamma_j} \frac{h'(w)}{h(w)}\,\mathrm{d}w = \frac{1}{2\pi i}\int_\Gamma \frac{h'(w)}{h(w)}\,\mathrm{d}w = \frac{1}{2\pi i}\int_\Gamma \frac{g'(w)}{g(w)}\,\mathrm{d}w - \frac{1}{2\pi i}\int_\Gamma \frac{f'(w)}{f(w)}\,\mathrm{d}w.$$

We derive that

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(w)}{f(w)}\,\mathrm{d}w = \frac{1}{2\pi i}\int_\Gamma \frac{g'(w)}{g(w)}\,\mathrm{d}w,$$

and (2.4.2) follows by applying Theorem 2.21 to both f and g. □

Corollary 2.24 (Rouché's Localization of Zeros). *Let* $\Omega \subset \mathbb{C}$ *be open, and* $f, g \in \mathcal{H}(\Omega)$ *functions that are not identically zero in any connected component of* Ω. *Let* Γ *be a cycle in* Ω *with* $\Gamma \simeq 0$ *in* Ω. *Assume further the conditions:*

(a) $|f(z) - g(z)| < |f(z)|$ *for all* $z \in \Gamma^*$, $\quad (b)$ $W(\Gamma, z) \in \{0, 1\}$ *for all* $z \in \mathbb{C} \setminus \Gamma^*$.

If we denote $\Omega_1 = \{z \in \Omega : W(\Gamma, z) = 1\}$, *then*

$$\mathcal{N}_1(f) = \mathcal{N}_1(g),$$

where $\mathcal{N}_1(f)$, $\mathcal{N}_1(g)$ *denote the number of zeros in* Ω_1 *of* f *and* g *respectively, and counted with multiplicity.*

Proof. All the conditions of Theorem 2.23 are satisfied. $\square$

Let us apply Corollary 2.24 to a concrete function.

Example 2.25. For the polynomial $P(z) = z^{10} - 3z^9 + 7z^4 + z - 1$, we want to find the number of zeros of P are in the open unit disk $\mathbb{D}$. Consider the function $f(z) = 7z^4$, and the path γ describing the unit circle $\partial D(0,1)$ traveled once and counterclockwise. Notice that, if $z \in \gamma^* = \partial D(0,1)$, then

$$|P(z) - f(z)| = |z^{10} - 3z^9 + z - 1| \leq |z|^{10} + 3|z|^9 + |z| + 1 \leq 6 < |7z^4| = |f(z)|.$$

All the assumptions of Corollary 2.24 are satisfied, and so P and f have the same number of zeros (counted with multiplicity) in the set $\{z \in \mathbb{D} : W(\gamma, z) = 1\} = \mathbb{D}$. Since f has exactly 4 zeros in $\mathbb{D}$, we may conclude that P has also 4 zeros in $\mathbb{D}$.

2.5 Hurwitz's Theorem

The following theorem says that the number of zeros of a sequence of holomorphic functions is essentially *stationary.*

Theorem 2.26 (Hurwitz's Theorem)**.** *Let* Ω *be open and connected,* D *a disk with* $\overline{D} \subset \Omega$, $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}(\Omega)$, *and* $f : \Omega \to \mathbb{C}$ *a function so that* $\{f_n\}_n$ *converges to* f *uniformly on compact subsets of* Ω. *Then* $f \in \mathcal{H}(\Omega)$, *and if* $\mathcal{Z}_\Omega(f) \cap \partial D = \emptyset$, *then there exists* $N \in \mathbb{N}$ *so that*

$$\mathcal{N}_D(f_n) = \mathcal{N}_D(f) \quad \textit{for all} \quad n \geq N;$$

where $\mathcal{N}_D(f_n)$, $\mathcal{N}_D(f)$ *denote the number of zeros of* f_n *and* f *within* D, *counted with multiplicity.*

Proof. That f is holomorphic is a consequence of a part of Weierstrass's Convergence Theorem. We recall the proof of this fact. Clearly f is continuous in Ω, as locally uniformly limit of continuous functions. And for every solid triangle $\Delta \subset \Omega$, we have that

$$\int_{\partial D} f = \lim_{n\to\infty} \int_{\partial D} f_n = 0;$$

where the first equality is due to the uniform convergence in Δ, and the latter to the Cauchy Theorem in a triangle for the f_n's (or by Corollary 1.14, as clearly $\partial\Delta \simeq 0$ in Ω). By Morera's Theorem, $f \in \mathcal{H}(\Omega)$.

We proceed with the second part of the theorem. Writing $D = D(w_0, r)$, there exists $\varepsilon > 0$ with the property that $\overline{D}(w_0, r+\varepsilon) \subset \Omega$, as $\overline{D} \subset \Omega$. Since $\mathcal{Z}_\Omega(f) \cap \partial D = \emptyset$, we have that

$$m := \min\{|f(z)| : z \in \partial D\} > 0.$$

The sequence $\{f_n\}_n$ converges uniformly to f in ∂D, and so we can find $N \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < m \leq |f(z)| \quad \text{for all} \quad z \in \partial D,\ n \geq N.$$

We can thus apply Corollary 2.24 for f, f_n, γ the path ∂D, the domain $D(w_0, r+\varepsilon)$, and $\Omega_1 = D$, to infer that $\mathcal{N}_D(f_n) = \mathcal{N}_D(f)$ for all $n \geq N$. $\square$

Corollary 2.27. *Let Ω be open and connected, $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}(\Omega)$, and $f : \Omega \to \mathbb{C}$ a function so that $\{f_n\}_n$ converges to f uniformly on compact subsets of Ω. If $\mathcal{Z}_\Omega(f_n) = \emptyset$ for all $n \in \mathbb{N}$, then either $f \equiv 0$ in Ω or $\mathcal{Z}_\Omega(f) = \emptyset$.*

Proof. The function f is holomorphic in Ω, as we saw in the proof of Theorem 2.26. Assume, for the sake of contradiction, that f is not null in Ω, but still there exists $z_0 \in \Omega$ with $f(z_0) = 0$. Since Ω is connected, and $f \not\equiv 0$ in Ω, by the Identity Principles for Analytic Functions, the zero z_0 of f is necessarily isolated. Thus there exists an open disk D centered at z_0, with $\overline{D} \subset \Omega$, and such that $\mathcal{Z}_{\partial D}(f) = \emptyset$. Applying Hurwitz's Theorem 2.26 to $\{f_n\}_n$, f, Ω, and the disk D, we deduce that f has the same number of zeros (counted with multiplicity) in D as f_n, for some $n \in \mathbb{N}$. But since z_0 is one of those zeros of f, this contradicts the fact that $\mathcal{Z}_\Omega(f_n) = \emptyset$. $\square$

2.6 Exercises

Exercise 2.1. *Let $f \in \mathcal{H}(\mathbb{C} \setminus \{0\})$ be a function with an isolated singularity at 0. Prove that the function $f(z) - \frac{\mathrm{Res}(f,0)}{z}$ has a primitive in $\mathbb{C} \setminus \{0\}$.*

Exercise 2.2. *Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant holomorphic in $\mathbb{C}$. Prove that the function $\mathbb{C} \ni z \mapsto e^{f(z)}$ has an essential singularity at ∞.*

Exercise 2.3. *Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $N \in \mathbb{N}$, and let $r > N$. Obtain a closed formula (depending on P and N) for the integral*

$$\int_{\partial D(0,r)} P(z) \sum_{n=0}^{N} e^{\frac{1}{z-n}}\, \mathrm{d}z.$$

Suggestion: Classify the singularities of the integrand, and apply the Cauchy Residues Theorem.

Exercise 2.4. *Use the Cauchy Residues Theorem to find*

$$\mathrm{pv} \int_{-\infty}^{\infty} \frac{e^{ix}}{x}.$$

Use this to deduce that

$$\int_0^{\infty} \frac{\sin x}{x}\, \mathrm{d}x = \frac{\pi}{2}.$$

Suggestion: Integrate the function $f(z) = e^{iz}/z$ in the path $\Gamma_{R,\varepsilon}$ given by an appropriate composition of

$$\gamma_R(t) = Re^{it},\ \gamma_\varepsilon(t) = \varepsilon e^{it} \quad t \in [0,\pi]; \quad L_{\varepsilon,R} := [-R,-\varepsilon], \quad H_{\varepsilon,R} := [\varepsilon, R].$$

You can use directly material from [5, Theorem 5.38, Remark 5.39].

Exercise 2.5. *Find the principal values*

$$\mathrm{pv} \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2}\, \mathrm{d}x.$$

Suggestion: Combine the outcome of Exercise 2.4 with integration by parts.

Exercise 2.6. *Find the principal value*

$$\mathrm{pv}\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2(1+x^2)}\,\mathrm{d}x.$$

Suggestion: Take into account that $2\sin^2 x = 1-\cos(2x)$, *and integrate* $f(z)=\frac{1-e^{2iz}}{z^2(1+z^2)}$ *in the same paths as in Exercise 2.4.*

Exercise 2.7. *Use the Cauchy Residues Theorem to calculate, for each* $n\in\mathbb{N}$, *the principal value*

$$\mathrm{pv}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{1+x^{2n}}.$$

Suggestion: Integrate the function $f(z)=(1+z^{2n})^{-1}$ *on the circular sector consisting of the segment* $[0,R]$, *the arc* $\{Re^{it} : t\in[0,\pi/n]\}$, *and the segment* $[0,Re^{i\pi/n}]$.

Exercise 2.8. *Use the Cauchy Residues Theorem to prove that*

$$\mathrm{pv}\int_{-\infty}^{\infty}\frac{e^{-2\pi i\xi x}}{e^{\pi x}+e^{-\pi x}}\,\mathrm{d}x=\frac{1}{e^{\pi\xi}+e^{-\pi\xi}},\qquad \xi\in\mathbb{R}.$$

Deduce that the Fourier Transform $\mathcal{F}(g)$ *of the real function* $x\mapsto g(x)=\mathrm{sech}(\pi x)=(\cosh(\pi x))^{-1}$ *is precisely* g.

Suggestion: For $\xi>0$, *integrate the function* $f(z)=\frac{e^{-2\pi i\xi z}}{e^{\pi z}+e^{-\pi z}}$ *in the rectangle of vertices* $-R$, R, $R+2i$, $-R+2i$. *For* $\xi=0$, *you can for example integrate the same function in the rectangle with half the height of the previous one.*

Exercise 2.9. *Use the Cauchy Residues Theorem to prove that*

$$\mathrm{pv}\int_{0}^{\infty}\frac{x^{\alpha-1}}{x+1}\,\mathrm{d}x=\frac{\pi}{\sin(\alpha\pi)},\qquad 0<\alpha<1.$$

Suggestion: Consider the holomorphic branch $\log z$ *of the logarithm in* $\mathbb{C}\setminus[0,+\infty)$, *the power function* $z^{\alpha-1}=e^{(\alpha-1)\log z}$, *and the function* $f(z)=z^{\alpha-1}/(z+1)$. *Integrate* f *in the closed path* $\Gamma_{R,\varepsilon,\delta}$ *given by an appropiate composition (possibly taking reverse paths) of the paths:*

$$\gamma_{R,\delta}(t):=Re^{it},\ \gamma_{\varepsilon,\delta}(t):=\varepsilon e^{it},\quad t\in[\delta,2\pi-\delta],\quad L_{R,\varepsilon,\delta}:=[\varepsilon e^{i\delta},Re^{i\delta}],\quad H_{R,\varepsilon,\delta}:=[\varepsilon e^{-i\delta},Re^{-i\delta}].$$

Then let $R\to\infty$, $\varepsilon,\delta\to 0^+$.

Exercise 2.10. *Use the Cauchy Residues Theorem to find*

$$\mathrm{pv}\int_{0}^{\infty}\frac{\log x}{(x+1)^3}\,\mathrm{d}x.$$

Suggestion: Consider the same branch of logarithm and paths as in Exercise 2.9.

Exercise 2.11. *Use the Cauchy Residues Theorem to find*

$$\mathrm{pv}\int_{0}^{\infty}\frac{\log x}{x^2-1}\,\mathrm{d}x.$$

Suggestion: Consider the same branch of logarithm and paths as those in Exercise 2.9, but replacing $H_{R,\varepsilon,\delta}:=[\varepsilon e^{-i\delta},Re^{-i\delta}]$ *with the composition of*

$$\ell^1_{\varepsilon,\delta,\eta}:=\left[\varepsilon e^{-i\delta},(1-\tfrac{\eta}{2})e^{-i\delta}\right],\quad \sigma_{\tau,\delta}(t)=(1-\tfrac{\eta}{2}e^{it})e^{-i\delta},\ t\in[0,\pi],\quad \ell^2_{\delta,\eta,R}=\left[(1+\tfrac{\eta}{2})e^{-i\delta},Re^{-i\delta}\right],$$

where $\eta,\varepsilon,\delta\to 0$ *and* $R\to\infty$.

Exercise 2.12. *Let $w \in \mathbb{C} \setminus \mathbb{Z}$. Given $N \in \mathbb{N}$ with $N \geq |w| + 100$, let R_N be the rectangle with vertices*

$$-(N+\tfrac{1}{2}) - i(N+\tfrac{1}{2}), \quad -(N+\tfrac{1}{2}) + i(N+\tfrac{1}{2}), \quad (N+\tfrac{1}{2}) + i(N+\tfrac{1}{2}), \quad (N+\tfrac{1}{2}) - i(N+\tfrac{1}{2}),$$

*and positive orientation. Also consider $\gamma_- \equiv \partial D(-w, \varepsilon)$ and $\gamma_+ \equiv \partial D(w, \varepsilon)$ with negative orientation, where $2\varepsilon = \operatorname{dist}(w, \mathbb{Z})$. For $\Gamma_N = \gamma_- \cup \gamma_+ \cup R_N$, use **Theorem 2.22** on $\Omega := \mathbb{C} \setminus \{\pm w\}$, to deduce that*

$$\frac{1}{2\pi i} \int_{\Gamma_N} \frac{\pi \cot(\pi z)}{w^2 - z^2} \, \mathrm{d}z = \sum_{n=-N}^{N} \frac{1}{w^2 - n^2}.$$

Then calculate the integral above, and let $N \to \infty$ to deduce that

$$\frac{\pi \cot(\pi w)}{w} = \sum_{n=-\infty}^{\infty} \frac{1}{w^2 - n^2}.$$

Suggestion: To calculate the integral, it is convenient to show first that $\sup\{|\cot(\pi z)| : z \in R_N,\ N \in \mathbb{N}\} < \infty$.

Exercise 2.13. *Find the number of solutions to the equation $z^7 - 4z^3 + z - 1 = 0$ within the open unit disk $D(0, 1)$.*

Exercise 2.14. *Find the number of solutions to the equation $z^4 - 5z + 1 = 0$ within the open sets:*

(a) $\Omega = D(0, 1)$.

(b) $\Omega = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

Exercise 2.15. *Given $0 < r < 1$, show that there exists $N \in \mathbb{N}$ so that, for each $n \geq N$, the function*

$$f_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}, \quad z \in \mathbb{C},$$

has no zeros inside the open disk $D(0, r)$.

Exercise 2.16. *Given $r > 0$, show that there exists $N \in \mathbb{N}$ so that, for each $n \geq N$, all the zeros of the function*

$$f_n(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots + \frac{1}{n!z^n}, \quad z \in \mathbb{C} \setminus \{0\},$$

are inside the open disk $D(0, r)$.

Suggestion: Compare $g_n(w) = f_n(1/w)$ with the exponential e^w in circles centered at the origin, and apply Rouché's Corollary 2.24.

Exercise 2.17. *Let $\Omega \subset \mathbb{C}$ be open, with $\overline{D}(0, 1) \subset \Omega$ and $f \in \mathcal{H}(\Omega)$ so that $|f(z)| < 1$ for all $z \in \partial D$. Find the number of solutions to the equation $f(z) = z^n$ in $D(0, 1)$.*

Exercise 2.18. *Prove that there exists no nonconstant polynomial $P : \mathbb{C} \to \mathbb{C}$ with $|P(z)| < 1$ for all $|z| = 1$.*

Exercise 2.19. *Given $\lambda > 1$, show that the equation $e^{-z} + z = \lambda$ has precisely one solution z_λ in the half-plane $\{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$, and this solution $z_\lambda \in \mathbb{R}$. In addition, prove that $\lim\limits_{\lambda \to 1^-} z_\lambda = 0$.*

Exercise 2.20. *Let* Ω *be open and connected, and* $\{f_n\}_{n\geq 1} \subset \mathcal{H}(\Omega)$ *converging uniformly on compact sets to a function* f *(necessarily holomorphic in* Ω*). Assume further that each* f_n *is injective in* Ω*. Prove that either* f *is constant in* Ω *or else* f *is injective in* Ω*.*

Suggestion: Use Corollary 2.27 for appropriate functions.

Exercise 2.21. *Let* Ω *be open and connected, and* $\{f_n\}_{n\geq 1} \subset \mathcal{H}(\Omega)$ *converging uniformly on compact sets to a nonconstant function* $f_0 \in \mathcal{H}(\Omega)$*. Prove that for every* $z_0 \in \Omega$ *there exists* $\{z_n\}_{n\geq 1} \subset \Omega$ *and* $N \in \mathbb{N}$ *so that*

$$\lim_{n\to\infty} z_n = z_0, \quad \textit{and} \quad f_n(z_n) = f_0(z_0) \quad \textit{for all} \quad n \geq N.$$

Suggestion: Use Hurwitz's Theorem 2.26 for appropriate functions.

Chapter 3

Convergence of Holomorphic Functions

Weierstrass's Convergente Theorem says that the *locally uniform limit* of holomorphic functions is holomorphic, and that the sequence of derivatives converges to the derivatives of the limit, also locally uniformly. We wish to determine whether a sequence $\{f_n\}_n$ of holomorphic functions has a (locally-uniform) convergent subsequence. The first observation is that, since every disk of $\mathbb{C}$ is relatively compact, the locally uniform convergence within an open set Ω is equivalent to the uniform convergence on each compact subset of Ω. Then, we will equip the space $C(\Omega, \mathbb{C})$ of all continuous functions $f : \Omega \to \mathbb{C}$ with a topology, on which the convergence is exactly the same as the convergence in compact subsets of Ω. This is achieved via the *Compact-Open Topology.* This topology is metrizable, meaning that its open sets can be described as union of open balls with respect to some distance ρ, thus allowing us to use sequential criteria for closures and compactness of subsets $\mathcal{F}$ of $C(\Omega, \mathbb{C})$. But the key theorem is that of *Arzelà-Ascoli*, which provides a characterization of the relatively compact subsets of $C(\Omega, \mathbb{C})$, in terms of *equicontinuity* and *local boundedness.* As shown by *Montel's Theorem* 3.16, a *locally bounded* family of holomorphic functions is equicontinuous, and therefore it has a subsequence converging uniformly on compact subsets. We also record *Vitali's Theorem* 3.19, where we obtain converge of the original sequence (instead of a subsequence). Finally, *Osgood's Theorem* 3.20 states that the pointwise limit of holomorphic functions is holomorphic in a dense open subset, and the converge is locally uniform within that set.

We begin by reminding the Weierstrass's Convergence Theorem.

Theorem 3.1 (Weierstrass Theorem)**.** *Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \to \mathbb{C}$ a function, and let $\{f_k : \Omega \to \mathbb{C}\}_k$ be sequence of holomorphic functions in Ω converging locally uniformly to f in Ω. Then,*

(i) f is holomorphic in Ω.

(ii) For every $n \in \mathbb{N}$, the sequence of n^{th}-derivatives $\{f_k^{(n)} : \Omega \to \mathbb{C}\}_k$ converges locally uniformly in Ω to the n^{th}-derivative $f^{(n)}$ of f.

Proof. See [5, Theorem 4.37] □

Again, it is worth pointing out that **locally uniform convergence** in Ω is exactly the same as the **uniform convergence on each compact subset** of Ω.

3.1 The Compact-Open Topology

In this section we construct the Compact-Open Topology, and the corresponding metrics. We allow for the continuous functions to the metric valued (X, d), instead of only $\mathbb{C}$-valued. We remind below the definitions of metric and pseudometric spaces.

3.1.1 Nested families of Compact Sets

Definition 3.2 (Nested Families of Compact Sets)**.** *Let $\Omega \subset \mathbb{C}$ be an open set. A* ***nested family of compact sets in*** *Ω is a sequence $\{K_n\}_{n\in\mathbb{N}}$ of compact subsets of Ω satisfying the properties*

- $K_n \subset \operatorname{int}(K_{n+1})$ *for all* $n \in \mathbb{N}$.
- $\bigcup_{n\in\mathbb{N}} K_n = \bigcup_{n\in\mathbb{N}} \operatorname{int}(K_n) = \Omega$.

Consequently, for any compact $K \subset \Omega$, there exists $n \in \mathbb{N}$ with $K \subset K_n$.

Although definitely not unique, each open set has a nested family of compact sets.

Proposition 3.3. *For every $\Omega \subset \mathbb{C}$ open, there exists a nested family of compact sets in Ω.*

Proof. We define

$$K_n := \overline{D}(0,n) \cap \{z \in \Omega \,:\, \operatorname{dist}(z, \mathbb{C}\setminus\Omega) \geq 1/n\}, \quad n \in \mathbb{N}.$$

Obviously $K_n \subset \Omega$ and K_n is bounded. That K_n is closed follows from the continuity of the distance function $\mathbb{C} \ni z \mapsto \operatorname{dist}(z, \mathbb{C}\setminus\Omega)$. Also observe that Also, since $\{z \in \Omega \,:\, \operatorname{dist}(z, \mathbb{C}\setminus\Omega) \geq 1/n\}$ is open, we have that

$$\begin{aligned}
\operatorname{int}(K_{n+1}) &= \operatorname{int}\left(\overline{D}(0,n+1)\right) \cap \operatorname{int}\left(\{z \in \Omega \,:\, \operatorname{dist}(z, \mathbb{C}\setminus\Omega) \geq 1/(n+1)\}\right) \\
&\supset \overline{D}(0,n) \cap \{z \in \Omega \,:\, \operatorname{dist}(z, \mathbb{C}\setminus\Omega) > 1/(n+1)\} \\
&\supset \overline{D}(0,n) \cap \{z \in \Omega \,:\, \operatorname{dist}(z, \mathbb{C}\setminus\Omega) \geq 1/n\} = K_n.
\end{aligned}$$

Finally, for every $z \in \Omega$, there exists $\varepsilon > 0$ with $D(z,\varepsilon) \subset \Omega$, whence $\operatorname{dist}(z, \mathbb{C}\setminus\Omega) \geq \varepsilon$. If $n \in \mathbb{N}$ is large enough so that $|z| \leq n$ and $\varepsilon > 1/n$, we have that $z \in K_n$. This proves the properties of Definition 3.2 for $\{K_n\}_{n\in\mathbb{N}}$. □

We will construct an appropriate distance between continuous functions associated with a nested family of compact sets. First we recall the definitions metric and pseudometric.

Definition 3.4 (Metric and Pseudometric)**.** *If A is a set, a* ***pseudometric or pseudodistance in*** *A is a function $\rho : A \times A \to [0, +\infty)$ so that*

- $\rho(x,x) = 0$ *for all $x \in A$. [Reflexivity]*
- $\rho(x,y) = \rho(y,x)$ *for all $x, y \in A$. [Symmetry]*
- $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ *for all $x, y, z \in A$. [Triangle Inequality]*

We often say that then (A,ρ) is a ***pseudometric space****.*

If, in addition, ρ has the property

$$\rho(x,y) = 0 \implies x = y, \quad x, y \in A;$$

then we say that ρ is a ***metric or distance in*** *A, and that (A,ρ) is a* ***metric space****.*

The following lemma permits to create bounded distances from an arbitrary metric.

Lemma 3.5. *Let (A,ρ) be a pseudometric space. Then*

$$\widetilde{\rho}(x,y) := \frac{\rho(x,y)}{1+\rho(x,y)}, \quad x, y \in A,$$

defines a pseudometric in A. If, in addition, ρ is a metric in A, then $\widetilde{\rho}$ is a metric as well.

Proof. Only the triangle inequality of $\widetilde{\rho}$ is non-trivial. Notice that the function $\varphi : [0, +\infty) \to [0, 1]$ given by $\varphi(t) = \frac{t}{1+t}$, $t \geq 0$, is increasing in $[0, +\infty)$. This property, along with the triangle inequality for ρ implies that, for all points $x, y, z \in A$, we have that

$$\widetilde{\rho}(x, z) = \frac{\rho(x, z)}{1 + \rho(x, z)} \leq \frac{\rho(x, y) + \rho(y, z)}{1 + \rho(x, y) + \rho(y, z)} \leq \frac{\rho(x, y)}{1 + \rho(x, y)} + \frac{\rho(y, z)}{1 + \rho(y, z)} = \widetilde{\rho}(x, y) + \widetilde{\rho}(y, z).$$

□

In the sequel, for an open set $\Omega \subset \mathbb{C}$ and a metric space (X, d), we denote the family of all continuous functions from Ω to X by $C(\Omega, X)$.

Proposition 3.6. *Let $\Omega \subset \mathbb{C}$ be open, $\mathcal{K} = \{K_n\}_{n\in\mathbb{N}}$ a nested family of compact sets in Ω, and (X, d) a metric space. We define, for all $f, g \in C(\Omega, X)$,*

$$\rho_{\mathcal{K}}(f, g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\rho_{K_n}(f, g)}{1 + \rho_{K_n}(f, g)}, \quad \textit{where} \quad \rho_{K_n}(f, g) := \max\{d(f(z), g(z)) \,:\, z \in K_n\}, \quad n \in \mathbb{N}. \tag{3.1.1}$$

Then $(C(\Omega, X), \rho_{K_n})$ is a pseudometric space for every $n \in \mathbb{N}$, and $(C(\Omega, X), \rho_{\mathcal{K}})$ is a metric space.

Proof. Using that d is a metric in X, taking into account Lemma 1.27, it is easily seen that both ρ_{K_n} and $\rho_{\mathcal{H}}$ are pseudometrics. In addition, if $f, g \in C(\Omega, F)$ are so that $\rho_{\mathcal{K}}(f, g) = 0$, we have that $\rho_{K_n}(f, g) = 0$ for all $n \in \mathbb{N}$. From the definition of ρ_{K_n}, this clearly implies that $f = g$ in K_n, and for each $n \in \mathbb{N}$. Since the union of $\{K_n\}_{n\in\mathbb{N}}$ is all of Ω, this implies that $f = g$ in Ω. □

3.1.2 Compact-Open Topology. Convergence and Metrizability

As we did in Proposition 3.6, for each compact set $K \subset \mathbb{C}$, and (X, d) a metric space, we define

$$\rho_K(f, g) := \sup\{d(f(z), g(z)) \,:\, z \in K\}, \quad f, g \in C(\Omega, X).$$

Proposition 3.7. *Let $\Omega \subset \mathbb{C}$ be open, (X, d) a metric space, and $\mathcal{K} := \{K_n\}_{n\in\mathbb{N}}$ be a nested family of compact sets in Ω. For the metric $\rho := \rho_{\mathcal{K}}$ associated with $\mathcal{K}$ as in* (3.1.1)*, and the metric space $(C(\Omega, X), \rho)$, the following properties hold.*

(i) For every $\varepsilon > 0$, we can find $\delta > 0$ and a compact set $K \subset \Omega$ so that, for all $f, g \in C(\Omega, X)$, one has
$$\rho_K(f, g) \leq \delta \implies \rho(f, g) \leq \varepsilon.$$

(ii) For every $\varepsilon > 0$ and every $K \subset \Omega$ compact, we can find $\delta > 0$ so that, for all $f, g \in C(\Omega, X)$, one has
$$\rho(f, g) \leq \delta \implies \rho_K(f, g) \leq \varepsilon.$$

(iii) If $A \subset C(\Omega, X)$, the following statements are equivalent.

(a) A is open in the metric space $(C(\Omega, X), \rho)$.

(b) For every $f \in A$ there exists $\delta > 0$ and a compact set $K \subset \Omega$ so that
$$\{g \in C(\Omega, X) \,:\, \rho_K(f, g) \leq \delta\} \subset A.$$

(iv) For functions $\{f_n\}_{n\geq 0} \subset C(\Omega, X)$, the following statements are equivalent.

(a) $\{f_n\}_{n\geq 1}$ converges to f_0 in the metric space $(C(\Omega, X), \rho)$.

(b) For every $j \in \mathbb{N}$, we have $\{f_n\}_{n\geq 1}$ converges uniformly to f_0 on K_j.

(c) For every compact $K \subset \Omega$, the sequence $\{f_n\}_{n\geq 1}$ converges to f_0 uniformly on K.

(v) $(C(\Omega,X),\rho)$ *is a metric space, which is complete if* (X,d) *is complete.*

(vi) *Let* $\mathcal{H} := \{H_n : n \in \mathbb{N}\}$ *be another nested family of compact sets in* Ω, *and* $\rho_{\mathcal{H}}$ *the associated metric as in* (3.1.1). *Then* ρ *and* $\rho_{\mathcal{H}}$ *are uniformly equivalent, that is, given* $\varepsilon > 0$ *there exists* $\delta > 0$ *so that if* $f, g \in C(\Omega, X)$ *are so that* $\rho_{\mathcal{H}}(f,g) \le \delta$, *then* $\rho(f,g) \le \varepsilon$.

Proof.

(i) Given $\varepsilon > 0$, we can find $N \in \mathbb{N}$ so that $\sum_{n\ge N} 2^{-n} \le \varepsilon/2$. Define $K := K_N$ and $\delta = \varepsilon/2$. If $f, g \in C(\Omega, X)$ are so that $\rho_K(f,g) \le \varepsilon/2$, then

$$\rho(f,g) = \sum_{n=1}^{N} \frac{1}{2^n}\frac{\rho_{K_n}(f,g)}{1+\rho_{K_n}(f,g)} + \sum_{n=N+1}^{\infty} \frac{1}{2^n}\frac{\rho_{K_n}(f,g)}{1+\rho_{K_n}(f,g)} \le \sum_{n=1}^{N} \frac{1}{2^n}\frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon.$$

(ii) By the properties of nested families, if $K \subset \Omega$ is compact, there exists $j \in \mathbb{N}$ with $K \subset K_j$. If $\varphi : [0,\infty) \to \mathbb{R}$ is the real function $\varphi(t) = \frac{t}{1+t}$, we define $\eta = \varphi(\varepsilon)$ and $\delta = \eta/2^j$. For functions $f, g \in C(\Omega, X)$ so that $\rho(f,g) \le \delta$, we have that

$$\varphi(\rho_{K_j}(f,g)) = \frac{\rho_{K_j}(f,g)}{1+\rho_{K_j}(f,g)} \le \rho(f,g) \le \delta,$$

that is, $\varphi(\rho_{K_j}(f,g)) \le \eta = \varphi(\varepsilon)$. Since φ is increasing, we may conclude

$$\rho_K(f,g) \le \rho_{K_j}(f,g) \le \varepsilon.$$

(iii) Assume that *(a)* holds. Since A is open, for each $f \in A$ there exists $\varepsilon > 0$ with the property

$$B_\rho(f,\varepsilon) := \{g \in C(\Omega,X) : \rho(f,g) \le \varepsilon\} \subset A.$$

Using *(i)*, we find $\delta > 0$ and a compact subset K of Ω for which

$$\{g \in C(\Omega,X) : \rho_K(f,g) \le \delta\} \subset B_\rho(f,\varepsilon) \subset A.$$

Thus we have shown *(b)*.

Conversely, assume that *(b)* holds, and let $f \in A$. By the assumption, there are $\delta > 0$ and $K \subset \Omega$ compact with

$$\{g \in C(\Omega,X) : \rho_K(f,g) \le \delta\} \subset A.$$

Using *(ii)*, we get $\varepsilon > 0$ so that

$$B_\rho(f,\varepsilon) := \{g \in C(\Omega,X) : \rho(f,g) \le \varepsilon\} \subset \{g \in C(\Omega,F) : \rho_K(f,g) \le \delta\} \subset A.$$

Thus the ball $B_\rho(f,\varepsilon) \subset A$.

(iv) Assuming *(a)*, for each $j \in \mathbb{N}$, one has

$$\lim_{n\to\infty} \frac{1}{2^j}\frac{\rho_{K_j}(f_n,f_0)}{1+\rho_{K_j}(f_n,f_0)} \le \lim_{n\to\infty} \rho(f_n,f_0) = 0,$$

whence $\lim\limits_{n\to\infty} \rho_{K_j}(f_n,f_0) = 0$. This means that $\{f_n\}_{n\ge1}$ converges uniformly on K_j to f, and *(b)* is proven.

The implication *(b)* $\implies$ *(c)* is a consequence of the properties of nested families of compact sets.

Finally, if *(c)* holds, and $\varepsilon > 0$, let $J \in \mathbb{N}$ so that $\sum_{j>J} 2^{-j} \le \varepsilon/2$. For the compact J-first compacts $\{K_1,\dots,K_J\}$ of the nested family, we cand find $N \in \mathbb{N}$ so that

$$\sum_{j=1}^{J} \frac{1}{2^j}\frac{\rho_{K_j}(f_n,f_0)}{1+\rho_{K_j}(f_n,f_0)} \le \frac{\varepsilon}{2}, \quad n \ge N.$$

Thus, for $n \ge N$, we may write

$$\rho(f_n,f_0) = \sum_{j=1}^{J} \frac{1}{2^j}\frac{\rho_{K_j}(f_n,f_0)}{1+\rho_{K_j}(f_n,f_0)} + \sum_{j=J+1}^{\infty} \frac{1}{2^j}\frac{\rho_{K_j}(f_n,f_0)}{1+\rho_{K_j}(f_n,f_0)} \le \frac{\varepsilon}{2} + \sum_{j=J+1}^{\infty} \frac{1}{2^j} \le \varepsilon.$$

(*v*) We know already from Proposition (3.6) that ρ is a metric. Assume that (X,d) is a complete metric space, and let $\{f_n\}_{n\geq 1} \subset C(\Omega, X)$ a Cauchy sequence in the metric ρ. Clearly, we get that, in particular, for each $z \in \Omega$, the sequence $\{f_n(z)\}_{n\geq 1} \subset F$ is Cauchy, and so there exists $f(z) := \lim_{n\to\infty} f_n(z)$. On the other hand, for each $j \in \mathbb{N}$, the sequence of restrictions $\{f_n\restriction_{K_j}\}_{n\geq 1}$ to K_j is also a Cauchy sequence in $(C(K_j, X), \rho_{K_j})$. Thus, $\{f_n\restriction_{K_j}\}_{n\geq 1}$ converges uniformly in K_j to some function, which necessarily must be $f\restriction_{K_j}$, from which we get the continuity of f in K_j. Since $\{K_j\}_j$ is a nested sequence, we may conclude that $f \in C(\Omega, X)$. Since we have shown that $\{f_n\}_{n\geq 1}$ converges to f uniformly on each K_j, by (*iv*), we get that $\{f_n\}_n$ converges to f with respect to the metric ρ.

(*vi*) Let $\varepsilon > 0$, and $N \in \mathbb{N}$ so that $\sum_{n>N} 2^{-n} \leq \varepsilon/2$. Since $\{H_m\}_{m\geq 1}$ is a nested family in Ω, we can find $M \in \mathbb{N}$ with $K_N \subset H_M$. Note that then

$$\rho_{K_n}(f,g) \leq \rho_{K_N}(f,g) \leq \rho_{H_M}(f,g), \quad n = 1, \ldots, N,\ f,g \in C(\Omega, X). \tag{3.1.2}$$

The function $\varphi(t) = \frac{t}{1+t} : [0,\infty) \to [0,1)$ is continuous and increasing, and so we can find $\eta > 0$ so that

$$f,g \in C(\Omega, X),\ \rho_{H_M}(f,g) \leq \eta \implies \frac{\rho_{H_M}(f,g)}{1+\rho_{H_M}(f,g)} \leq \frac{\varepsilon}{2}.$$

Together with (3.1.2), this implies that

$$f,g \in C(\Omega, X),\ \rho_{H_M}(f,g) \leq \eta \implies \sum_{n=1}^{N} \frac{1}{2^n} \frac{\rho_{K_n}(f,g)}{1+\rho_{K_n}(f,g)} \leq \frac{\rho_{H_M}(f,g)}{1+\rho_{H_M}(f,g)} \sum_{j=1}^{N} \frac{1}{2^n} \leq \frac{\varepsilon}{2}. \tag{3.1.3}$$

But clearly

$$\rho_{\mathcal{H}}(f,g) \geq \frac{1}{2^M} \frac{\rho_{H_M}(f,g)}{1+\rho_{H_M}(f,g)} \geq \frac{\rho_{H_M}(f,g)}{2^M}.$$

So, letting $\delta := 2^{-M}\eta$ one has that $\rho_{\mathcal{H}}(f,g) \leq \delta$ implies $\rho_{H_M}(f,g) \leq \eta$, and therefore, using (3.1.3) and the choice of N, we may conclude

$$\rho_{\mathcal{K}}(f,g) = \sum_{n=1}^{N} \frac{1}{2^n} \frac{\rho_{K_n}(f,g)}{1+\rho_{K_n}(f,g)} + \sum_{n=N+1}^{\infty} \frac{1}{2^n} \frac{\rho_{K_n}(f,g)}{1+\rho_{K_n}(f,g)} \leq \frac{\varepsilon}{2} + \sum_{n=N+1}^{\infty} \frac{1}{2^n} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Proposition 3.7 tells us how to define the *topology of the uniform convergence in compact sets.*

Definition 3.8 (Compact-Open Topology)**.** *Let $\Omega \subset \mathbb{C}$ and (X,d) a metric space. The* ***compact-open topology*** *$\tau_{\rm co}$ in $C(\Omega, X)$ is the topology associated with the distance $\rho_{\mathcal{K}}$; where $\mathcal{K} = \{K_n\}_{n\in\mathbb{N}}$ is a nested family of compact sets in Ω. In other words, if $B_{\rho_{\mathcal{K}}}(f,\varepsilon)$ denotes the open ball respect to the metric $\tau_{\rm co}$, centered at f and with radius ε, we have that*

$$\tau_{\rm co} := \{U \subset C(\Omega, X) \,:\, \textit{for every } f \in U \textit{ there is } \varepsilon > 0 \textit{ with } B_{\rho_{\mathcal{K}}}(f,\varepsilon) \subset U\}.$$

The topologycal space $(C(\Omega, X), \tau_{\rm co})$ is metrizable, and does not depend on the chosen nested family $\mathcal{K}$. Moreover, if (X,d) is complete, $(C(\Omega, X), \rho_{\mathcal{K}})$ is complete as well.

3.2 The Arzelà-Ascoli Theorem

Throughout this section, if Ω is open and (X,d) is a metric space, $C(\Omega, X)$ is equipped with the Compact-Open topology from Definition 3.8.

3.2.1 Equicontinuous Families

Definition 3.9 (Equicontinuity). *Let $A \subset \mathbb{C}$ be a set, and (X, d) a metric space. A set $\mathcal{F} \subset C(A, X)$ is **equicontinuous at** $z_0 \in A$ if*

$$\text{for every } \varepsilon \text{ there exists } \delta > 0 \text{ so that } \sup\{d(f(z), f(z_0)) : z \in D(z_0, \delta) \cap A,\ f \in \mathcal{F}\} \leq \varepsilon.$$

*We also say that $\mathcal{F}$ **is equicontinuous** if $\mathcal{F}$ is equicontinuous at every $z_0 \in A$.*

The statement of Arzelà-Ascoli theorem contains several topological concepts in metric spaces, that are convenient to refresh.

Remark 3.10 (Topology in Metric Spaces). Let (Y, d) be any metric space, which in the sequel will play the role of (X, d) or $(C(\Omega, X), \rho)$. We recall the following.

(1) A subset $A \subset (Y, d)$ is **totally bounded** if for every $\varepsilon > 0$ there are finitely-many (open) balls $B(y_1, \varepsilon), \ldots, B(y_n, \varepsilon)$ in Y whose union contains A.

(2) A subset $A \subset (Y, d)$ is **compact** if for every collection $\{U_\alpha\}_{\alpha \in \Lambda}$ of open subsets of Y whose union contains A, there is a finite subcollection $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ of $\{U_\alpha\}_{\alpha \in \Lambda}$ whose union still contains A.

(3) A subset $A \subset (Y, d)$ is **relatively compact** if there exists $K \subset (Y, d)$ compact, with $A \subset K$. Since closed subsets of compact sets are also compact, A is relatively compact if and only if the closure $\overline{A}$ of A is compact in (Y, d).

(4) The **Bolzano-Weierstrass Theorem:** A set $A \subset (Y, d)$ is compact if and only A is **sequentially compact.** The latter means that every sequence $\{y_n\}_{n \geq 1} \subset A$ has a convergent subsequence to some $y \in A$.

(5) In the metric space (Y, d), closures of sets can be characterize via sequences. More precisely, if $A \subset (Y, d)$, we have that $y \in \overline{A}$ if and only if there exists a sequence $\{y_n\}_{n \geq 1} \subset A$ with $\lim\limits_{n \to \infty} d(y_n, y) = 0$.
Since the closure of A is, by the definition, the smallest closed subset containing A, the above implies that a set A is closed if and only if for every sequence $\{y_n\}_{n \geq 1} \subset A$ convergent to $y \in Y$, we have that $y \in A$ as well.

(6) The metric space (Y, d) is **complete** if every Cauchy sequence $\{y_n\}_n \subset (Y, d)$ is convergent (in the metric d) to some $y \in Y$. **Assuming that** (Y, d) **is complete**, the following equivalences for a set $A \subset (Y, d)$ are a consequence of Bolzano-Weierstrass Theorem:

    (6a) A is compact $\iff$ A is closed and totally bounded.

    (6b) A is relatively compact $\iff$ A is totally bounded.

We now state some properties involving equicontinuity and these topological concepts.

Proposition 3.11. *Let $\Omega \subset \mathbb{C}$ be open, (X, d) a metric space, and $\mathcal{F} \subset C(\Omega, X)$. Then*

(i) The following statements are equivalent.

  (a) $\mathcal{F}$ is totally bounded.

  (b) For every compact $K \subset \Omega$ and $\varepsilon > 0$ there are $f_1, \ldots, f_N \in \mathcal{F}$ so that, for all $f \in \mathcal{F}$ we can find $j \in \{1, \ldots, N\}$ with $\sup\{d(f(z), f_j(z)) : z \in K\} \leq \varepsilon$.

(ii) $\mathcal{F}$ is equicontinuous if and only if $\overline{\mathcal{F}}$ is equicontinuous.

(iii) If $z_0 \in \Omega$, the following statements are equivalent.

(a) $\mathcal{F}(z_0) := \{f(z_0) : f \in \mathcal{F}\}$ *is relatively compact in* X.

(b) $\overline{\mathcal{F}}(z_0)$ *is relatively compact in* X.

Proof.

(i) Assume that (a) holds, and fix $\varepsilon > 0$ and $K \subset \Omega$ compact. By Proposition 3.7(ii) there exists $\delta > 0$ so that

$$f, g \in C(\Omega, X),\ \rho(f, g) \leq \delta \implies \sup\{d(f(z), g(z)) : z \in K\} \leq \varepsilon. \tag{3.2.1}$$

If $\mathcal{F}$ is totally bounded, there are functions $f_1, \ldots, f_n \in \mathcal{F}$ so that for every $f \in \mathcal{F}$ we can find $j \in \{1, \ldots, n\}$ with $\rho(f, f_j) \leq \delta$, and then $\sup\{d(f(z), f_j(z)) : z \in K\} \leq \varepsilon$, thanks to (3.2.1). This shows (b).

Now assume that (b) holds, and let $\varepsilon > 0$. By Proposition 3.7(i), there are $\delta > 0$ and $K \subset \Omega$ compact with

$$f, g \in C(\Omega, X),\ \sup\{d(f(z), g(z)) : z \in K\} \leq \delta \implies \rho(f, g) \leq \varepsilon. \tag{3.2.2}$$

By the assumption, there are functions $f_1, \ldots, f_n \in \mathcal{F}$ so that for every $f \in \mathcal{F}$ one has $\sup\{d(f(z), f_j(z)) : z \in K\} \leq \delta$ for one of those f_j. We get from (3.2.2) that $\rho(f, f_j) \leq \varepsilon$, which shows that $\mathcal{F}$ is totally bounded.

(ii) Since $\mathcal{F} \subset \overline{\mathcal{F}}$, if $\overline{\mathcal{F}}$ is equicontinuous, then $\mathcal{F}$ is equicontinuous as well. Conversely, if $\mathcal{F}$ is equicontinuous, $z \in \Omega$ and $\varepsilon > 0$, there are $\delta > 0$ so that $\overline{D}(z, \delta) \subset \Omega$ and

$$\sup\{d(g(w), g(z)) : g \in \mathcal{F},\ w \in \overline{D}(z, \delta)\} \leq \varepsilon. \tag{3.2.3}$$

For each $f \in \overline{\mathcal{F}}$, there is a sequence $\{f_n\}_{n\geq 1} \subset \mathcal{F}$ convergent to f in the metric ρ. But this implies, in particular, that $\lim_{n\to\infty} d(f_n(w), f(w)) = 0$ for all $w \in \Omega$. Since (3.2.3) holds replacing g with f_n, taking the limit we get that also

$$\sup\{d(f(w), f(z)) : w \in \overline{D}(z, \delta)\} \leq \varepsilon.$$

This proves the equicontinuity of $\overline{\mathcal{F}}$, since $f \in \overline{\mathcal{F}}$ is arbitrary, and δ depends only on z.

(iii) Notice that $\mathcal{F}(z_0) \subset \overline{\mathcal{F}}(z_0)$, and so the former is relatively compact when then latter is.

Also observe that $\overline{\mathcal{F}}(z_0) \subset \overline{\mathcal{F}(z_0)}$. Indeed, if $f \in \overline{\mathcal{F}}$, we can find a sequence $\{f_n\}_{n\geq 1} \subset \mathcal{F}$ convergent to f in the metric ρ. Consequently, $\lim_{n\to\infty} d(f_n(z_0), f(z_0)) = 0$, and thus $f(z_0) \subset \overline{\mathcal{F}(z_0)}$. This proves that $\overline{\mathcal{F}}(z_0) \subset \overline{\mathcal{F}(z_0)}$.

Now, if $\mathcal{F}(z_0)$ is relatively compact, then $\overline{\mathcal{F}(z_0)}$ is compact, and by what we have just proved, $\overline{\mathcal{F}}(z_0)$ is relatively compact. □

3.2.2 The Arzelà-Ascoli Theorem and Consequences

We are ready to prove the Arzelà-Ascoli Theorem, which characterizes the compact subsets of $(C(\Omega, X), \rho)$, for $\Omega \subset \mathbb{C}$ open, and (X, d) a complete metric space.

Theorem 3.12 (Arzelà-Ascoli Theorem). *Let* $\Omega \subset \mathbb{C}$ *be open,* (X, d) *a complete metric space, and* $\mathcal{F} \subset C(\Omega, X)$. *Equipping* $C(\Omega, X)$ *with the open compact topology, the following statements are equivalent.*

(i) $\mathcal{F}$ *is closed, equicontinuous, and* $\mathcal{F}(K)$ *is a compact subset of* (X, d) *for each compact subset* $K \subset \Omega$.

(ii) $\mathcal{F}$ *is closed, equicontinuous, and* $\mathcal{F}(z)$ *is a compact subset of* (X, d) *for all* $z \in \Omega$.

(iii) $\mathcal{F}$ *is closed, equicontinuous, and* $\mathcal{F}(z)$ *is relatively compact in* (X, d) *for all* $z \in \Omega$.

(iv) $\mathcal{F}$ *is compact.*

Proof. The implications $(i) \implies (ii) \implies (iii)$ are obvious.

Now, let ρ be a metric in $(C(\Omega, X), \tau_{\mathrm{co}})$, associated with a nested family of compact sets in Ω. By Proposition 3.6(v), we know that $(C(\Omega, X), \rho)$ is a complete metric space.

$(iii) \implies (iv)$: Since $\mathcal{F}$ is closed, and $(C(\Omega, X), \rho)$ is complete, $\mathcal{F}$ is compact if and only if $\mathcal{F}$ is totally bounded. By Proposition 3.11(i), it suffices to show that $\mathcal{F}$ satisfies property $(ii)(b)$ of Proposition 3.11.

Let $\varepsilon > 0$ and $K \subset \Omega$ compact. Since $\mathcal{F}$ is equicontinuous, for each $z \in K$, we can find a disk $D(z, \delta_z) \subset \Omega$ so that

$$\sup\{d(f(z), f(w)) \,:\, w \in D(z, \delta_z),\ f \in \mathcal{F}\} \leq \frac{\varepsilon}{3}. \tag{3.2.4}$$

By the compactness of K, we can find $z_1, \ldots, z_n \in K$ so that

$$K \subset \bigcup_{k=1}^{n} D(z_k, \delta_k); \tag{3.2.5}$$

where we have abbreviated $\delta_k = \delta_{z_k}$. We define new elements

$$L := \overline{\bigcup_{k=1}^{n} \mathcal{F}(z_k)}; \quad \varphi : \mathcal{F} \to L^n, \quad \varphi(f) := (f(z_1), \ldots, f(z_n)), \text{ for all } f \in \mathcal{F}.$$

By the assumption, each $\mathcal{F}(z_k)$ is relatively compact, and so L is a compact subset of X. With the distance

$$d^n\left(\{x_k\}_{k=1}^n, \{x'_k\}_{k=1}^n\right) := \max\{d(x_k, x'_k) \,:\, k = 1, \ldots, n\}, \quad \text{for all} \quad \{x_k\}_{k=1}^n, \{x'_k\}_{k=1}^n \in L^n,$$

the space (L^n, d^n) is metric and compact. Since $\varphi(\mathcal{F})$ is totally bounded with the metric d^n (as a subset of L^n), we can find functions $f_1, \ldots, f_N \in \mathcal{F}$ so that

$$\varphi(\mathcal{F}) \subset \bigcup_{j=1}^{N} B_{d^n}(\varphi(f_j), \varepsilon/3).$$

In other words, given any $f \in \mathcal{F}$, we can find $f_j \in \mathcal{F}$ for which

$$\max\{d\left(f(z_k), f_j(z_k)\right) \,:\, k = 1, \ldots, n\} \leq \frac{\varepsilon}{3}. \tag{3.2.6}$$

Now, for any $z \in K$, by (3.2.5) there exists $k \in \{1, \ldots, n\}$ with $z \in D(z_k, \delta_k)$. Using (3.2.4) for f and for f_j and the points z and z_j, and also (3.2.6), we get, by the triangle inequality, that

$$d(f(z), f_j(z)) \leq d(f(z), f(z_k)) + d(f(z_k), f_j(z_k)) + d(f_j(z_k), f_j(z)) \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

$(iv) \implies (i)$: Clearly $\mathcal{F}$ is closed, because $\mathcal{F}$ is compact.

Let us now check that if $K \subset \Omega$ is compact, then $\mathcal{F}(K)$ is a compact subset of $(C(\Omega, X), \rho)$. In metric spaces, the compact sets are precisely the sequentially compact sets. So, given a sequence $\{f_n(z_n)\}_{n\geq 1} \subset \mathcal{F}(K)$; where $\{z_n\}_{n\geq 1} \subset K$ and $\{f_n\}_{n\geq 1} \subset \mathcal{F}$, we need to find $f_0 \in \mathcal{F}$ and $z_0 \in K$ and a subsequence of $\{f_n(z_n)\}_{n\geq 1}$ converging to $f_0(z_0)$ in the metric d. By the assumption, $\mathcal{F}$ is compact in $(C(\Omega, X), \rho)$ and K is compact, passing to subsequences if necessary, we may assume that there exists $f_0 \in \mathcal{F}$ and $z_0 \in K$ with

$$\lim_{n\to\infty} \rho(f_n, f_0) = 0, \quad \text{and} \quad \lim_{n\to\infty} |z_n - z_0| = 0. \tag{3.2.7}$$

Our claim is that $\{f_n(z_n)\}_{n\geq 1}$ converges to $f_0(z_0)$ in the metric d. Indeed, by (3.2.7) and Proposition 3.7, the convergence of f_n to f_0 is uniformly on K. But using also the continuity of $f_0 : \Omega \to X$ and that $\{z_n\}_{n\geq 1}$, we can find $N \in \mathbb{N}$ so that simultaneously

$$\max\{d(f_n(z), f_0(z)) \,:\, z \in K\} \leq \frac{\varepsilon}{2}, \quad \text{and} \quad d\,(f_0(z_n), f_0(z_0)) \leq \frac{\varepsilon}{2}, \quad n \geq N.$$

By the triangle inequality,

$$d(f_n(z_n), f_0(z_0)) \leq d(f_n(z_n), f_0(z_n)) + d\,(f_0(z_n), f_0(z_0)) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \text{whenever} \quad n \geq N.$$

This proves our claim, and not it only remains to check that $\mathcal{F}$ is equicontinuous. Indeed, otherwise, we can find $z_0 \in \Omega$, $\varepsilon > 0$ and sequences $\{f_n\}_{n\geq 1} \subset \mathcal{F}$, $\{z_n\}_{n\geq 1} \subset \Omega$ satisfying

$$\lim_{n\to\infty} z_n = z_0 \quad \text{and yet} \quad d\,(f_n(z_n), f_n(z_0)) \geq \varepsilon, \quad \text{for all} \quad n \geq 1. \tag{3.2.8}$$

As in the previous argument, we may assume that $\{f_n\}_{n\geq 1}$ converges to f_0 in the metric ρ, and that $\lim\limits_{n\to\infty} d(f_n(z_n), f_0(z_0)) = 0$. But then (3.2.8) yields

$$0 = \lim_{n\to\infty} d(f_n(z_n), f_0(z_0)) \geq \limsup_{n\to\infty} d(f_n(z_n), f_n(z_0)) - \lim_{n\to\infty} d(f_n(z_0), f_0(z_0)) \geq \varepsilon,$$

a contradiction. Observe that we used that $\lim\limits_{n\to\infty} d(f_n(z_0), f_0(z_0))$, as $\{f_n\}_{n\geq 1}$ converges to f_0 uniformly on each compact susbet of Ω. □

Corollary 3.13. *Let $\Omega \subset \mathbb{C}$ be open, (X, d) a complete metric space, and $\mathcal{F} \subset C(\Omega, X)$. Equipping $C(\Omega, X)$ with the open compact topology, the following statements are equivalent.*

(i) $\mathcal{F}$ is relatively compact.

(ii) $\mathcal{F}$ is equicontinuous and $\mathcal{F}(w)$ is a relatively compact subset of (X, d) for all $w \in \Omega$.

Proof. The set $\mathcal{F}$ is relatively compact if and only if (by definition) $\overline{\mathcal{F}}$ is compact. From the equivalence $(iii) \iff (iv)$ of Theorem 3.12 and Proposition 3.11, we get the desired equivalence. □

Corollary 3.14. *Let $\Omega \subset \mathbb{C}$ be open, (X, d) a complete metric space, and $\mathcal{F} := \{f_n\}_{n\geq 1} \subset C(\Omega, X)$ a sequence. The following hold.*

(i) If $\mathcal{F}$ is equicontinuous, and all the sets $\mathcal{F}(z)$, $z \in \Omega$, are relatively compact in (X, d), then there exists is a subsequence $\{f_{n_k} \,:\, k \in \mathbb{N}\}$ convergent to some $f \in C(\Omega, X)$ in the compact-open topology.

(ii) If $\mathcal{F}$ is equicontinuous and pointwise convergent, then $\{f_n\}_{n\geq 1}$ converges to some $f \in C(\Omega, X)$ in the compact-open topology.

Proof.

(i) By Corollary 3.13, $\mathcal{F}$ is relatively compact in the metric space $(C(\Omega, X)$, and then (i) follows.

(ii) The sequence (set) $\mathcal{F}$ is pointwise convergent, and that means, for all $w \in \Omega$, the $\mathcal{F}(w)$ is convergent sequence in X. Therefore, $\mathcal{F}(z)$ is relatively compact for all $z \in \Omega$. Applying (i), we find a subsequence $\{f_{n_k} \,:\, k \in \mathbb{N}\}$ of $\{f_n\}_{n\geq 1}$ convergent to $f \in C(\Omega, X)$.

We need to show that the original sequence $\{f_n\}_{n\geq 1}$ converges to f in the metric ρ. Assume for the sake of contradiction, that there exists $\varepsilon > 0$ and a subsequence $\{f_{m_j}\}_{j\geq 1}$ with $\rho(f_{m_j}, f) \geq \varepsilon$ for all $j \in \mathbb{N}$. The set $\widetilde{\mathcal{F}} := \{f_{m_j} \,:\, j \in \mathbb{N}\}$ is equicontinuous (as $\mathcal{F}$ is), and $\widetilde{\mathcal{F}}(z)$ is relatively compact in (X, d) for all $z \in \Omega$. By (i), we can find a subsequence, which we keep denoting by $\{f_{m_j}\}_{j\in\mathbb{N}}$, converging to some $g \in C(\Omega, X)$ in the metric ρ. By the continuity of the metric ρ, we get that $\rho(f, g) \geq \varepsilon$. However, for $\{f_{m_j}\}_{j\in\mathbb{N}}$ converges pointwise to g, and must coincide with the pointwise limit of $\{f_{n_k}\}_{k\in\mathbb{N}}$, which is the function f. We obtain that $f = g$, a contradiction.

□

3.3 Normal Families and Montel's Theorem

Definition 3.15 (Normal Family). *Let $\Omega \subset \mathbb{C}$ be open, and $\mathcal{F} \subset C(\Omega, \mathbb{C})$ be a family of functions. We equip $C(\Omega, \mathbb{C})$ with the metric ρ from Definition 3.8. We say that*

- *$\mathcal{F}$ is a* ***a normal family in*** *Ω if $\overline{\mathcal{F}}$ is compact in $(C(\Omega, \mathbb{C}), \rho)$, that is, if $\mathcal{F}$ is relatively compact in $(C(\Omega, \mathbb{C}), \rho)$.*
- *$\mathcal{F}$ is* ***locally bounded in*** *Ω if for every $z_0 \in \Omega$ there exists $r > 0$ so that $\overline{D}(z_0, r) \subset \Omega$ and $\mathcal{F}(\overline{D}(z_0, r))$ is a bounded subset of $\mathbb{C}$. Here,*

$$\mathcal{F}(\overline{D}(z_0, r)) := \{f(w) \,:\, f \in \mathcal{F},\, w \in \overline{D}(z_0, r)\}.$$

Theorem 3.16 (Montel's Theorem). *Let $\Omega \subset \mathbb{C}$ be open, and $\mathcal{F} \subset \mathcal{H}(\Omega)$ a family of holomorphic functions in Ω. The following statements are equivalent.*

(i) $\mathcal{F}$ is normal.

(ii) $\mathcal{F}$ is locally bounded.

Proof.

$(i) \implies (ii)$: Since $\mathcal{F}$ is normal, $\overline{\mathcal{F}}$ is compact, and by the Arzelà-Ascoli Theorem 3.12, we get that $\mathcal{F}(K)$ is relatively compact (thus bounded) in $\mathbb{C}$, for each compact $K \subset \Omega$.

$(ii) \implies (i)$: Since $\mathcal{F}$ is locally bounded, all the sets $\mathcal{F}(z)$, $z \in \Omega$, are relatively compact in $\mathbb{C}$. By Corollary 3.13, it suffices to check that $\mathcal{F}$ is equicontinuous in Ω. Given $z_0 \in \Omega$, by the assumption, we can find $r > 0$ be so that $\overline{D}(z_0, 2r) \subset \Omega$ and $\mathcal{F}(\overline{D}(z_0, 2r))$ is bounded. Thus there exists $M \geq 1$ so that $\mathcal{F}(\overline{D}(z_0, 2r)) \subset D(0, M)$. We can apply the Cauchy Integral Formula (for example, Corollary 1.2) to $f \in \mathcal{F}$ over the circle $\gamma \equiv \partial D(z_0, 2r)$ to obtain, for each $z \in \overline{D}(z_0, r)$,

$$\begin{aligned}
|f(z) - f(z_0)| &= \left| \frac{1}{2\pi i} \int_\gamma \left(\frac{f(w)}{w - z} - \frac{f(w)}{w - z_0} \right) \mathrm{d}w \right| = \left| \frac{1}{2\pi i} \int_\gamma f(w) \frac{z - z_0}{(w - z)(w - z_0)} \mathrm{d}w \right| \\
&= \left| \frac{1}{2\pi i} \int_0^{2\pi} f(z_0 + 2re^{it}) \frac{z - z_0}{(z_0 + 2re^{it} - z) 2re^{it}} \cdot 2ire^{it} \, \mathrm{d}t \right| \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(z_0 + 2re^{it})|}{2r - |z - z_0|} |z - z_0| \, \mathrm{d}t \leq \frac{M}{r} |z - z_0|. \qquad (3.3.1)
\end{aligned}$$

Given $\varepsilon > 0$, we take $0 < \delta \leq \varepsilon r / M$, and if $z \in \overline{D}(z_0, r)$ we get that

$$|f(z) - f(z_0)| \leq \varepsilon,$$

by virtue of (3.3.1). Since the choice of δ does not depend on $f \in \mathcal{F}$, this shows that $\mathcal{F}$ is equicontinuous at z_0, thus equicontinuous in Ω because $z_0 \in \Omega$ is arbitrary. ☐

Corollary 3.17. *Let $\Omega \subset \mathbb{C}$ be open, and $\mathcal{F} \subset \mathcal{H}(\Omega)$ a set. The following statements are equivalent.*

(i) $\mathcal{F}$ is compact.

(ii) $\mathcal{F}$ is closed and locally bounded.

Proof. It is immediate from Montel's Theorem 3.16. ☐

For sequences, Montel's Theorem 3.16 read us follows.

Corollary 3.18. *Let $\Omega \subset \mathbb{C}$ be open, and $\{f_n\}_{n \geq 1} \subset \mathcal{H}(\Omega)$ a locally bounded sequence. Then there exists a subsequence $\{f_{n_k}\}_{k \geq 1}$ converging to some $f \in \mathcal{H}(\Omega)$ uniformly on compact subsets of Ω.*

Proof. By Montel's Theorem 3.16, $\{f_n\}_{n \geq 1}$ is a normal family, that is, a relatively compact subset of the metric space $(C(\Omega, \mathbb{C}), \rho)$. By Bolzano-Weierstrass theorem, there exists a subsequence $\{f_{n_k}\}_{k \geq 1}$ convergent to some $f \in C(\Omega, \mathbb{C})$ in the metric ρ. But this is precisely the uniform convergence in compact subsets of Ω. And by Weierstrass Theorem 3.1, $f \in \mathcal{H}(\Omega)$. ☐

3.3.1 Vitali's Theorem and Osgood's Theorem

Concerning the convergence of the original sequence (instead of a subsequence), we have the following theorem due to Vitali. See also Exercise 3.8.

Theorem 3.19 (Vitali)**.** *Let $\Omega \subset \mathbb{C}$ be open and connected, $f \in \mathcal{H}(\Omega)$ and $\{f_n\}_{n\geq 1} \subset \mathcal{H}(\Omega)$ a locally bounded sequence. For the set*

$$A := \{z \in \Omega \,:\, \lim_{n\to\infty} f_n(z) = f(z)\},$$

assume that $A' \cap \Omega \neq \emptyset$. Then $\{f_n\}_{n\geq 1}$ converges to f uniformly on compact subsets of Ω.

Proof. Suppose, for the sake of contradiction, that there exists a compact $K \subset \Omega$ so that $\{f_n\}_n$ does not converge to f uniformly on K. Then we can find a subsequence $\{f_{n_j}\}_j$ of $\{f_n\}_n$ with

$$\sup_{z\in K} |f_{n_j}(z) - f(z)| \geq \varepsilon \quad \text{for all} \quad j \in \mathbb{N}. \tag{3.3.2}$$

But $\{f_{n_j}\}_j$ is locally bounded in Ω (as $\{f_n\}_n$ is), and by Corollary 3.18, we can find a further subsequence $\{f_{n_{m_j}}\}_j$ of $\{f_{n_j}\}_j$ converging uniformly on compact subsets of Ω to some $g \in \mathcal{H}(\Omega)$. Clearly $g = f$ in the set A. Since Ω is connected and $A' \cap \Omega \neq \emptyset$, by the Identity Principles for Analytic Functions, we deduce that $f = g$ in Ω. In particular, we get that $\{f_{n_{m_j}}\}_j$ converges uniformly on K to f, which contradicts (3.3.2). □

Finally, we record the following result of Osgood, showing that pointwise convergence of holomorphic functions implies uniformly convergence in compact subsets of a dense open subset.

Theorem 3.20 (Osgood)**.** *Let $\Omega \subset \mathbb{C}$ be open and $\{f_n\}_{n\geq 1} \subset \mathcal{H}(\Omega)$ converging pointwise to a function $f : \Omega \to \mathbb{C}$ in Ω. Then there exists an open set $V \subset \Omega$ dense in Ω so that $f \in \mathcal{H}(V)$ and $\{f_n\}_{n\geq 1}$ converges to f uniformly on compact subsets of V.*

Proof. We will first prove that given any open subset U of Ω with $\overline{U} \subset \Omega$, we can find a nonempty open subset W_U of U so that $\{f_n\}_{n\geq 1}$ converges to f uniformly on compact subsets of $\overline{W_U}$. Indeed, consider the subsets of $\overline{U}$ given by

$$E_k := \{z \in \overline{U} \,:\, |f_n(z)| \leq k \;\text{ for all }\; n \in \mathbb{N}\}, \quad k \in \mathbb{N}.$$

Because of the pointwise convergence of $\{f_n\}$ (to f) in Ω, we have that $\overline{U} = \bigcup_{k\in\mathbb{N}} E_k$. Note also that $E_k \subset E_{k+1}$ for all $k \in \mathbb{N}$. Therefore we can find $k_0 \in \mathbb{N}$ so that $E_k \neq \emptyset$ for all $k \geq k_0$. Therefore, we can write

$$\overline{U} = \bigcup_{k=k_0}^{\infty} E_k.$$

Since $\overline{U}$ is complete (as a closed subset of $\mathbb{C}$), and $\{E_k\}_{k\geq k_0}$ are nonempty closed subsets whose union is $\overline{U}$, by Baire's Category Theorem, at least one E_k must have nonempty interior. That is E_k contains an open set W_U. But also note that

$$|f_n(z)| \leq k \quad \text{for all} \quad z \in W_U,\, n \in \mathbb{N}.$$

In particular $\{f_n\}_n$ is locally (actually globally) bounded in the set W_U. By Montel's Theorem 3.16, $\{f_n\}_n$ is a normal family in W_U, thus relatively compact in $(C(W_U, \mathbb{C}), \rho)$. By the pointwise convergence to f, combining Corollaries 3.13–3.14 we deduce that $\{f_n\}_{n\geq 1}$ converges to f uniformly on compact subsets of W_U.

Now, given any $z \in \Omega$, there exists $j_z \in \mathbb{N}$ so that $\overline{D}(z, 1/j) \subset \Omega$ for all $j \geq j_z$. Let $W_{z,j} := W_{D(z,1/j)}$ denote the open set associated with $D(z, 1/j)$ as above. We define

$$V := \bigcup_{z\in\Omega} \bigcup_{j=j_z}^{\infty} W_{z,j}.$$

Being the union of open sets, V is open. Also, given $z \in \Omega$ and $\varepsilon > 0$, we can find $j \geq j_z$ so that $1/j \leq \varepsilon/2$. Then, since $W_{z,j}$ is a nonempty subset of $D(z, 1/j)$, there exists $\xi \in W_{z,j} \subset D(z, 1/j)$, implying that $|z - \xi| \leq 1/j < \varepsilon$. This shows that V is dense in Ω. Finally, if $K \subset V$ is compact, then K is contained in the union of finitely-many of those $W_{z,j}$, $z \in \Omega$, $j \geq j_z$, $z \in \Omega$. Relabelling those (z, j) as $\alpha_1, \ldots, \alpha_m$, we have that

$$K \subset \bigcup_{l=1}^{N} W_{\alpha_l}.$$

But recall that $\{f_n\}_n$ converges uniformly to f in compact subsets of each W_{α_l}, thus converging in K. And by Weierstrass Theorem 3.1, we deduce that $f \in \mathcal{H}(V)$. □

3.4 Exercises

We will denote by ρ a metric associated with a nested family of compact sets, as in Definition 3.8. Recall that, by Proposition 3.7, the convergence of continuous functions f_n is equivalent to the uniform convergence on compact subsets.

Exercise 3.1. *Let $\Omega := \{z \in \mathbb{C} \ \ \mathrm{Im}(z) > 0\}$, and the sequence of functions $\{f_n(z) = \tan(nz),\ z \in \Omega\}_{n\geq 1}$. Prove that for every $\varepsilon > 0$ the sequence $\{f_n\}_{n\geq 1}$ converges to the constant function $f(z) = i$ uniformly on $\Omega_\varepsilon := \{z \in \mathbb{C} : \mathrm{Im}(z) \geq \varepsilon\}$. Deduce that $\{f_n\}_{n\geq 1}$ converges to i with respect to the metric ρ. Finally, show that the convergence is not uniform in Ω.*

Exercise 3.2. *Let $K \subset \mathbb{C}$ be compact, and $\{f_n : K \to \mathbb{R}\}_n$ a sequence of real-valued and continuous functions on K such that $\{f_n\}_n$ converges pointwise to a continuous $f : K \to \mathbb{R}$, and that $f_n(z) \leq f_{n+1}(z)$ for all $z \in K$, $n \in \mathbb{N}$. Prove that $\{f_n\}_n$ converges to f uniformly.*

Exercise 3.3. *Consider the series of functions*

$$\sum_{n=1}^{\infty} \frac{\sin(nz)}{n^2}, \quad z \in \mathbb{C}.$$

Prove that:

(a) The series converges uniformly in $z \in \mathbb{R}$.

(b) For each $z \in \mathbb{C} \setminus \mathbb{R}$, the numerical series $\sum_{n=1}^{\infty} \frac{\sin(nz)}{n^2}$ diverges.

Exercise 3.4. *Let $\Omega \subset \mathbb{C}$ be open, $z \in \Omega$, $\{z_n\}_{n\geq 1} \subset \Omega$ with $\lim_{n\to\infty} z_n = z$, and $f \in C(\Omega, \mathbb{C})$, $\{f_n\}_n \subset \mathcal{H}(\Omega)$ with $\{f_n\}_n$ converging to f in the metric ρ. Prove that*

$$\lim_{n\to\infty} f_n(z_n) = f(z).$$

Exercise 3.5. *Let $\Omega \subset \mathbb{C}$ be open, and $\{f_n : \Omega \to \mathbb{C}\}_{n\in\mathbb{N}}$ a sequence of functions so that there exist $0 < \alpha \leq 1$, $L, M > 0$ so that*

$$\sup\{|f_n(z)| : n \in \mathbb{N},\, z \in \Omega\} \leq M, \quad |f_n(z) - f_n(w)| \leq L|z - w|^\alpha, \quad z, w \in \Omega,\, n \in \mathbb{N}.$$

Show that $\{f_n : \Omega \to \mathbb{C}\}_{n\in\mathbb{N}}$ is equicontinuous in Ω. Use the Arzelà-Ascoli Theorem 3.12 to deduce that $\{f_n : \Omega \to \mathbb{C}\}_{n\in\mathbb{N}}$ has a subsequent convergent to some $f \in C(\Omega, \mathbb{C})$ satisfying that

$$|f(z) - f(w)| \leq L|z - w|^\alpha, \quad z, w \in \Omega.$$

Exercise 3.6. *Let $\Omega \subset \Omega$ be open, $f \in \mathcal{H}(\Omega)$ and $\{f_n\}_{n\geq 1} \subset \mathcal{H}(\Omega)$. Prove that the following statements are equivalent.*

(a) $\{f_n\}_{n\geq 1}$ *converges to* f *in the metric* ρ.

(b) For every closed and piecewise C^1-path γ in Ω, $\{f_n\}_{n\geq 1}$ converges to f uniformly on γ^*.

Exercise 3.7. *Find $U, \Omega \subset \mathbb{C}$ open, $g \in \mathcal{H}(U)$, and $\mathcal{F} \subset \mathcal{H}(\Omega)$ a normal family with $f(\Omega) \subset U$ for all $f \in \mathcal{F}$, so that $g \circ \mathcal{F} := \{g \circ f \,:\, f \in \mathcal{F}\}$ is not normal.*

Exercise 3.8. *Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \to \mathbb{C}$ a function, and a locally bounded sequence $\{f_n\}_{n\geq 1} \subset \mathcal{H}(\Omega)$ converging pointwise to f in Ω. Prove that $f \in \mathcal{H}(\Omega)$ and that $\{f_n\}_{n\geq 1}$ converges to f uniformly on compact subsets of Ω.*

Exercise 3.9. *Let $\Omega \subset \mathbb{C}$ be open, and $\mathcal{F} \subset \mathcal{H}(\Omega)$. Consider the family of derivatives $\mathcal{F}' := \{f' :$ $f \in \mathcal{F}$. Prove the following.*

(a) If $\mathcal{F}$ is normal, then $\mathcal{F}'$ is normal.

(b) $\mathcal{F}$ is normal if and only if $\mathcal{F}'$ is normal and for each connected component Ω_j of Ω there exists a point $w_j \in \Omega_j$ for which $\mathcal{F}(w_j)$ is bounded.

Exercise 3.10. *If $\Omega \subset \mathbb{C}$ is open, and $\mathcal{F} \subset \mathcal{H}(\mathbb{D})$, show that the following are equivalent.*

(a) $\mathcal{F}$ is normal.

(b) For every $\varepsilon > 0$ there exists $c > 0$ so that

$$c\mathcal{F} = \{cf \,:\, f \in \mathcal{F}\} \subset B_\rho(0, \varepsilon);$$

where $B_\rho(0, \varepsilon)$ is the open ball centered at 0 and with radius ε with respect to the metric ρ.

Exercise 3.11. *For $\mathbb{D} := D(0,1)$ and $\mathcal{F} \subset \mathcal{H}(\mathbb{D})$, show that the following are equivalent.*

(a) $\mathcal{F}$ is normal.

(b) There exist positive constants $\{M_n\}_{n\geq 0}$ with $\limsup\limits_{n\to\infty} M_n^{1/n} \leq 1$, and so that if $f \in \mathcal{F}$ has the expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{D},$$

then $|a_n| \leq M_n$ for all $n \geq 0$.

Exercise 3.12. *Let $\Omega \subset \mathbb{C}$ be open, and $f \in \mathcal{H}(\Omega)$. Prove that, for every closed disk $\overline{D}(z_0, R) \subset \Omega$,*

$$|f(z_0)|^2 \leq \frac{1}{\pi R^2} \int_0^R \int_0^{2\pi} |f(z_0 + re^{it})|^2 \, r \,\mathrm{d}r \,\mathrm{d}t.$$

Assume further that Ω is connected, $\mathcal{F} \subset \mathcal{H}(\Omega)$, and there exists $M > 0$ so that

$$\|f\|^2_{L^2(D(z_0,r))} := \int_{D(z_0,r)} |f(x+iy)|^2 \,\mathrm{d}x \,\mathrm{d}y \leq M, \quad \text{whenever} \quad \overline{D}(z_0, r) \subset \Omega,\ f \in \mathcal{F}.$$

Prove that $\mathcal{F}$ is a normal family.

Chapter 4

Conformal Mappings

A *conformal* mapping between two open sets of $\mathbb{C}$ is a holomorphic bijection whose inverse is holomorphic as well. This chapter is an exposition of some of the most elementary results for conformal maps, which we will refer to as *biholomorphic*. First we show how to construct local inverses of holomorphic maps around, assuming that the derivative does not vanish; see Theorem 4.1. The next main theorem is the *Open Mapping* Theorem 4.4, which shows that non-constant holomorphic maps carry open sets to open sets. This is actually one of the main ingredients to prove the *Global Inverse Function* Theorem 4.5, from which we obtain that any holomorphic and injective map has non-zero derivative at each point, and admits a global holomorphic inverse.

In Section 4.2, we review the topology of the Extended Complex Plane $\mathbb{C}_\infty$ given by the image of the Riemann Sphere under the stereographic projection. When a biholomorphic map transforms an open set Ω onto itself, it is called an *automorphism* of Ω. We will see how to characterize all the automorphisms of the unit disk in Theorem 4.25, via rotations of certain fractional-linear transformations. Generalizing these transformations gives raise to the *Möbius Transformations* or *Automorphisms* of $\mathbb{C}_\infty$. In Section 4.4, we classify this transformations, study their the fixed points, and define the *cross-ratio* of 4 points in $\mathbb{C}_\infty$, which permits to show that Möbius Transformations map circles of $\mathbb{C}_\infty$ into circles of $\mathbb{C}_\infty$, preserving a given orientation, thus preserving the *left-sides* and *right-sides* of a circle.

One of the key theorems of this course is the *Riemann Mapping* Theorem 4.43, which tells us that any proper open and simply connected subset of $\mathbb{C}$ is *conformally-equivalent* to the unit disk, meaning that there is a biholomorphic map between Ω and $\mathbb{D}$. The proof of this theorem is very far from easy, and requires the usage of Hurwitz's Theorem 2.26, Montel's Theorem 3.16, the Inverse Function Theorem 4.5, and the Characterization of the automorphisms of $\mathbb{D}$; Theorem 4.25. Among other applications, Riemann Mapping Theorem allows to prove that simple-connectedness is equivalent to the validity of the Cauchy Global Theorem; compare with Corollary 1.29.

4.1 The Inverse Function Theorems and Open Mapping Theorem

4.1.1 Local Inverse Function Theorem

We begin with local holomorphic inverses around points with non-zero derivative. In addition, we use the Argument Principle 2.22 to derive a path-integral formula for the derivative.

Theorem 4.1 (Local Inverse Function Theorem)**.** *Let $\Omega \subset \mathbb{C}$ an open set, $f : \Omega \to \mathbb{C}$ holomorphic in Ω, and $z_0 \in \Omega$ so that $f'(z_0) \neq 0$. Then there exists an open set $U \subset \Omega$ with $z_0 \in U$ such that $V := f(U)$ is open, $f'(z) \neq 0$ for all $z \in U$, the restriction of $f\restriction_U \colon U \to f(U)$ is a bijection, and its inverse $f^{-1} : V \to U$ is holomorphic in V, with*

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))} \quad \textit{for all} \quad w \in V. \tag{4.1.1}$$

Furthermore, for every closed disk $\overline{D}(z_0,r) \subset U$, *we have*

$$f^{-1}(w) = \frac{1}{2\pi i}\int_{\partial D(z_0,r)} \frac{zf'(z)}{f(z)-w}\,\mathrm{d}z, \quad \textit{for all} \quad w \in f\left(D(z_0,r)\right). \tag{4.1.2}$$

Proof. The first part and formula (4.1.1) can be deduced using the Inverse Function Theorem for C^1 functions in $\mathbb{R}^2$. Indeed, writing $z_0 = x_0 + iy_0$, the assumption $f'(z_0) \neq 0$ leads us to $\det(Df(x_0,y_0)) \neq 0$, meaning that $Df(x_0,y_0)$ is invertible. Here $Df(x_0,y_0)$ is the differential map of f at (x_0,y_0), regarding f as a C^1 function $\Omega \to \mathbb{R}^2$. Since f' (or Df) is continuous in Ω, by the Inverse Function Theorem in $\mathbb{R}^n$, there exists an open subset U of Ω containing z_0, with $f'(z) \neq 0$ for all $z \in U$, with $f(U)$ open, and with $f_{|U} : U \to f(U)$ being a bijection whose inverse $f^{-1} : f(U) \to U$ is also of class $C^1(f(U),\mathbb{R}^2)$. Moreover, $Df(x,y)$ is invertible for every $(x,y) \in U$ and the differential of f^{-1} at $w \in f(U)$ satisfies

$$D(f^{-1})(w) = \left(Df(f^{-1}(w))\right)^{-1}. \tag{4.1.3}$$

Let us check that $f^{-1} : f(U) \to \mathbb{C}$ is holomorphic in $f(U)$ and prove (4.1.1). Let $w \in f(U)$ and $z \in U$ with $f(z) = w$. Since $f'(z) \neq 0$, considering the limit of the inverse, given $\varepsilon > 0$ there exists $\delta > 0$ so that $0 < |u-z| < \delta$, $u \in U$, implies

$$\left|\frac{u-z}{f(u)-f(z)} - \frac{1}{f'(z)}\right| < \varepsilon. \tag{4.1.4}$$

Now, by the continuity of f^{-1} on $f(U)$, there exists $\eta > 0$ so that $|\xi - w| < \eta$, $\xi \in f(U)$, implies $|f^{-1}(\xi) - f^{-1}(w)| < \delta$. We can thus apply (4.1.4) with $f^{-1}(\xi)$ in place of u to obtain

$$\left|\frac{f^{-1}(\xi)-f^{-1}(w)}{\xi - w} - \frac{1}{f'(z)}\right| < \varepsilon.$$

Finally, to check (4.1.2), assume $\overline{D}(z_0,r) \subset U$, and let $w \in f(D(z_0,r))$, $\xi := f^{-1}(w) \in D(z_0,r)$. Consider the function $f_w(z) = f(z) - w$, defined for $z \in U$. We have that $f_w(\xi) = 0$, and $f_w'(\xi) = f'(\xi) \neq 0$, as f' never vanishes in U. This shows that f_w has a zero at ξ of order 1. Also, notice that f_w has no more zeros in U, since the existence of another zero ξ' in U of f_w would imply that $f(\xi) = f(\xi')$, and the injectivity of f in U would yield $\xi = \xi'$. Applying Theorem 2.22 with the functions $g(z) = z$, f_w, and the path $\partial D(z_0,r)$, we obtain

$$\begin{aligned}\xi = g(\xi) &= \sum_{z \in \mathcal{Z}_U(f_w)} g(z) m_0(f_w,z) \\ &= \frac{1}{2\pi i}\int_{\partial D(z_0,r)} \frac{f_w'(z)}{f_w(z)} g(z)\,\mathrm{d}z = \frac{1}{2\pi i}\int_{\partial D(z_0,r)} \frac{zf'(z)}{f(z)-w}\,\mathrm{d}z.\end{aligned}$$

Since $\xi = f^{-1}(w)$, the above is precisely (4.1.2). □

4.1.2 The Open Mapping Theorem

Our next goal is to show that non-constant holomorphic functions in connected open sets are open. The following proposition is the key ingredient.

Proposition 4.2. *Let* $\Omega \subset \mathbb{C}$ *be open and connected,* $f \in \mathcal{H}(\Omega)$ *non-constant, and* $z_0 \in \Omega$. *Let* $m \in \mathbb{N}$ *be the order of the zero of the function* $\Omega \ni z \mapsto f(z) - f(z_0)$. *Then there exist* $W \subset \Omega$ *containing* z_0, *and a function* $\varphi \in \mathcal{H}(W)$ *with*

- $f(z) = f(z_0) + (\varphi(z))^m$ *for all* $z \in W$.
- $\varphi'(z) \neq 0$ *for all* $z \in W$, φ *is bijective in* $W \to \varphi(W)$ *and* $\varphi^{-1} : \varphi(W) \to W$ *is holomorphic.*

- $\varphi(W) = D(0,r)$ *and* $f(W) = D(f(z_0), r^m)$ *for some* $r > 0$.

Proof. Since f is non-constant, by the Identity Principles for Analytic Functions, there exists a closed disk $\overline{D}(z_0, \varepsilon)$ contained in Ω with $f(z) \neq f(z_0)$ for all $z \in \overline{D}(z_0, \varepsilon) \setminus \{z_0\}$. Thus we can find $g \in \mathcal{H}(\Omega)$ satisfying that

$$f(z) - f(z_0) = (z - z_0)^m g(z) \quad \text{and} \quad g(z) \neq 0 \quad \text{for all} \quad z \in D(z_0, \varepsilon). \tag{4.1.5}$$

Since $D(z_0, \varepsilon)$ is simply connected and g never vanishes in $D(z_0, \varepsilon)$, by Corollary 1.29 g has an m^{th} root in $D(z_0, \varepsilon)$. This means that there exists $h \in \mathcal{H}(D(z_0, \varepsilon))$ with $g(z) = (h(z))^m$ for all $z \in D(z_0, \varepsilon)$. Defining $\varphi(z) := (z - z_0)h(z)$ in $z \in D(z_0, \varepsilon)$, then (4.1.5) gives

$$f(z) - f(z_0) = (\varphi(z))^m \quad \text{for all} \quad z \in D(z_0, \varepsilon). \tag{4.1.6}$$

Also, clearly $\varphi'(z_0) = h(z_0) \neq 0$, because $g(z_0) \neq 0$. We can apply Theorem 4.1 to φ in the open set $D(z_0, \varepsilon)$, thus finding an open set $U \subset D(z_0, \varepsilon)$ containing z_0 as in the mentioned theorem. In particular $\varphi(U)$ is open and contains $\varphi(z_0) = 0$, and $\varphi : U \to \varphi(U)$ is a bijective. Thus we can find $r > 0$ with the property that $D(0, r) \subset \varphi(U)$. The desired open set is defined by

$$W := \varphi^{-1}\left(D(0, r)\right) \cap U.$$

Notice that $\varphi(W) = D(0, r) \cap \varphi(U) = D(0, r)$, which in combination with formula (4.1.6) gives the third claim. Now, the first claim follows from (4.1.6), because W is contained in U. Since $\varphi : U \to \varphi(U)$ is bijective with holomorphic inverse, the same applies for the restriction of φ to W, which proves the second of our claims. $\square$

The next topological definition is of course standard.

Definition 4.3 (Open Mapping). *Let $\Omega \subset \mathbb{C}$ be open, and $f : \Omega \to \mathbb{C}$ a function. We say that f is **open** if $f(U)$ is an open subset of $\mathbb{C}$ for all open subset $U \subset \Omega$.*

Now, the Open Mapping Theorem reads as follows.

Theorem 4.4 (Open Mapping Theorem). *Let $\Omega \subset \mathbb{C}$ be open and connected, and $f \in \mathcal{H}(\Omega)$ non-constant. Then f is open.*

Proof. If $U \subset \Omega$ is an open set, and we check that for every $z \in U$ we can find and open set U_z with $z \in U_z \subset U$ and $f(U_z)$ is open, then we will have that

$$f(U) = f\left(\bigcup_{z \in U} U_z\right) = \bigcup_{z \in U} f\left(U_z\right),$$

is open as well.

Let $z_0 \in U$ and $\delta > 0$ with $D(z_0, \delta) \subset U$. Applying Proposition 4.2 to f, z_0, and the connected open set $D(z_0, \delta)$, we obtain a new open set $W \subset D(z_0, \delta)$ containing z_0 so that $f(W)$ is the open disk $D(w_0, r^m)$ for some $r > 0$.

$\square$

4.1.3 Global Inverse Function Theorem

Finally, we show that if a holomorphic function on a domain is injective, then the derivative never vanishes and the function is biholomorphic onto its image.

Theorem 4.5 (Global Inverse Function and Open Mapping Thorem). *Let $\Omega \subset \mathbb{C}$ open and connected, and $f \in \mathcal{H}(\Omega)$ an injective function in Ω. Then $f'(z) \neq 0$ for all $z \in \Omega$, the image $f(\Omega)$ is open, and the inverse $f^{-1} : f(\Omega) \to \Omega$ is holomorphic in $f(\Omega)$.*

Proof. For the sake of contradiction, assume that $f'(z_0) = 0$ for some $z_0 \in \Omega$. This implies that the function $\Omega \ni z \mapsto f(z) - f(z_0)$ has a zero of order $m \geq 2$ at z_0. For these m, f, Ω, and z_0, let φ and W be as those of Proposition 4.2. If $\varphi(W) = D(0, r)$, let $w \in D(0, r^m) \setminus \{0\}$, and let $\xi_1, \ldots, \xi_m \in D(0, r)$ the (distinct) solutions to the equation $z^m = w$. Since $\varphi : W \to D(0, r)$ is a bijection, there are m points $z_1, \ldots, z_m \in W$ with $\varphi(z_j) = \xi_j$ for all $j = 1, \ldots, m$. Consequently,

$$f(z_j) = f(z_0) + (\varphi(z_j))^m = f(z_0) + w, \quad \text{for all} \quad j = 1, \ldots, m.$$

Since $m \geq 2$, this contradicts that f is injective in Ω. Therefore, we must have $f'(z) \neq 0$ for all $z \in \Omega$.

Now, applying Theorem 4.1, we obtain, for each $z \in \Omega$, an open set $U_z \subset \Omega$ with $z \in U_z$, with $f(U_z)$ open and $f{\restriction_{U_z}}\colon U_z \to f(U_z)$ has a holomorphic inverse. This shows that $f(\Omega)$ is open, and that $f : \Omega \to f(\Omega)$ has a holomorphic inverse $f^{-1} : f(\Omega) \to \Omega$.

□

In the sequel, by a **biholomorphic map** between two open sets U and V, we understand a bijective function $f : U \to V$ so that both f and f^{-1} are holomorphic on U and V respectively.

4.2 Topology in the Extended Complex Plane

Let us recall the definition of the Extended Complex Plane.

Definition 4.6 (Extended Complex Plane)**.** *If ∞ denotes a point at infinity for $\mathbb{C}$, meaning that $\infty \notin \mathbb{C}$, we define the* ***extended complex plane*** *by $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$.*

4.2.1 The Metric Induced by the Stereographic Projection

There is a bijection between between $\mathbb{C}_\infty$ and $\mathbb{S}^2$ via the *stereographic projection.*

Definition 4.7 (The Stereographic Projection)**.** *Denote by $N = (0, 0, 1) \in \mathbb{R}^3$, the* north pole. *The* ***Stereographic Projection*** *onto $\mathbb{C}$ is the mapping $\Pi : \mathbb{S}^2 \to \mathbb{C}_\infty$ given by*

$$\Pi(P) = \begin{cases} \text{the unique point } z \in L_{N,P} \cap \mathbb{C} & \text{if } P \in \mathbb{S}^2 \setminus \{N\} \\ \infty & \text{if } P = N. \end{cases} \tag{4.2.1}$$

Here $L_{N,P}$ denotes the affine line in $\mathbb{R}^3$ passing through N and P.

The mapping $\Pi : \mathbb{S}^2 \to \mathbb{C}_\infty$ in (4.2.1) is well-defined, and below we recall the formulae for the stereographic projection.

Proposition 4.8. *The mapping $\Pi : \mathbb{S}^2 \to \mathbb{C}_\infty$ defined in (4.2.1) satisfies*

$$\Pi(X, Y, Z) = \frac{X + iY}{1 - Z} \equiv \left(\frac{X}{1 - Z}, \frac{Y}{1 - Z}\right) \quad \text{for all} \quad (X, Y, Z) \in \mathbb{S}^2 \setminus \{N\}. \tag{4.2.2}$$

Moreover, $\Pi : \mathbb{S}^2 \to \mathbb{C}_\infty$ is a bijection whose inverse $\Pi^{-1} : \mathbb{C}_\infty \to \mathbb{S}^2$ is given by

$$\Pi^{-1}(z) = \begin{cases} \dfrac{1}{|z|^2 + 1} \left(2\,\mathrm{Re}(z), 2\,\mathrm{Im}(z), |z|^2 - 1\right) & \text{if } z \in \mathbb{C} \\ N = (0, 0, 1) & \text{if } z = \infty. \end{cases} \tag{4.2.3}$$

Proof. See [5, Proposition 1.20]. □

By Proposition 4.8, the stereographic projection Π defines a bijection between $\mathbb{S}^2$ and $\mathbb{C}_\infty$. In fact, we can use Π to define a distance function in $\mathbb{C}_\infty$, and so a topology in $\mathbb{C}_\infty$.

Definition 4.9 (Spherical Metric)**.** *The **spherical or chordal metric** in $\mathbb{C}_\infty$ is the function $\widehat{d}$: $\mathbb{C}_\infty \times \mathbb{C}_\infty \to [0,+\infty)$ given by*

$$\widehat{d}(z,w) := \frac{1}{2}\|\Pi^{-1}(z) - \Pi^{-1}(w)\|_2 = \frac{1}{2}\sqrt{(X-X')^2 + (Y-Y')^2 + (Z-Z')^2}, \tag{4.2.4}$$

whenever $z, w \in \mathbb{C}_\infty$, $\Pi^{-1}(z) = (X,Y,Z) \in \mathbb{S}^2$, $\Pi^{-1}(w) = (X',Y',Z') \in \mathbb{S}^2$.

Note that $\widehat{d}(z,w) \leq \frac{1}{2}\operatorname{diam}(\mathbb{S}^2) = 1$ for all $z, w \in \mathbb{C}_\infty$. We can express $\widehat{d}(z,w)$ solely in terms of $z, w \in \mathbb{C}_\infty$.

Proposition 4.10. *The function $\widehat{d} : \mathbb{C}_\infty \times \mathbb{C}_\infty \to [0,+\infty)$ defines a distance in $\mathbb{C}_\infty$ and*

$$\widehat{d}(z,w) = \begin{cases} \dfrac{|z-w|}{\sqrt{(1+|z|^2)\,(1+|w|^2)}} & \text{if } z,w \in \mathbb{C} \\ \dfrac{1}{\sqrt{|z|^2+1}} & \text{if } z \in \mathbb{C},\ w = \infty \\ 0 & \text{if } z = w = \infty. \end{cases} \tag{4.2.5}$$

Proof. See [5, Proposition 1.22]. □

Corollary 4.11. *The Stereographic Projection $\Pi : \big(\mathbb{S}^2, \|\cdot\|_2\big) \to (\mathbb{C}_\infty, \widehat{d}\,)$ is an homeomorphism between metric spaces. Consequently, $(\mathbb{C}_\infty, \widehat{d}\,)$ is a compact metric space.*

Proof. Actually, by Definition 4.9, Π defines a 1/2-isometry:

$$\widehat{d}(\Pi(u),\Pi(v)) = \frac{1}{2}\|\Pi^{-1}(\Pi(u)) - \Pi^{-1}(\Pi(v))\|_2 = \frac{1}{2}\|u-v\|_2, \quad u,v \in \mathbb{S}^2.$$

The surjectivity of Π is given, for example, by Proposition 4.8.

□

Proposition 4.12. *The metrics $(\mathbb{C}, |\cdot|)$ and $(\mathbb{C}, \widehat{d}\,)$ are equivalent in $\mathbb{C}$. More precisely, one has that*

$$\widehat{d}(z,w) \leq |z-w|, \quad z,w \in \mathbb{C},$$

and for every $z \in \mathbb{C}$ and $\varepsilon > 0$ there exists $\delta > 0$ so that

$$z \in \mathbb{C},\ \widehat{d}(z,w) < \delta \implies |z-w| < \varepsilon.$$

Proof. The first inequality follows from formula (4.2.5) for $\widehat{d}$. For the second property, notice that $\Pi : \big(\mathbb{S}^2 \setminus \{N\}, \|\cdot\|_2\big) \to (\mathbb{C}, |\cdot|)$ is a continuous bijection, by formula (4.2.2). Thus given $\varepsilon > 0$ and $u \in \mathbb{S}^2$, there exists $\delta > 0$ so that $\|u-v\|_2 < \delta$, with $v \in \mathbb{S}^2$ implies $\|\Pi(u) - \Pi(v)\| < \varepsilon$. Now the definition (4.2.4) proves the assertion. □

4.2.2 Balls, Open, Closed, Compact Sets, Closures and Interiors

Our goal is to describe the topoogy induced by the metric $\widehat{d}$.

Remark 4.13. Let us make several observations.

(1) Let us consider balls in $(\mathbb{C}_\infty, \widehat{d}\,)$ centered at ∞. If $r > 0$, by formula (4.2.5), we have that $\widehat{d}(z,\infty) < r$, for $z \in \mathbb{C}$, if and only if

$$|z|^2 > \frac{1}{r^2} - 1.$$

Denoting by $B_{\widehat{d}}(\infty, r)$ the open ball in $(\mathbb{C}_\infty, \widehat{d}\,)$ centered at ∞ and with radius r, the above shows that

$$B_{\widehat{d}}(\infty, r) := \begin{cases} \mathbb{C}_\infty & \text{if } \; r \geq 1, \\ \mathbb{C}_\infty \setminus \overline{D}\left(0, \sqrt{\frac{1}{r^2} - 1}\right) & \text{if } \; r < 1. \end{cases}$$

Similarly, the corresponding closed balls are

$$\overline{B_{\widehat{d}}}(\infty, r) := \begin{cases} \mathbb{C}_\infty & \text{if } \; r \geq 1, \\ \mathbb{C}_\infty \setminus D\left(0, \sqrt{\frac{1}{r^2} - 1}\right) & \text{if } \; r < 1. \end{cases}$$

To examine the balls centered at points of $\mathbb{C}$, observe that by Proposition 4.12 one has that

$$D(z, r) \subset B_{\widehat{d}}(z, r), \quad \overline{D}(z, r) \subset \overline{B_{\widehat{d}}}(z, r), \quad z \in \mathbb{C}, r > 0,$$

and for every $\varepsilon > 0$ and $z_0 \in \mathbb{C}$ there exists $\delta > 0$ so that

$$\overline{B_{\widehat{d}}}(z_0, \delta) \subset D(z_0, \varepsilon).$$

(2) Consequently, if $A \subset \mathbb{C}$, then

$$A \; \text{ is open in } \; (\mathbb{C}, |\cdot|) \iff A \; \text{ is open in } \; (\mathbb{C}_\infty, \widehat{d}\,).$$

In particular, $\mathbb{C}$ is open in $(\mathbb{C}_\infty, \widehat{d}\,)$.

(3) If $F \subset \mathbb{C}$, then by the previous remark,

$$F \; \text{ is closed in } \; (\mathbb{C}, |\cdot|) \iff F \cup \{\infty\} \; \text{ is closed in } \; (\mathbb{C}_\infty, \widehat{d}\,).$$

(4) Concerning convergence of sequences, observe that if $\{z_n\}_n \subset \mathbb{C}$, then (4.2.5) shows that

$$\lim_{n\to\infty} \widehat{d}(z_n, \infty) = 0 \iff \lim_{n\to\infty} |z_n| = \infty.$$

In other words, the convergence $\{z_n\}_n \to \infty$ in the metric $\widehat{d}$ is equivalent to the convergence to infinity of the real numbers $\{|z_n|\}_n$.

On the other hand, if $z \in \mathbb{C}$, and $\{z_n\}_n \subset \mathbb{C}$, then

$$\lim_{n\to\infty} |z_n - z| = 0 \iff \lim_{n\to\infty} \widehat{d}(z_n, z) = 0.$$

This is a consequence of part (1).

(5) If $K \subset \mathbb{C}$ is compact in $(\mathbb{C}, |\cdot|)$, then K is closed in $(\mathbb{C}_\infty, \widehat{d}\,)$, thus compact in $(\mathbb{C}_\infty, \widehat{d}\,)$.

Indeed, if $\{z_n\}_n$ is a sequence contained in K, which converges to $z \in \mathbb{C}_\infty$ with respect to the metric $\widehat{d}$, then by the previous remark, $z \neq \infty$, as the sequence $\{z_n\}_n$ must be bounded. Thus $z \in \mathbb{C}$, and since $\lim_{n\to\infty} \widehat{d}(z_n, z) = 0$, we get that also $\{z_n\}_n$ converges to z in $(\mathbb{C}, |\cdot|)$, by (4). By the compactness of K in $(\mathbb{C}, |\cdot|)$, we get that $z \in K$. This shows that K is closed in $(\mathbb{C}_\infty, \widehat{d}\,)$, and thus compact, because $\mathbb{C}_\infty$ is compact itself.

We can now describe the open sets, the interior, closure, and boundary in $(\mathbb{C}_\infty, \widehat{d}\,)$.

Theorem 4.14. *The open sets of* $(\mathbb{C}_\infty, \widehat{d}\,)$ *are*

$$\mathcal{T}_{\mathbb{C}_\infty} := \{U \subset \mathbb{C} \,:\, U \text{ is open in } (\mathbb{C}, |\cdot|)\} \cup \{\mathbb{C}_\infty \setminus K, \text{ with } K \text{ compact in } (\mathbb{C}, |\cdot|)\}\,.$$

Also, if $A \subset \mathbb{C}$*, the following holds.*

(i) $\operatorname{int}_{\widehat{d}}(A) = \operatorname{int}(A)$.

(ii) If A is bounded, then $\overline{A}^{\widehat{d}} = \overline{A}$ *and* $\partial^{\widehat{d}}(A) = \partial A$.

(iii) If A is unbounded, then $\overline{A}^{\widehat{d}} = \overline{A} \cup \{\infty\}$ *and* $\partial^{\widehat{d}}(A) = \partial A \cup \{\infty\}$.

Proof. By Remark 4.13(2), if $U \subset (\mathbb{C}, |\cdot|)$ is open, then U is also open in $(\mathbb{C}_\infty, \widehat{d})$. Also, if $U = \mathbb{C}_\infty \setminus K$, with $K \subset \mathbb{C}$ compact, we know from Remark 4.13(5) that K is closed in $(\mathbb{C}_\infty, \widehat{d})$, and then U is open in $(\mathbb{C}_\infty, \widehat{d})$.

Conversely, if $U \subset \mathbb{C}_\infty$ is open in $(\mathbb{C}_\infty, \widehat{d})$, we have two possibilities. In the case where $\infty \notin U$, we have that $U \subset \mathbb{C}$, and then U is open in $(\mathbb{C}, |\cdot|)$ by Remark 4.13(2). And if $\infty \in U$, and $0 < r < 1$ is so that $B_{\widehat{d}}(\infty, r) \subset U$, then by Remark 4.13(1), there exists $\delta > 0$ so that $\mathbb{C}_\infty \setminus \overline{D}(0,\delta) \subset U$. Defining $V := U \setminus \{\infty\} \subset \mathbb{C}$, we see that V is open in $(\mathbb{C}_\infty, \widehat{d})$, as U is open and $\{\infty\}$ is closed in $(\mathbb{C}_\infty, \widehat{d})$. Again by Remark 4.13(2), V is open in $(\mathbb{C}, |\cdot|)$, and thus $\mathbb{C} \setminus V$ is closed there. But since $\mathbb{C} \setminus V \subset \overline{D}(0,\delta)$, we get that $K := \mathbb{C} \setminus V$ is a compact subset of $(\mathbb{C}, |\cdot|)$. And clearly $U = \{\infty\} \cup V = C_\infty \setminus K$.

Now, (i) is a consequence of the fact that the interior of a set (in any topological space) is the union of its open subsets, and the characterization of open sets we obtained in Remark 4.13.

Onto (ii), if A is bounded, then $\overline{A}$ is compact in $(\mathbb{C}, |\cdot|)$, and thus closed in $(\mathbb{C}_\infty, \widehat{d})$ by Remark 4.13(5). This shows the inclusion $\overline{A}^{\widehat{d}} \subset \overline{A}$. For the reverse inclusion, if $z \in \overline{A} \subset \mathbb{C}$, then we can find $\{z_n\}_n \subset A$ converging to z in the metric $(\mathbb{C}, |\cdot|)$, and thus with respect to $(\mathbb{C}_\infty, \widehat{d})$ by Remark 4.13(4). This shows that $z \in \overline{A}^{\widehat{d}}$, as desired. The identity for the boundaries follows by recalling that $\partial_\tau(A) = \overline{A}^\tau \cup \operatorname{int}_\tau(A)$ for any topology τ, in combination with the identity for the closures and the identity for the interiors (i).

To show (iii), note that if A is unbounded, then every set E closed in $(\mathbb{C}_\infty, \widehat{d})$ that contains A must contained ∞ as well. Thus, each of those E are of the form $E = \{\infty\} \cup F$, with $F \subset \mathbb{C}$ closed in $(\mathbb{C}, |\cdot|)$ by Remark 4.13(3). Therefore, we may write

$$\begin{aligned}\overline{A}^{\widehat{d}} &= \bigcap \{E \,:\, A \subset E,\, E \text{ closed in } (\mathbb{C}_\infty, \widehat{d})\} \\ &= \{\infty\} \cup \Big(\bigcap \{F \,:\, A \subset F,\, F \text{ closed in } (\mathbb{C}, |\cdot|)\} \Big) = \{\infty\} \cup \overline{A}.\end{aligned}$$

And the identity between boundaries follows from the ones for closures and interiores. □

Finally, we comment on the continuity of functions at points of $\mathbb{C}_\infty$. In general, the continuity of an $f : (X, d_X) \to (Y, d_Y)$ at a point x_0, between metric spaces $(X, d_X), (Y, d_Y)$ is defined (or can be characterized) as:

$$\{x_n\}_n \subset X \setminus \{x\}, \quad \lim_{n\to\infty} d_X(x_n, x) = 0 \implies \lim_{n\to\infty} d_Y(f(x_n), f(x)) = 0.$$

Remark 4.15. Let $U \subset (\mathbb{C}_\infty, \widehat{d})$ be open, and $f : U \to \mathbb{C}_\infty$, both the source and target equipped with the metric $\widehat{d}$. Let also $z_0 \in U$. Using Remark 4.13(2),(4), it is easy to verify that

$$f \text{ is continuous at } z_0 \iff \begin{cases} \lim\limits_{z\to z_0} f(z) = f(z_0) & \text{if } \; z_0 \in U \setminus \{\infty\},\ f(z_0) \in \mathbb{C}, \\ \lim\limits_{z\to z_0} |f(z)| = +\infty & \text{if } \; z_0 \in U \setminus \{\infty\},\ f(z_0) = \infty, \\ \lim\limits_{|z|\to+\infty} f(z) = f(\infty) & \text{if } \; z_0 = \infty,\ f(z_0) \in \mathbb{C}, \\ \lim\limits_{|z|\to+\infty} |f(z)| = +\infty & \text{if } \; z_0 = f(z_0) = \infty. \end{cases}$$

The first and third limit are in the usual sense between complex numbers.

For instance, we consider the function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ given by $f(z) = 1/z$, $f(0) = \infty$, $f(\infty) = 0$. Taking into account the previous characterizations and that

$$\lim_{z\to 0} |f(z)| = \lim_{z\to 0} \frac{1}{|z|} = +\infty, \qquad \lim_{|z|\to+\infty} f(z) = \lim_{|z|\to+\infty} \frac{1}{z} = 0 = f(\infty).$$

Thus, f is continuous at all points of $\mathbb{C}_\infty$.

4.3 Automorphisms of the Unit Disk

Definition 4.16 (Automorphism)**.** *If $\Omega \subset \mathbb{C}$ is open, an **automorphism of** Ω is any bijection $f : \Omega \to \Omega$ with both and f and f^{-1} holomorphic.*

We denote the family of all automorphisms of Ω by $\operatorname{Aut}(\Omega)$.

4.3.1 Functions φ_w : Definition and Properties

We are interested in finding all automorphisms of $\mathbb{D} := D(0,1)$. The key functions are the following.

Definition 4.17. *For each $w \in \mathbb{D}$, we define the mappings $\varphi_w : \mathbb{C} \setminus \{1/\overline{w}\} \to \mathbb{C}$ by*

$$\varphi_w(z) = \frac{z-w}{1-\overline{w}z}, \quad z \in \mathbb{C} \setminus \{1/\overline{w}\}. \tag{4.3.1}$$

We extend φ_w to $\mathbb{C}_\infty$ is $\widetilde{\varphi}_w : \mathbb{C}_\infty \to \mathbb{C}_\infty$, defined by

$$\widetilde{\varphi}_w(z) = \begin{cases} \varphi_w(z) & \text{if } z \in \mathbb{C} \setminus \{1/\overline{w}\}, \\ \infty & \text{if } z = 1/\overline{w}, \\ -1/\overline{w} & \text{if } z = \infty. \end{cases} \tag{4.3.2}$$

In the following proposition, $\mathbb{T}$ will denote the unit circle of $\mathbb{C}$, that is, $\mathbb{T} = \partial D(0,1)$.

Proposition 4.18. *For $w \in \mathbb{D}$, the following properties hold.*

(i) *If $w \neq 0$, then $\varphi_w \in \mathcal{H}(\mathbb{C} \setminus \{1/\overline{w}\})$, φ_w has a pole of order 1 at $1/\overline{w}$, and $\varphi_w(\mathbb{C} \setminus \{1/\overline{w}\}) = \mathbb{C} \setminus \{-1/\overline{w}\}$.*

And if $w = 0$, φ_0 is the identity map in $\mathbb{C}$, with ∞ being a pole of order 1 of φ_0.

(ii) *$\varphi_{-w} \circ \varphi_w$ is the identity in $\mathbb{C} \setminus \{1/\overline{w}\}$, $\varphi_w \circ \varphi_{-w}$ is the identity in $\mathbb{C} \setminus \{-1/\overline{w}\}$, and $\varphi_\omega : \mathbb{C} \setminus \{1/\overline{w}\} \to \mathbb{C} \setminus \{-1/\overline{w}\}$ is biholomorphic.*

(iii) *$\widetilde{\varphi}_w : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a homeomorphism.*

(iv) *$\varphi_w(\mathbb{D}) = \mathbb{D}$ and $\varphi_w(\mathbb{T}) = \mathbb{T}$.*

(v) *$\varphi_w'(z) = \dfrac{1-|w|^2}{(1-\overline{w}z)^2}$ for all $z \in \mathbb{C} \setminus \{1/\overline{w}\}$. In particular, we have $\varphi_w'(0) = 1 - |w|^2$ and $\varphi_w'(w) = \dfrac{1}{1-|w|^2}$.*

Proof. In property (i), for $w \neq 0$, all the properties are immediate or follow very easily, except, perhaps, the part about the pole. But notice that $\varphi_w(z) = g(z)/h(z)$, with $g(z) = z - w$ and $h(z) = 1 - \overline{w}z$, with $g, h \in \mathcal{H}(\mathbb{C})$, $h(1/\overline{w}) = 0$, $g(1/\overline{w})$, $h'(1/\overline{w}) \neq 0$. This shows that φ_w has a pole of order 1 at $1/\overline{w}$. And property (ii) is very easy to check.

Concerning property (iii), that $\widetilde{\varphi}_w : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a bijection follows from (ii) and the formula (4.3.2). The continuity of $\widetilde{\varphi}_w$ and its inverse is clear from the characterization of continuity at points in $\mathbb{C}_\infty$ that we learnt in Remark 4.15.

To prove (iv), observe that

$$1+|w|^2|z|^2-\overline{w}z-w\overline{z}=|1-\overline{w}z|^2, \quad |z-w|^2=|z|^2+|w|^2-\overline{w}z-w\overline{z}.$$

From which it is very easy to verify that $|\widetilde{\varphi}_w(z)|<1$ if and only if $z\in\mathbb{D}$, and $|\widetilde{\varphi}_w(z)|=1$ if and only if $z\in\mathbb{T}$.

Finally, (v) is merely a computation.

□

4.3.2 The Maximum Modulus Principles

Let us recall the Maximum Modulus Principles and the Schwarz Lemma.

Theorem 4.19 (Maximum Modulus Principle I)**.** *Let $\Omega\subset\mathbb{C}$ be open and connected, $f:\Omega\to\mathbb{C}$ be holomorphic in Ω, and $z_0\in\Omega$, $r>0$ so that $\overline{D}(z_0,r)\subset\Omega$. Then*

$$|f(z_0)|\leq \max\{|f(z)|\,:\,z\in\partial D(z_0,r)\}. \tag{4.3.3}$$

Moreover, the inequality (4.3.3) *becomes equality if and only if f is constant in Ω.*

Proof. See [5, Theorem 4.48]. □

As a consequence, holomorphic maps in domains having local maxima are constant.

Corollary 4.20. *Let $\Omega\subset\mathbb{C}$ be open and connected, $f:\Omega\to\mathbb{C}$ be holomorphic in Ω, and assume there exists $z_0\in\Omega$ with $|f(z_0)|\geq|f(z)|$ for all $z\in\Omega$. Then f is constant in Ω.*

Proof. See [5, Theorem 4.49]. □

The second maximum principle tells us that that maximums of holomorphic maps in domains are attained at the boundary.

Theorem 4.21 (Maximum Modulus Principle II)**.** *Let $\Omega\subset\mathbb{C}$ be open, connected, and* ***bounded****. Let $f:\overline{\Omega}\to\mathbb{C}$ be continuous in $\overline{\Omega}$ and holomorphic in Ω. Then, the maximum of f in $\overline{\Omega}$ is attained in the boundary:*

$$\max\{|f(z)|\,:\,z\in\overline{\Omega}\}=\max\{|f(z)|\,:\,z\in\partial\Omega\}. \tag{4.3.4}$$

Proof. See [5, Theorem 4.50]. □

Using what we learnt about the topology in $\mathbb{C}_\infty$ in Section 4.2, we can generalize a bit Theorem 4.21 so that now it holds (in a different form) for unbounded domains.

Corollary 4.22. *Let $\Omega\subset\mathbb{C}$ be open and connected, and $f\in\mathcal{H}(\Omega)$ so that there exists $0\leq M<\infty$ so that*

$$\limsup_{\Omega\ni z\to w}|f(z)|\leq M, \quad \text{for all} \quad w\in\partial_\infty\Omega.$$

Then $|f(z)|\leq M$ for all $z\in\Omega$.

Proof. For $\delta>0$, consider the open set $W_\delta:=\{z\in\Omega\,:\,|f(z)|>M+\delta\}$. We claim that $\overline{W_\delta}^{\mathbb{C}_\infty}\subset\Omega$. Indeed, otherwise there is $w\in\overline{W_\delta}^{\mathbb{C}_\infty}\setminus\Omega$. Since $W_\delta\subset\Omega$, this tells us that

$$w\in\overline{\Omega}^{\mathbb{C}_\infty}\setminus\Omega=\overline{\Omega}^{\mathbb{C}_\infty}\setminus\operatorname{int}_\infty(\Omega)=\partial_\infty\Omega.$$

But then the assumption and at the same time the definition of W_δ lead us to

$$M+\delta\leq\limsup_{W_\delta\ni z\to w}|f(z)|\leq M,$$

a contradiction. Consequently, $\overline{W_\delta}^{\mathbb{C}_\infty} \subset \Omega \subset \mathbb{C}$, and so, by Theorem 4.14, W_δ is bounded and $\overline{W_\delta} \subset \mathbb{C}$.

The next claim is that $W_\delta = \emptyset$. Indeed, otherwise the compactness of $\overline{W_\delta}$ gives $z_0 \in \overline{W_\delta} \subset \Omega$ with

$$M + \delta \leq |f(z_0)| \leq \max\{|f(w)| \,:\, w \in \overline{W_\delta}\} = \max\{|f(z)| \,:\, z \in \Omega\},$$

where the last equality follows from the fact that if $z \in \Omega \setminus \overline{W_\delta}$, then $|f(z)| \leq M + \delta$. Since $z_0 \in \Omega$, Corollary 4.20 gives that f is constant in Ω, thus $|f(z)| = |f(z_0)| \geq M + \delta$ for all $z \in \Omega$, contradicting the hypothesis.

We conclude that $W_\delta = \emptyset$ for $\delta > 0$, as desired. □

Using the Maximum Modulus Principles, one can show Schwarz's Lemma.

Theorem 4.23 (Schwarz Lemma). *Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function with $f(0) = 0$ and $\sup\{|f(z)| \,:\, z \in \mathbb{D}\} \leq 1$. Then*

(i) $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$.

(ii) $|f'(0)| \leq 1$.

(iii) If either (i) holds with equality for some $z \in \mathbb{D} \setminus \{0\}$ or (ii) holds with equality, then there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ so that

$$f(z) = \lambda z \quad \textit{for all} \quad z \in \mathbb{D}.$$

Proof. See [5, Theorem 4.51]. □

4.3.3 Key Inequalities and Characterizations of $\mathrm{Aut}(\mathbb{D})$

We now state some estimates that are useful also in the proof of the Riemann Mapping Theorem 4.43 of the next section.

Proposition 4.24. *Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic with $|f(z)| \leq 1$ for all $z \in \mathbb{D}$. Then, if $w, \xi \in \mathbb{D}$ are so that $f(w) = \xi$, then*

$$|f'(w)| \leq \frac{1 - |\xi|^2}{1 - |w|^2}. \tag{4.3.5}$$

Moreover, the inequality (4.3.5) *is an identity if and only if there exists $\lambda \in \mathbb{T}$ with*

$$f(z) = \varphi_{-\xi}\left(\lambda \varphi_w(z)\right), \quad \textit{for all} \quad z \in \mathbb{D}. \tag{4.3.6}$$

Proof. We consider function

$$g(z) = \varphi_\xi \circ f \circ \varphi_{-w}(z), \quad z \in \mathbb{D}. \tag{4.3.7}$$

Since $w, \xi \in \mathbb{D}$, by Proposition 4.18, $g \in \mathcal{H}(\mathbb{D})$, and $|g(z)| \leq 1$ for all $z \in \mathbb{D}$. Also, because $\varphi_{-w}(0) = w$, $f(w) = \xi$, and $\varphi_\xi(\xi) = 0$, we get that $g(0) = 0$. By Theorem 4.23, $|g'(0)| \leq 1$. On the other hand, Proposition 4.18(v) gives

$$g'(0) = \varphi_\xi'(\xi) \cdot f'(w) \cdot \varphi_{-w}'(0) = f'(w) \cdot \frac{1 - |w|^2}{1 - |\xi|^2}, \tag{4.3.8}$$

which implies (4.3.5) by the estimate $|g'(0)| \leq 1$.

Now, assume that (4.3.5) holds with equality. By formula (4.3.8), this implies that $|g'(0)| = 1$. By Theorem 4.23, there exists $\lambda \in \mathbb{T}$ with $g(z) = \lambda z$ for all $z \in \mathbb{D}$. Since the inverse of φ_ξ is $\varphi_{-\xi}$ (see Proposition 4.18(ii)), by the definition of g in (4.3.7), we derive (4.3.6).

Conversely, if for some $\lambda \in \mathbb{T}$, formula (4.3.6) holds, then differentiating f at w and using Proposition 4.18(v) lead us to

$$f'(w) = \varphi'_{-\xi}(\lambda\varphi_w(w)) \cdot \lambda\varphi'_w(w) = \lambda\frac{1-|\xi|^2}{1-|w|^2},$$

and clearly (4.3.5) holds with equality. □

The following theorem shows that all automorphisms of the unit disk are, up to a rotation, one of the functions φ_w, $w \in \mathbb{D}$.

Theorem 4.25 (Characterization of the Automorphisms of $\mathbb{D}$)**.** *Let $f \in \mathrm{Aut}(\mathbb{D})$ and $w \in \mathbb{D}$ so that $f^{-1}(0) = w$. Then there exists $\lambda \in \mathbb{T}$ so that $f(z) = \lambda\varphi_w(z)$ for all $z \in \mathbb{D}$. Consequently,*

$$\mathrm{Aut}(\mathbb{D}) = \{\lambda \cdot \varphi_w \,:\, w \in \mathbb{D},\, \lambda \in \mathbb{T}\}.$$

Proof. Consider the inverse $g = f^{-1} : \mathbb{D} \to \mathbb{D}$ of f, and bear in mind tht $f'(w) \neq 0$ and $g'(0) = 1/f'(w)$. Applying Proposition 4.24 to both f and g, we get

$$|f'(w)| \leq \frac{1}{1-|w|^2}, \quad |g'(0)| \leq 1-|w|^2,$$

and since $g'(0) = 1/f'(w)$, this shows that

$$|f'(w)| = \frac{1}{1-|w|^2}, \quad |g'(0)| = 1-|w|^2.$$

And by the second part of Proposition 4.24, and bearing in mind that φ_0 is the identity map, there exists $\lambda \in \mathbb{T}$ so that $f(z) = \lambda\varphi_w(z)$ for all $z \in \mathbb{D}$. □

4.4 The Möbius Transformations

The Möbius Transformations are fractional-lineal transformations more general than those from Definition 4.17. They can be used, for example, to transform homeomorphically the unit circle into $\mathbb{R} \cup \{\infty\}$ and the same time transform biholomorphically the interior of the circle to the upper half-plane and the exterior to the lower plane.

4.4.1 Definition, Basic Properties and Examples

Definition 4.26 (Möbius Transformations)**.** *A **Möbius Transformation** is any mapping $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ of the form*

$$T(z) = \frac{az+b}{cz+d}, \text{ for all } z \in \mathbb{C}, \quad T(\infty) = \frac{a}{c}, \quad \text{where } a,b,c,d \in \mathbb{C}, \text{ with } ad-bc \neq 0.$$

In the case $c = 0$, then necessarily $a \neq 0$ and we understand that $T(\infty) = \frac{a}{0} = \infty$. Also, if $c \neq 0$, we understand that

$$T\left(\frac{-d}{c}\right) = \infty.$$

We will denote the family of all Möbius Transformations by $\mathrm{Aut}(\mathbb{C}_\infty)$.

Here are some of the most basic properties of the Möbius Transformations.

Proposition 4.27. *Let $T \in \mathrm{Aut}(\mathbb{C}_\infty)$ with $T(z) = \frac{az+b}{cz+d}$. Then, the following hold.*

(i) $T \in \mathcal{M}(\mathbb{C}_\infty)$ with a pole of order 1 at $-d/c$.

(ii) $T : \mathbb{C} \setminus \{-d/c\} \to \mathbb{C} \setminus \{a/c\}$ *is bijective and holomorphic in* $\mathbb{C} \setminus \{-d/c\}$, *with the inverse* $T^{-1} : \mathbb{C} \setminus \{a/c\} \to \mathbb{C} \setminus \{-d/c\}$ *being a Möbius Transformation as well, and*

$$T^{-1}(z) = \frac{dz - b}{-cz + a}.$$

(iii) $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ *is an homeomorphism.*

Proof. Property (i) is immediate, observing that T is a rational function and recalling Theorem 2.19. The part about the pole is also immediate.

To prove (ii), a simple computation shows the desired formula for T^{-1}, from which the rest of the properties follows at once. And (iii) is a consequence of (ii) together with the fact that T attains the value ∞ either at ∞ (when $c = 0$) or at $-d/c$ when $c = 0$. $\square$

There are four special (and fundamental) examples of Möbius Transformations.

Definition 4.28 (Translation, Dilation, Rotation, Inversion)**.** *Let* $T \in \mathrm{Aut}(\mathbb{C}_\infty)$. *Then,*

- T *is a* ***translation*** *if* $T(z) = z + a$, $z \in \mathbb{C}_\infty$, $a \in \mathbb{C}$.
- T *is a* ***rotation*** *if* $T(z) = \lambda z$ $z \in \mathbb{C}_\infty$, $\lambda \in \mathbb{T}$.
- T *is a* ***dilation*** *if* $T(z) = az$, $z \in \mathbb{C}_\infty$, $a \in \mathbb{C} \setminus \{0\}$.
- T *is an* ***inversion*** *if* $T(z) = 1/z$, $z \in \mathbb{C}_\infty$.

All $T \in \mathrm{Aut}(\mathbb{C}_\infty)$ can be written as the composition of 4 of of those from Definition 4.28.

Proposition 4.29. *If* $T \in \mathrm{Aut}(\mathbb{C}_\infty)$, *then there are* $T_1, T_2, T_3, T_4 \in \mathrm{Aut}(\mathbb{C}_\infty)$ *translation, rotations, dilations, or inversions so that*

$$T = T_4 \circ T_3 \circ T_2 \circ T_1.$$

Proof. Let $T(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$.

In the case where $c = 0$, it is clear that $T = T_2 \circ T_1$, where

$$T_1(z) = \frac{a}{d} z, \quad T_2(z) = z + \frac{b}{d}.$$

Clearly T_1 is a dilation, and T_2 a translation.

In the case where $c \neq 0$, it is straightforward to verify that $T = T_4 \circ T_3 \circ T_2 \circ T_1$, where

$$T_1(z) = z + \frac{d}{c}, \quad T_2(z) = \frac{1}{z}, \quad T_3(z) = \frac{bc - ad}{c^2} z, \quad T_4(z) = z + \frac{a}{c}.$$

We see that T_1 is a translation, T_2 is a inversion, T_3 is a dilation, and T_4 again a translation. $\square$

Consequently, $(\mathrm{Aut}(\mathbb{C}_\infty), \circ)$ is a (non-abelian) group.

Proposition 4.30. *If* $T, S \in \mathrm{Aut}(\mathbb{C}_\infty)$, *then* $S \circ T \in \mathrm{Aut}(\mathbb{C}_\infty)$.

Proof. Using Proposition 4.29, it suffices to check that $S \circ T \in \mathrm{Aut}(\mathbb{C}_\infty)$ when S is one of the transformations of Definition 4.28. But this verification is straightforward. $\square$

4.4.2 Fixed Points and Cross Ratio

We next show that the identity is the only Möbius map with more than two fixed points.

Proposition 4.31 (Fixed Points of $\mathrm{Aut}(\mathbb{C}_\infty)$). *If $T \in \mathrm{Aut}(\mathbb{C}_\infty)$ and T is not the identity map, then there are at most two points $z \in \mathbb{C}_\infty$ with $T(z) = z$.*

Proof. Let $T(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$, and consider first the case where $c \neq 0$. Then $T(\infty) = a/c$, thus ∞ is not a fixed point of T. And for $z \in \mathbb{C}$, $T(z) = z$ if and only if

$$\frac{az+b}{cz+d} = z, \quad \text{that is,} \quad cz^2 + (d-a)z - b = 0,$$

which have at most two solutions.

Now, in the case $c = 0$, clearly $T(\infty) = \infty$. The other fixed points are obtained by solving $z = \frac{a}{d}z + \frac{b}{d}$. If $b = 0$, then $z = \frac{a}{d}z$ only admits the solution $z = 0$, as $a \neq d$ (because T is not the identity map). And if $b \neq 0$, since $d \neq 0$ (as otherwise $ad - bc = 0$), we have the solution $z = b/(d-a)$. In any case, there is at most one fixed point in addition to ∞. □

Consequenly, $T \in \mathrm{Aut}(\mathbb{C}_\infty)$ is completely determined by its action on three distinct points.

Corollary 4.32. *If $T, S \in \mathrm{Aut}(\mathbb{C}_\infty)$ and $z_1, z_2, z_3 \in \mathbb{C}_\infty$ are pairwise distinct points with $S(z_i) = T(z_i)$ for all $i = 1, 2, 3$, then $T = S$ in $\mathbb{C}_\infty$.*

Proof. By Propositions 4.27 and 4.30, we have that $R := T^{-1} \circ S \in \mathrm{Aut}(\mathbb{C}_\infty)$ satisfies

$$R(z_1) = z_1, \quad R(z_2) = z_2, \quad R(z_3) = z_3.$$

Thus R has three distinct fixed points in $\mathbb{C}_\infty$. By Proposition 4.31, R is the identity map, whence $T = S$. □

The following definition will lead us to a practical way to construct Möbius Maps passing through three distinct points of $\mathbb{C}_\infty \times \mathbb{C}_\infty$.

Definition 4.33 (Cross-Ratio). *Let $z_2, z_3, z_4 \in \mathbb{C}_\infty$ three pairwise distinct points. Denote by $S := S_{(z_2,z_3,z_4)} : \mathbb{C}_\infty \to \mathbb{C}_\infty$ the Möbius Transformation defined by*

$$S(z) = \begin{cases} \frac{z-z_3}{z-z_4}\frac{z_2-z_4}{z_2-z_3} & \textit{if } z_2, z_3, z_4 \in \mathbb{C}_\infty, \\ \frac{z-z_3}{z-z_4} & \textit{if } z_2 = \infty, \\ \frac{z_2-z_4}{z-z_4} & \textit{if } z_3 = \infty, \\ \frac{z-z_3}{z_2-z_3} & \textit{if } z_4 = \infty. \end{cases} \tag{4.4.1}$$

Notice that S is the unique $S \in \mathrm{Aut}(\mathbb{C}_\infty)$ so that $S(z_2) = 1$, $S(z_3) = 0$, $S(z_4) = \infty$.

*Now, if $z_1 \in \mathbb{C}_\infty \setminus \{z_2, z_3, z_4\}$, we define the **cross-ratio of** (z_1, z_2, z_3, z_4) of z_1, z_2, z_3, z_4 (where the order matters), by the number*

$$(z_1, z_2, z_3, z_4) := S(z_1) = S_{(z_2,z_3,z_4)}(z_1) \in \mathbb{C}_\infty.$$

The following result is often called the *Symmetry Principle*, and shows that Möbius maps always preserve cross-ratios.

Proposition 4.34. *Let $z_2, z_3, z_4 \in \mathbb{C}_\infty$ be distinct points and $T \in \mathrm{Aut}(\mathbb{C}_\infty)$. Then,*

$$(z, z_2, z_3, z_4) = (T(z), T(z_2), T(z_3), T(z_4)), \quad \textit{for all} \quad z \in \mathbb{C}_\infty.$$

Proof. Define $S = S_{(z_2,z_3,z_4)}$ as in (4.4.1), and $R = S \circ T^{-1} \in \mathrm{Aut}(\mathbb{C}_\infty)$. We have that

$$R(T(z_2)) = 1, \quad R(T(z_3)) = 0, \quad R(T(z_4)) = \infty.$$

Thus, according to the notation in Definition 4.33, $R = S_{(T(z_2),T(z_3),T(z_4))}$, as Corollary 4.32 guarantees that there is only one transformation mapping $T(z_2), T(z_3), T(z_4)$ to $1, 0, \infty$ respectively. Consequenly, for each $z \in \mathbb{C}_\infty$, one has

$$(z, z_2, z_3, z_4) = S(z) = S \circ T^{-1}(T(z)) = R(T(z)) = (T(z), T(z_2), T(z_3), T(z_4)).$$

□

We now easily obtained the mentioned determination of Möbius Maps.

Corollary 4.35. *Let $z_2, z_3, z_4 \in \mathbb{C}_\infty$ be three pairwise distinct points and $w_2, w_3, w_4 \in \mathbb{C}_\infty$ another three pairwise distinct. Then there exists a unique $T \in \mathrm{Aut}(\mathbb{C}_\infty)$ so that $T(z_i) = w_i$ for all $i = 2, 3, 4$.*

Proof. Define $S_1, S_2 \in \mathrm{Aut}(\mathbb{C}_\infty)$ by the pointwise cross-ratio:

$$S_1(z) = (z, z_2, z_3, z_4), \quad S_2(z) = (z, w_2, w_3, w_4), \quad z \in \mathbb{C}_\infty.$$

Taking $T = S_2^{-1} \circ S_1$ we get a desired Möbius transformation, which is unique by Corollary 4.32.

□

4.4.3 Preservation of Circles and Orientations

Let us extend a bit the notion of circle in the extended plane $\mathbb{C}_\infty$.

Definition 4.36 (Circles of $\mathbb{C}_\infty$)**.** *A* ***circle of*** $\mathbb{C}_\infty$ *is a set $\Gamma \subset \mathbb{C}_\infty$ so that either Γ is a (usual) circle of $\mathbb{C}$ or $\Gamma = \ell \cup \{\infty\}$, with $\ell \subset \mathbb{C}$ a line of $\mathbb{C}$.*

By a line of $\mathbb{C}$ we mean a 1-dimensional affine subspace of $\mathbb{R}^2$.

The reason we call those lines through ∞ *circles* is that their image under the inverse of the stereographic projection is precisely a circle of $\mathbb{S}^2$ passing through the north pole.

In the sequel, denote $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$.

Lemma 4.37. *Let $T \in \mathrm{Aut}(\mathbb{C}_\infty)$. Then $T^{-1}(\mathbb{R}_\infty)$ is a circle of $\mathbb{C}_\infty$.*

Proof. If $T(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$, and $w \in \mathbb{C}_\infty$, then $T(w) \in \mathbb{R}_\infty$ is equivalent to

$$\frac{aw+b}{cw+d} = T(w) = \overline{T(w)} = \frac{\overline{a}\overline{w} + \overline{b}}{\overline{c}\overline{w} + \overline{d}},$$

or, equivalently

$$(a\overline{c} - \overline{a}c)|w|^2 + (a\overline{d} - \overline{b}c)w + (b\overline{c} - d\overline{a})\overline{w} + (b\overline{d} - \overline{b}d) = 0. \tag{4.4.2}$$

Now, consider first the case where $a\overline{c} \in \mathbb{R}$. Then, letting $i\xi = a\overline{d} - \overline{b}c$, we have $i\overline{\xi} = b\overline{c} - d\overline{a}$, and, after some simplications, (4.4.2) becomes

$$\mathrm{Re}(\xi w) = \mathrm{Im}(\overline{b}d).$$

This defines a line of $\mathbb{C}$, if we verify that $\xi \neq 0$. Indeed, otherwise $a\overline{d} = \overline{b}c$. On the other hand either $a \neq 0$ or $c \neq 0$ (by the general condition $ad - bc \neq 0$). In the first case $a \neq 0$, then using that $a\overline{c} = \overline{a}c$, we get

$$a\overline{d} = \overline{b}c \implies \overline{a}a\overline{d} = \overline{b}\overline{a}c = \overline{b}a\overline{c} \implies \overline{a}\overline{d} = \overline{b}\overline{c} \implies ad = bc,$$

a contradiction. Similarly, we arrive at a contradiction assuming that $c \neq 0$, and therefore we have shown that $\xi \neq 0$.

Now, assume $a\bar{c} \notin \mathbb{R}$, thus $a\bar{c} - \bar{a}c \neq 0$. Letting $\xi \in \mathbb{C}$ be so that $i\xi = a\bar{d} - \bar{b}c$, (4.4.2) becomes

$$|w|^2 + \gamma w + \overline{\gamma w} = K, \quad \text{where} \quad \gamma = \frac{\xi}{2\,\mathrm{Im}(a\bar{c})}, \quad K = \frac{\mathrm{Im}(\bar{b}c)}{\mathrm{Im}(a\bar{c})}.$$

This describes a circle of $\mathbb{C}$, if we verify that $K + |\gamma|^2 > 0$. Indeed,

$$\begin{aligned} K + |\gamma|^2 &= \frac{\mathrm{Im}(\bar{b}c)}{\mathrm{Im}(a\bar{c})} + \frac{|\xi|^2}{4\,\mathrm{Im}(a\bar{c})^2} = \frac{1}{4\,\mathrm{Im}(a\bar{c})^2}\left(|\xi|^2 - (\bar{b}d - b\bar{d})(a\bar{c} - \bar{a}c)\right) \\ &= \frac{1}{4\,\mathrm{Im}(a\bar{c})^2}\left(|a|^2|d|^2 + |b|^2|d|^2 - ad\overline{bc} - bc\overline{ad}\right) = \frac{|ad - bc|^2}{4\,\mathrm{Im}(a\bar{c})^2} > 0. \end{aligned}$$

□

The condition "real cross-ratio" can be used to determine the membership to circles of $\mathbb{C}_\infty$.

Proposition 4.38. *Let $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ be four distinct points. The following are equivalent:*

(a) $(z_1, z_2, z_3, z_4) \in \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$.

(b) There exists a circle of $\mathbb{C}_\infty$ passing through z_1, z_2, z_3, z_4.

Proof. The three points z_2, z_3, z_4 determine a unique circle of $\mathbb{C}_\infty$, which we denote by Γ. Defining $S := S_{(z_2,z_3,z_4)}$ as in (4.4.1) we have that $S(z_2) = 1 \in \mathbb{R}_\infty$, $S(z_3) = 0 \in \mathbb{R}_\infty$, $S(z_4) \in \mathbb{R}_\infty$. By Lemma 4.37, $S^{-1}(\mathbb{R}_\infty)$ is a circle containing z_2, z_3, z_4, thus $S^{-1}(\mathbb{R}_\infty) = \Gamma$. Thus $(z_1, z_2, z_3, z_4) = S(z_1) \in \mathbb{R}_\infty$ if and only if $z_1 \in \Gamma$.

□

We are now ready to show that Möbius transformations map circles to circles.

Proposition 4.39. *Let $T \in \mathrm{Aut}(\mathbb{C}_\infty)$ and Γ be a circle of $\mathbb{C}_\infty$, Then $T(\Gamma)$ is a circle of $\mathbb{C}_\infty$.*

Proof. Let Γ be a circle of $\mathbb{C}_\infty$, and $z_2, z_3, z_4 \in \Gamma$ be three distinct points. If $T \in \mathrm{Aut}(\mathbb{C}_\infty)$, denote $w_i = T(z_i)$ for $i = 2, 3, 4$. Since $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is bijective, w_2, w_3, w_4 are three distinct points which determine a circle $\Gamma' \subset \mathbb{C}_\infty$. If $z \in \mathbb{C}_\infty$, then by Proposition 4.38, $z \in \Gamma$ if and only if $(z, z_2, z_3, z_4) \in \mathbb{R}_\infty$. By Proposition 4.34, this is equivalent to saying that $(T(z), w_2, w_3, w_4)$. And again by Proposition 4.38, this is equivalent to $T(z) \in \Gamma'$. □

A consequence of Proposition 4.39 is that the mapping from Corollary 4.35 is so that it maps the circle determine by the three point input to the one determined to by the three point output.

Corollary 4.40. *Given two circles Γ, Γ' of $\mathbb{C}_\infty$, there exists $T \in \mathrm{Aut}(\mathbb{C}_\infty)$ so that $T(\Gamma) = \Gamma'$.*

In fact, given three distinct points $(z_2, z_3, z_4) \in \Gamma$ and three distinct points $(w_2, w_3, w_4) \in \Gamma'$, there exists a unique $T \in \mathrm{Aut}(\mathbb{C}_\infty)$ with $T(\Gamma) = \Gamma'$ and $T(z_i) = w_i$ for $i = 2, 3, 4$.

Proof. It is immediate from Corollary 4.35 and Proposition 4.39. □

We next focus on preserving not only circles of $\mathbb{C}_\infty$ but also their *interiors* and *exteriors* with respect to a given *orientation.*

Definition 4.41 (Right and Left Side of Circles)**.** *Given a circle Γ of $\mathbb{C}_\infty$, and three distinct points $z_2, z_3, z_4 \in \Gamma$, the **orientation determined by** z_2, z_3, z_4 (where the order matters) if ordered triple $\mathcal{O} = (z_2, z_3, z_4)$. The **right side of** Γ **with respect to** $\mathcal{O}$ is the set*

$$R(\Gamma, \mathcal{O}) = \{z \in \mathbb{C}_\infty \,:\, \mathrm{Im}(z, z_2, z_3, z_4) > 0\}.$$

*The **left side of** Γ **with respect to** $\mathcal{O}$ is the set*

$$L(\Gamma, \mathcal{O}) = \{z \in \mathbb{C}_\infty \,:\, \mathrm{Im}(z, z_2, z_3, z_4) < 0\}.$$

With the notation of Definition 4.41, one has, by virtue of Proposition 4.38,

$$\Gamma = \{z \in \mathbb{C}_\infty : \operatorname{Im}(z, z_2, z_3, z_4) = 0\}.$$

Since the mapping $\mathbb{C}_\infty \mapsto (z, z_2, z_3, z_4)$ is a Möbius Transformation, in particular is an homeomorphism $\mathbb{C}_\infty \to \mathbb{C}_\infty$.

The Möbius Transformations map right (left) sides of circles to right (left) sides of circles.

Theorem 4.42 (Orientation Principle). *Let $\Gamma_1, \Gamma_2 \subset \mathbb{C}_\infty$ be circles and $T \in \operatorname{Aut}(\mathbb{C}_\infty)$ so that $T(\Gamma_1) = \Gamma_2$. Then, for the orientations $\mathcal{O}_1 := (z_2, z_3, z_4)$ of Γ_1 and $\mathcal{O}_2 := (T(z_2), T(z_3), T(z_4))$ of Γ_2, we have that*

$$T\left(R(\Gamma_1, \mathcal{O}_1)\right) = R(\Gamma_2, \mathcal{O}_2) \quad \text{and} \quad T\left(L(\Gamma_1, \mathcal{O}_1)\right) = L(\Gamma_2, \mathcal{O}_2).$$

Proof. It is immediate from Definition 4.41 and Proposition 4.34. □

4.5 The Riemann Mapping Theorem

We split the proof of this theorem into five claims.

Theorem 4.43 (Riemann Mapping Theorem). *Let $\Omega \subset \mathbb{C}$ be open and connected with $\Omega \neq \mathbb{C}$ and so that $\Gamma \simeq 0$ in Ω for all cycles Γ in Ω. Then there exists a biholomorphic map $F : \Omega \to \mathbb{D}$.*

Moreover, given $z_0 \in \Omega$ and $\theta_0 \in (-\pi, \pi]$, there exists a unique $F : \Omega \to \mathbb{D}$ with

$$F(z_0) = 0, \quad \operatorname{Arg}(F'(z_0)) = \theta_0.$$

Proof. We define the family of maps

$$\mathcal{F} := \{\psi : \Omega \to \mathbb{D} : \psi \in \mathcal{H}(\Omega) \text{ and } \psi \text{ is injective in } \Omega\}.$$

We split the proof into three main steps.

Claim 1: $\mathcal{F} \neq \emptyset$.

Consider a point $w_0 \in \mathbb{C} \setminus \Omega$ and define $h : \Omega \to \mathbb{C}$ by formula $h(z) = z - w_0$, $z \in \Omega$. By the assumption on Ω and Theorem 1.19, we can find $\varphi \in \mathcal{H}(\Omega)$ with $\varphi^2 = h$ in Ω. Note that φ is injective and $\varphi(z) \neq 0$ for all $z \in \Omega$, because so is h. By the Open Mapping Theorem 4.4, one has that $\varphi(\Omega)$ is open, and thus contains a closed disk $\overline{D}(w, r)$.

And notice that if $z \in \Omega$ and $\varphi(z) \in \overline{D}(-w, r)$, then $-\varphi(z) \in \overline{D}(w, r) \subset \varphi(\Omega)$, and thererefore we can find $z_1 \in \Omega$ with $-\varphi(z) = \varphi(z_1)$. The injectivity of h implies that $z = z_1$, and so $\varphi(z) = 0$, a contradiction. This shows that $\varphi(\Omega) \cap \overline{D}(-w, r) = \emptyset$, allowing us to define

$$\psi : \Omega \to \mathbb{D}, \quad \psi(z) = \frac{r}{\varphi(z) + w}, \quad z \in \Omega.$$

The image of ψ is indeed contained in $\mathbb{D}$, as $|\varphi(z) + w| > r$. And clearly ψ is injective, as so is φ. Hence **Claim 1** is proven.

Claim 2: Given $\psi \in \mathcal{F}$ so that $\psi(\Omega) \subsetneq \mathbb{D}$ and $z_0 \in \Omega$, there exists $\psi_1 \in \mathcal{F}$ with $|\psi_1'(z_0)| > |\psi'(z_0)|$.

Let $w \in \mathbb{D} \setminus \psi(\Omega)$, and consider the map $\varphi_w \circ \psi$; where φ_w is that of (4.3.1). Since $\varphi_w : \mathbb{D} \to \mathbb{D}$ is biholomorphic (see e.g. Proposition 4.18), we clearly have that $\varphi_w \circ \psi$ is holomorphic and injective $\Omega \to \mathbb{D}$. Moreover, $\varphi_w \circ \psi(z) \neq 0$ for all $z \in \Omega$, as $(\psi_w)^{-1}(0) = \{w\}$ and $w \notin \psi(\Omega)$. Making use again of the assumption on Ω and Theorem 1.19, we can find $g \in \mathcal{H}(\Omega)$ so that $g^2 = \varphi_w \circ \psi$ in Ω.

Since $|g(z)| < 1$ for all $z \in \Omega$, and g is injective in Ω (as so are φ_w and ψ), the map $\psi_1 : \Omega \to \mathbb{D}$ given $\psi_1 := \varphi_{g(z_0)} \circ g$, is well-defined, holomorphic and injective, that is, $\psi_1 \in \mathcal{F}$. Now, if $s : \mathbb{C} \to \mathbb{C}$ is the square function $s(z) = z^2$, then we can write

$$\psi = \varphi_{-w} \circ g^2 = \varphi_{-w} \circ s \circ \varphi_{-g(z_0)} \circ \varphi_{g(z_0)} \circ g = h \circ \psi_1, \quad h := \varphi_{-w} \circ s \circ \varphi_{-g(z_0)}.$$

This implies that

$$\psi'(z_0) = h'(\psi_1(z_0)) \cdot \psi_1'(z_0) = h'(0) \cdot \psi_1'(z_0). \tag{4.5.1}$$

Also, observe that $h : \mathbb{D} \to \mathbb{D}$ and is holomorphic, and that h cannot be injective in $\mathbb{D}$, as the $\varphi_{-w}, \varphi_{-g(z_0)}$ are bijections, and s is not injective in $\mathbb{D}$. We can apply Proposition 4.24 to derive

$$|h'(0)| \leq 1 - |h(0)|^2.$$

Moreover the first inequality is strict because otherwise Proposition 4.24 would imply that $h(z) = \varphi_{-h(0)}(\lambda z)$ for all $z \in \mathbb{D}$ and some $\lambda \in \mathbb{T}$, obtaining that h is injective in $\mathbb{D}$, a contradiction. Therefore, we must have

$$|h'(0)| < 1 - |h(0)|^2 \leq 1 \tag{4.5.2}$$

Combining (4.5.1) and (4.5.2) gives

$$|\psi_1'(z_0)| = \frac{|\psi'(z_0)|}{|h'(0)|} < |\psi'(z_0)|.$$

Claim 3: There exists $h \in \mathcal{F}$ with $h(\Omega) = \mathbb{D}$.

Fix $z_0 \in \Omega$ and $r_0 > 0$ so that $\overline{D}(z_0, r_0) \subset \Omega$, and by the Cauchy Inequalities we have that

$$|\psi'(z_0)| \leq \frac{\sup\{|\psi(z)| \,:\, z \in \partial D(z_0, r_0)\}}{r_0} \leq \frac{1}{r_0}, \quad \psi \in \mathcal{F}.$$

Thus $\{|\psi'(z_0)| \,:\, \psi \in \mathcal{F}\}$ is a (nonempty, by **Claim 1**) bounded subset of $\mathbb{R}$, and so we can find a sequence $\{\psi_n\}_n \subset \mathcal{F}$ with

$$\lim_{n\to\infty} |\psi_n'(z_0)| = a := \sup\{|\psi'(z_0)| \,:\, \psi \in \mathcal{F}\} \in \mathbb{R}. \tag{4.5.3}$$

Observe that, since each $\psi \in \mathcal{F}$ is injective, by Theorem 4.5, $\psi'(z_0) \neq 0$ for all $\psi \in \mathcal{F}$, and so $a > 0$. Also, notice $|\psi_n(z)| \leq 1$ for all $n \in \mathbb{N}$, and so $\{\psi_n\}_n$ is a locally bounded family of $\mathcal{H}(\Omega)$. By Montel's Theorem 3.16 (or Corollary 3.18), there exists a subsequence of $\{\psi_n\}_n$, which we keep denoting by $\{\psi_n\}_n$, converging locally uniformly in Ω to some $h \in \mathcal{H}(\Omega)$, and still satisfying (4.5.3). By the pointwise convergence, $|h(z)| \leq 1$ for all $z \in \Omega$.

Since ψ_n is holomorphic and injective, and Ω is connected, a consequence of Hurwitz's Theorem 2.26 (see Exercise 2.20), is that h is either constant or injective in Ω. But by Weierstrass Theorem 3.1, $\{\psi_n'\}_n$ also converges locally uniformly to h' in Ω. In particular

$$|h'(z_0)| = \lim_{n\to\infty} |\psi_n'(z_0)| = a > 0.$$

Therefore h cannot be a constant, whence h is injective. Also, by the Open Mapping Theorem 4.4, $h(\Omega) \subset \overline{\mathbb{D}}$ implies that $h(\Omega) \subset \mathbb{D}$. Consequently $h \in \mathcal{F}$.

It only remains to show that $h(\Omega) = \mathbb{D}$. Assume, for the sake of contradiction, that $h(\Omega) \subsetneq \mathbb{D}$. By **Claim 2**, we can find another function $h_1 \in \mathcal{F}$ with $|h_1'(z_0)| > |h'(z_0)| = a$. But this contradicts the definition of a, see (4.5.3).

We have shown **Claim 3**, and so the first part of the theorem.

Claim 4: Given $z_0 \in \Omega$ and $\theta_0 \in (-\pi, \pi]$, there exists $F : \Omega \to \mathbb{D}$ biholomorphic with $F(z_0) = 0$ and $\operatorname{Arg}(F'(z_0)) = \theta_0$.

We already know that there is $F : \Omega \to \mathbb{D}$ biholomorphic. Define $G := \varphi_{F(z_0)} \circ F$, which is also biholomorphic between Ω and $\mathbb{D}$, but additionally satisfies $G(z_0) = 0$, and $G'(z_0) \neq 0$ by Theorem 4.5. Now it is enough to multiply G by a suitable constant $\lambda \in \mathbb{T}$, obtaining $F_1 := \lambda \cdot G : \Omega \to \mathbb{D}$ biholomorphic, with $F_1(z_0) = 0$ and $\operatorname{Arg}(F_1'(z_0)) = \theta_0$.

Claim 5: Given $z_0 \in \Omega$ and $\theta_0 \in (-\pi, \pi]$, there exists a *unique* $F : \Omega \to \mathbb{D}$ biholomorphic with $F(z_0) = 0$ and $\operatorname{Arg}(F'(z_0)) = \theta_0$.

We know that at least one of those F's exists by **Claim 4**. Let F_1 and F_2 be two such functions, and define $\psi := F_1 \circ F_2^{-1} : \mathbb{D} \to \mathbb{D}$, which is an automorphism of $\mathbb{D}$. Therefore, by Theorem 4.25, there are $\lambda \in \mathbb{T}$ and $w \in \mathbb{D}$ so that

$$\psi(z) = \lambda \frac{z - w}{1 - \overline{w}z}, \quad z \in \mathbb{D}.$$

Since $F_1(z_0) = F_2(z_0) = 0$, we have that $\psi(0) = 0$, and therefore $w = 0$. Consequently, $\psi(z) = \lambda z$ for all $z \in \mathbb{D}$. In addition,

$$\lambda = \psi'(0) = F_1'(F_2^{-1}(0)) \cdot (F_2^{-1})'(0) = \frac{F_1'(z_0)}{F_2'(z_0)} = \frac{|F_1'(z_0)|}{|F_2'(z_0)|} > 0,$$

where the last inequality follows from the fact that $F_1'(z_0)$ and $F_2'(z_0)$ have the same principal argument. The above implies $\lambda > 0$, and so $\lambda = 1$. $\square$

Using Riemann's Mapping Theorem 4.43, we can show simple-connectedness actually characterizes the validity of the Cauchy Global Theorem, thus characterizing also all the stamtements from Corollary 1.29.

Corollary 4.44. *Let $\Omega \subset \mathbb{C}$ be open and connected. The following statements are equivalent.*

(i) *Ω is simply connected.*

(ii) *$\Gamma \simeq 0$ in Ω for all cycles Γ in Ω.*

Proof. We can assume that $\Omega \neq \mathbb{C}$. If Ω is simply connected, then Ω satisfies (ii) by Corollary 1.29. Conversely, if (ii) holds, then by Theorem 4.43 there exists a biholomorphic map $F : \Omega \to \mathbb{D}$. Since $\mathbb{D}$ is simply-connected and F is a homeomorphism, we have that Ω is simply connected. $\square$

4.6 Exercises

Exercise 4.1. *Let $\Omega, U \subset \mathbb{C}$ be two open sets, and functions $f : \Omega \to U$, $g : U \to \mathbb{C}$ with $f(\Omega) = U$, $g \in \mathcal{H}(U)$, and $g(f(z)) = z$ for all $z \in \Omega$. Prove that $f \in \mathcal{H}(\Omega)$, $g'(f(z)) \neq 0$ for all $z \in \Omega$, and that*

$$f'(z) = \frac{1}{g'(f(z))}, \quad z \in \Omega.$$

Exercise 4.2. *Denoting $\mathbb{D} = D(0,1)$ and $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. Prove that*

$$f(z) = i\frac{1 - z}{1 + z}$$

is holomorphic and injective in $\mathbb{D}$, with $f(\mathbb{D}) = \mathbb{H}$, and find the inverse $f^{-1} : \mathbb{H} \to \mathbb{C}$ of f.

Exercise 4.3. *Let $\Omega \subset \mathbb{C}$ be open and connected, and $f \in \mathcal{H}(\Omega)$ so that $\mathrm{Re}(f)(z) \geq 0$ for all $z \in \Omega$. Use the Open Mapping Theorem to show that actually $\mathrm{Re}(f)(z) > 0$ for all $z \in \mathbb{C}$.*

Exercise 4.4. *Let $\Omega \subset \mathbb{C}$ be open, $z_0 \in \Omega$, and $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ be holomorphic and injective in $\Omega \setminus \{z_0\}$. Prove that f cannot have an essential singularity at z_0.*

Exercise 4.5. *Let $f : \mathbb{C} \to \mathbb{C}$ holomorphic and injective in $\mathbb{C}$. Prove that f must be of the form $f(z) = az + b$, with $a, b \in \mathbb{C}$, $a \neq 0$.*

Suggestion: Apply the previous exercise to show that f has either a removable singularity or pole at ∞. Which holomorphic functions in $\mathbb{C}$ have that behavior at ∞?

Exercise 4.6. *Let $f \in \mathcal{H}(\mathbb{C})$ so that $f(f(1/n)) = 1/n$ for all $n \in \mathbb{N}$. Prove that f is of the form $f(z) = z$ or $f(z) = w - z$ for some $w \in \mathbb{C}$.*

Suggestion: Show that f is injective and apply the previous exercise.

Exercise 4.7. *Determine whether there exists $f \in \operatorname{Aut}(\mathbb{D})$ with the following restrictions.*

(a) $f(0) = 0$ *and* $f(1/2) = i/2$.

(b) $f(0) = 0$ *and* $f(1/3) = 1/4$.

(c) $f(0) = 1/2$ *and* $f'(0) = 3/4$.

Exercise 4.8. *Prove that if $f \in \operatorname{Aut}(\mathbb{D})$ with $f(0) = 0$ and $f'(0) > 0$, then $f(z) = z$ for all $z \in \mathbb{D}$.*

Exercise 4.9. *Prove that if $f \in \operatorname{Aut}(\mathbb{D})$ and f has two fixed points, then $f(z) = z$ for all $z \in \mathbb{D}$.*

Exercise 4.10. *Prove that there is no $f : \mathbb{D} \to \mathbb{D}$ holomorphic with $f(1/2) = 3/4$ and $f'(1/2) = 2/3$.*

Exercise 4.11. *Prove that there is a unique $f : \mathbb{D} \to \mathbb{D}$ holomorphic with $f(0) = 1/2$ and $f'(0) = 3/4$, and find the formula for f.*

Exercise 4.12. *Let $\Omega \subset \mathbb{C}$ be open, $r_2 > r_1 > 0$ with $\{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\} \subset \Omega$ and $f \in \mathcal{H}(\Omega)$. Denoting*

$$M(r) := \sup\{|f(z)| : |z| = r\}, \quad r_1 \leq r \leq r_2,$$

prove the inequalities

$$\log M(r) \leq \left(\frac{\log r_2 - \log r}{\log r_2 - \log r_1}\right) \log M(r_1) + \left(\frac{\log r - \log r_1}{\log r_2 - \log r_1}\right) \log M(r_2), \quad r_1 \leq r \leq r_2.$$

Exercise 4.13. *Let $\Omega \subset \mathbb{C}$ be open and connected, $R > 0$ so that $\overline{D}(0, R) \subset \Omega$ and $f \in \mathcal{H}(\Omega)$. We denote,*

$$M(r) := \sup\{|f(z)| : |z| = r\}, \quad A(r) := \sup\{|\operatorname{Re} f(z)| : |z| = r\}, \quad 0 \leq r \leq R.$$

Prove the inequalities

$$M(r) \leq \frac{R + r}{R - r} \left(A(R) + |f(0)|\right), \quad 0 \leq r \leq R.$$

Exercise 4.14. *Let $\Omega \subset \mathbb{C}$ be open and connected, and U open, connected and bounded with $\overline{U} \subset \Omega$. If $f \in \mathcal{H}(\Omega)$ is so that $|f|$ is constant in ∂U, show that either f is constant in Ω or $|f|$ has (at least) one zero in $\overline{U}$.*

Exercise 4.15. *Let $\Omega \subset \mathbb{C}$ be open, bounded, and connected, and $\{f_n\}_n$ a sequence of holomorphic functions in Ω, continuous in $\overline{\Omega}$, which converges uniformly in $\partial\Omega$. Prove that $\{f_n\}_n$ converges uniformly in $\overline{\Omega}$.*

Exercise 4.16. *Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic. Prove that there exists $\{z_n\}_n \subset \mathbb{D}$ with $\lim_{n \to \infty} |z_n| = 1$ and $\{f(z_n)\}_n$ is bounded.*

Exercise 4.17. *Prove that if* $|w| < 1$ *and* $|\xi| > 1$, *then*

$$\frac{|w| - |\xi|}{1 - |w\xi|} \leq \left| \frac{w + \xi}{1 + w\overline{\xi}} \right| \leq \frac{|w| + |\xi|}{1 + |w\xi|}.$$

Exercise 4.18. *Let* $f \in \mathcal{H}(\mathbb{D})$ *with* $|f(z)| \leq 1$ *for all* $z \in \mathbb{D}$. *Prove that, for all* $z \in \mathbb{D}$,

$$\frac{|f(0)| - |z|}{1 - |f(0)z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 + |f(0)z|}.$$

Exercise 4.19. *Let* $f \in \mathcal{H}(\mathbb{D})$ *be non-constant, with* $\mathrm{Re}(f)(z) \geq 0$ *for all* $z \in \mathbb{D}$. *Use the Möbius Transformation* $T(z) = \frac{z-1}{z+1}$ *and Schwarz's Lemma (Theorem 4.23) to show that*

$$\frac{1 - |z|}{1 + |z|}|f(0)| \leq |f(z)| \leq \frac{1 + |z|}{1 - |z|}|f(0)|, \quad \textit{for all} \quad z \in \mathbb{D}.$$

Exercise 4.20. *Let* $f \in \mathcal{H}(\mathbb{D})$ *with* $|f(z)| \leq 1$ *for all* $z \in \mathbb{D}$, *and let* $z_1, \ldots, z_n \in \mathbb{D}$ *be points with* $f(z_k) = 0$ *for all* $k = 1, \ldots, n$. *Prove that*

$$|f(z)| \leq \prod_{k=1}^{n} \left| \frac{z - z_k}{1 - \overline{z_k} z} \right|, \quad z \in \mathbb{D}.$$

Exercise 4.21. *Let* $f \in \mathcal{H}(\mathbb{D})$, *and* $z_1, \ldots, z_n \in \mathbb{D} \setminus \{0\}$ *so that*

(i) $|f(z)| \leq 1$ *for all* $z \in \mathbb{D}$.

(ii) $f(z_k) = 0$ *for all* $k = 1, \ldots, n$.

(iii) $f(0) = \lambda \cdot z_1 z_2 \cdots z_n$ *for some* $\lambda \in \mathbb{T}$.

Find a explicit formula for f.

Exercise 4.22. *Let* $\Omega \subset \mathbb{C}$ *be open and connected, with* $\overline{\mathbb{D}} \subset \Omega$, *and let* $f \in \mathcal{H}(\Omega)$ *with* $|f(z)| = 1$ *for all* $z \in \mathbb{T}$. *Find a general formula for* f *in terms of the zeros of* f *in* $\mathbb{D}$.

Exercise 4.23. *Let* $\Omega \subset \mathbb{C}$ *be open and connected with* $\overline{D}(0, 1) \subset \Omega$, *and* $f \in \mathcal{H}(\Omega)$ *with* $|f(z)| = 1$ *for all* $z \in \mathbb{T}$. *With the additional information* $f(0) = 2^{-5/2}$ *and that the zeros of* $\mathbb{D}$ *are* $\frac{1}{4}(1 + i)$ *(of order* 1*) and* $1/2$ *(of order* 2*), use the previous exercise to find the explicit formula for* f.

Exercise 4.24. *For the inversion* $T(z) = 1/z$, *show the following.*

(a) $T(\partial D(0, r)) = \partial D(0, 1/r)$.

(b) If $\gamma \subset \mathbb{C}$ *is a circle with* $0 \notin \gamma$, *then* $T(\gamma)$ *is a circle of* $\mathbb{C}$ *with* $0 \notin T(\gamma)$.

(c) If $\gamma \subset \mathbb{C}$ *is a circle with* $0 \in \gamma$, *then* $T(\gamma) = \ell \cup \{\infty\}$ *with* $\ell \subset \mathbb{C}$ *a line so that* $0 \notin \ell$.

(d) If $\gamma = \ell \cup \{\infty\}$ *with* $\ell \subset \mathbb{C}$ *a line so that* $0 \notin \ell$, *then* $T(\gamma)$ *is a circle of* $\mathbb{C}$ *with* $0 \notin T(\gamma)$.

(e) If $\gamma = \ell \cup \{\infty\}$ *with* $\ell \subset \mathbb{C}$ *a line so that* $0 \in \ell$, *then* $T(\gamma) = \ell' \cup \{\infty\}$ *with* $\ell' \subset \mathbb{C}$ *a line so that* $0 \in \ell'$.

Exercise 4.25. *Find the Möbius Transformation* T *in the following cases.*

(a) T *maps* $0, i, 1-i$ *to* $-1, 0, i$.

(b) $T(1+\frac{i}{2}) = 0$ *and* $T(D(1,1)) = D(0,1)$.

(c) $T(\partial D(0,2)) = \mathbb{R}$ *and* $T(0) = -i$.

Exercise 4.26. *Let* $T(z) = \frac{az+b}{cz+d}$ *a Möbius Transformation with* $T(\mathbb{R}) = \mathbb{R}$. *Show that we can assume that* $a, b, c, d \in \mathbb{R}$.

Exercise 4.27. *Let* $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ *and* T *a Möbius Transformation with* $T(\mathbb{H}) = \mathbb{D}$. *Show that* T *can be written as*

$$T(z) = \lambda \frac{z - z_0}{z - \overline{z_0}}, \quad z \in \mathbb{H},$$

for $\lambda \in \mathbb{T}$, $T(z_0) = 0$.

Exercise 4.28. *Find all Möbius Transformations* T *so that* $T(\partial\mathbb{D}) = \partial\mathbb{D}$.

Exercise 4.29. *Find a biholomorphic map between* $U = \{z \in \mathbb{C} : \operatorname{Re}(z), \operatorname{Im}(z) > 0\}$ *and* $V = \{z \in \mathbb{C} : \operatorname{Im}(z)\}$.

Exercise 4.30. *Verify that the function* $f(z) = \dfrac{(z+1)^2 - 2}{(z-1)^2 + 2}$ *is biholomorphic between* $\{z \in \mathbb{C} : |z| < 1, \operatorname{Re}(z) > 0\}$ *and* $\mathbb{D}$.

Chapter 5

Harmonic Functions

This chapter is devoted to analytical properties of *harmonic functions* in the plane. These are defined as C^2-smooth functions satisfying the Laplace Equation. We first show in Theorem 5.5 that simple-connectedness characterizes the existence of *harmonic conjugates*, or equivalently, the fact every harmonic function can be written as the real part of some holomorphic functions. In particular, every harmonic function in an arbitrary open set can be locally written as the real part of holomorphic map; see Corollary 5.7. From this observation one can derive plenty of regularity properties for harmonic functions such as their real-analyticity and the open-map property; see Exercise 5.6.

But the goal is to characterize the harmonicity by an integral property over circles, without the need to a priori show the Laplace equation or the C^2 regularity. This integral property is the *Mean Value Property.* While it is easy to see from the Cauchy Integral Formula that harmonic functions satisfy this property, the converse is very non-trivial, and requires to prove the *Maximum Principles* (Theorems 5.10 and 5.14) for functions with the Mean Value Property. A consequence of these principles is that two functions having this property that agree (continuously) in the boundary of a bounded domain must coincide also in the interior; see Corollary 5.15.

The Poisson Extension of a continuous function in a circle is a continuous extension to the closure of the disk, that is harmonic in the interior. The Maxmimum Principles permit to prove that these extensions are unique. In other words, the Dirichlet Problem in a disk has a unique solution; see Theorem 5.16. This allows to characterize harmonic functions via the Mean Value Property (Theorem 5.18).

Using this characterization, one can show that the locally uniform limit of harmonic functions is harmonic; see Corollary 5.19. The argument is in the same spirit as that of Weierstrass Convergence Theorem 3.1, where an integral property over triangles that characterizes holomorphic maps (via Morera's Theorem) is used to prove that locally uniform limits of holomorphic maps are harmonics.

Furthermore, the Mean Value Property leads to the Harnack Inequalities (Theorem 5.20), yielding estimates between the values of a (nonnegative) harmonic map in the interior of a disk and the values at the boundary. Based on these inequalities, Harnack's Convergence Theorem 5.22 classifies completely the convergence of monotonically non-decreasing sequences of harmonic function.

Definition 5.1 (Real Harmonic Function)**.** *Let $\Omega \subset \mathbb{R}^2$ be open and $u : \Omega \to \mathbb{R}$ a function of class $C^2(\Omega)$. We say that u is **harmonic** if u satisfies the Laplace Equation:*

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \textit{on} \quad \Omega. \tag{5.0.1}$$

We denote the family of all harmonic functions in Ω by $\mathrm{Har}(\Omega)$.

By saying that $u \in C^2(\Omega)$ we of course mean that u has partial derivatives up to order two, and are continuous in Ω.

Definition 5.2 (Harmonic Conjugate). *Let $\Omega \subset \mathbb{C}$ be open and let $u : \Omega \to \mathbb{R}$ be a harmonic function. We say that $v : \Omega \to \mathbb{R}$ is* ***a harmonic conjugate of*** *u in Ω if the function $u + iv$ is holomorphic in Ω.*

Real and Imaginary Parts of holomorphic maps are always harmonic.

Proposition 5.3. *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ holomorphic. Then $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are harmonic in Ω.*

Proof. The functions $\mathrm{Re}(f), \mathrm{Im}(f) : \Omega \to \mathbb{R}$ are of class $C^\infty(\Omega)$ and the Laplace Equation is satisfied thanks to the Cauchy-Riemann Equations. $\square$

5.1 Harmonic Conjugates in Simply Connected Domains

The existence of a harmonic conjugate can be characterized as follows.

Proposition 5.4. *Let $\Omega \subset \mathbb{C}$ be open and $u \in \mathrm{Har}(\Omega)$. Then, the following statements are equivalent.*

(i) u has a harmonic conjugate in Ω.

(ii) The function $f := u_x - iu_y$ has a primitive in Ω.

Proof. Assume that u has a harmonic conjugate v, so that $g := u + iv \in \mathcal{H}(\Omega)$. By the Cauchy Riemann Equations, we have that

$$g' = u_x + iv_x = u_x - iu_y = f,$$

thus showing that g is a primitive of f.

Conversely, let $g \in \mathcal{H}(\Omega)$ be so that $g' = f$ in Ω. We can write $g = w + iv$, where $w = \mathrm{Re}(g)$ and $v = \mathrm{Im}(g)$. We get that

$$u_x - iu_y = g' = w_x + iv_x = w_x - iw_y.$$

We get that $u_x = w_x$ and $u_y = w_y$ in Ω. From the first equation, we obtain that $w(x,y) = u(x,y) + \varphi(y)$; for a one-variable differentiable function φ. Inserting this into the second equation, we derive $u_y(x,y) + \varphi'(y) = u_y(x,y)$. This implies that φ is a constant $c \in \mathbb{R}$, and so $w = u + c$ in Ω. But since $g \in \mathcal{H}(\Omega)$, and $w = \mathrm{Re}(g)$ and $v = \mathrm{Im}(g)$, the function v is a harmonic conjugate of w, and so of u as well. $\square$

Since domains with whose cycles are all null-homologous are characterized by the existence of primitives, we obtained the following.

Theorem 5.5 (Existence of Harmonic Conjugates). *Let $\Omega \subset \mathbb{C}$ be open. Then, the following statements are equivalent.*

(i) $\Gamma \simeq 0$ in Ω for all cycles Γ in Ω.

(ii) Every $u \in \mathrm{Har}(\Omega)$ has a harmonic conjugate in Ω.

Proof. Assume that (i) holds and let $u \in \mathrm{Har}(\Omega)$. The function $f = u_x - iu_y$ is holomorphic in Ω since u_x, $-u_y$ satisfy the Cauchy-Riemann Equations as a consequence of the Laplace Equation (5.0.1) for u. By Theorem 1.19, f has a primitive in Ω. By Proposition 5.4, u has a harmonic conjugate.

Conversely, assume that (ii) holds and let $f \in \mathcal{H}(\Omega)$ with $f(z) \neq 0$ for all $z \in \Omega$. By Exercise 5.1, the function $u := \log|f|$ is harmonic in Ω. So by (ii), there exists $v \in \mathrm{Har}(\Omega)$ so that the function $g := u + iv$ is holomorphic in Ω. Taking the modulus of e^g, we get

$$|e^g| = e^{\mathrm{Re}(g)} = e^u = |f|,$$

in all of Ω. But then the function $h = e^{-g} \cdot f$ has modulus constantly equal to 1 in Ω. Consequently, for each connected component U of Ω, there exists a constant $c_U \in \mathbb{C} \setminus \{0\}$ with $f = c_U e^g$ in U. Taking $\xi_U \in \mathbb{C}$ with $c_U = e^{\xi_U}$, one gets that $f = e^{\xi_U + g}$ in U. Thus, on each connected component of Ω, the function f has a holomorphic logarithm. By Theorem 1.19, we get (i). □

Consequently, simple-connectedness has a new characterization.

Corollary 5.6 (Harmonic Conjugates in Simply Connected Domains)**.** *Let $\Omega \subset \mathbb{C}$ be open and connected. Then, the following statements are equivalent.*

(i) *Ω is simply connected.*

(ii) *Every $u \in \mathrm{Har}(\Omega)$ has a harmonic conjugate in Ω.*

Proof. It suffices to combine Theorem 5.5 with Corollary 4.44. □

We conclude this section by noticing that then holomorphic functions are locally real parts of holomorphic functions.

Corollary 5.7. *Let $\Omega \subset \mathbb{C}$ be open, and $u \in \mathrm{Har}(\Omega)$. Then, for every $z_0 \in \Omega$ there exist $r > 0$ and $f \in \mathcal{H}(D(z_0, r))$ so that $D(z_0, r) \subset \Omega$ and $u = \mathrm{Re}(f)$.*

Proof. If $z_0 \in \Omega$ and $r > 0$ is so that $D(z_0, r) \subset \Omega$, we can apply Corollary 5.6 to the simply connected set $D(z_0, r)$ to obtain that u has a harmonic conjugate v in $D(z_0, r)$. Thus $f := u + iv$ is holomorphic in $D(z_0, r)$ and $u = \mathrm{Re}(f)$. □

5.2 The Mean Value Property and the Maximum Principles

We focus our attention on the following integral condition for a continuous function.

Definition 5.8 (Mean Value Property)**.** *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be continuous in Ω. We say that f has the* ***Mean Value Property in*** *Ω if for every $z_0 \in \Omega$ and $r > 0$ so that $\overline{D}(z_0, r) \subset \Omega$, one has*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\,\mathrm{d}t.$$

All harmonic maps enjoy the Mean Value Property.

Proposition 5.9. *Let $\Omega \subset \mathbb{C}$ be open and $u \in \mathrm{Har}(\Omega)$. Then u has the Mean Value Property in Ω.*

Proof. Given $\overline{D}(z_0, r) \subset \Omega$, there exists $\varepsilon > 0$ so that $\overline{D}(z_0, r) \subset D(z_0, r + \varepsilon) \subset \Omega$. By Corollary 5.6, there exists $f \in \mathcal{H}(D(z_0, r + \varepsilon))$ with $\mathrm{Re}(f) = u$ in $D(z_0, r + \varepsilon)$. Therefore, f has the Mean Value Property for holomorphic maps,

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\,\mathrm{d}t.$$

It suffices now to take real parts on both sides. □

The first Maximum Principles shows that non-constant functions with the Mean Value Property cannon have local maximum.

Theorem 5.10 (Maximum Principle I)**.** *Let $\Omega \subset \mathbb{C}$ be open and connected, and $u \in C(\Omega, \mathbb{R})$ with the Mean Value Property in Ω. Then, if there exists $z_0 \in \Omega$ with $u(z_0) \geq u(z)$ for all $z \in \Omega$, then u is constant in Ω.*

Proof. Consider the set of points $A = \{z \in \Omega : u(z) = u(z_0)\}$. Obviously $\emptyset \neq A \subset \Omega$. By the continuity of u in Ω, it is clear that u is a closed relative to Ω. Since Ω is connected, we will show that $A = \Omega$ as soon as we prove that A is open. To see this, let $z \in A$ and $R > 0$ so that $\overline{D}(z, R) \subset \Omega$. By the Mean Value Property of u, we have that, for all $0 < r \leq R$,

$$u(z_0) = u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it})\, \mathrm{d}t.$$

We can rewrite this as

$$0 = \frac{1}{2\pi} \int_0^{2\pi} \left(u(z_0) - u(z + re^{it})\right)\, \mathrm{d}t,$$

where the integrand is a continuous function in $t \in [0, 2\pi]$, which is nonnegative by the assumption on u and z_0. Therefore $u(z_0) = u(z + re^{it})$ for all $t \in [0, 2\pi]$ and all $0 < r \leq R$. We conclude that $u = u(z_0)$ in $D(z, R)$, showing that $D(z, R) \subset A$. Therefore, A is an open set. $\square$

Analogously, we deduce the following.

Corollary 5.11 (Minimum Principle)**.** *Let $\Omega \subset \mathbb{C}$ be open and connected, and $u \in C(\Omega, \mathbb{R})$ with the Mean Value Property in Ω. Then, if there exists $z_0 \in \Omega$ with $u(z_0) \leq u(z)$ for all $z \in \Omega$, then u is constant in Ω.*

Proof. It follows by applying Theorem 5.10 to $-u$. $\square$

From the previous principles, we see that functions with the Mean Value Property in bounded domains attained their extraema at the boundary.

Corollary 5.12. *Let $\Omega \subset \mathbb{C}$ be open, connected and bounded, and $u \in C(\overline{\Omega}, \mathbb{R})$ with the Mean Value Property in Ω. Then,*

$$\max\{u(z) : z \in \overline{\Omega}\} = \max\{u(z) : z \in \partial\Omega\} \quad \textit{and} \quad \min\{u(z) : z \in \overline{\Omega}\} = \min\{u(z) : z \in \partial\Omega\}.$$

Proof. Assume that the first identity does not hold. Then there exists $z_0 \in \Omega$ with $u(z_0) \geq u(z)$ for all $z \in \overline{\Omega}$ and $u(z_0) > \max\{u(z) : z \in \partial\Omega\}$. By Theorem 5.10, u is constant in Ω. The continuity of u in $\overline{\Omega}$ implies that u is constant in $\overline{\Omega}$ as well, contradicting that $u(z_0) > u(z)$ for all $z \in \partial\Omega$.

As concerns the identity for the minimum, it suffices to apply the first identity to $-u$ instead of u. $\square$

The following technical lemma is the key for the second Maximum Principle, and allows for unbounded domains.

Lemma 5.13. *Let $\Omega \subset \mathbb{C}$ be open and connected, and $\psi : \Omega \to \mathbb{R}$ continuous and with the Mean Value Property in Ω. Assume further that*

$$\limsup_{z \to w,\, z \in \Omega} \psi(z) \leq 0, \quad \textit{for all} \quad w \in \partial_{\mathbb{C}_\infty}(\Omega). \tag{5.2.1}$$

Then either $\psi(z) < 0$ for all $z \in \Omega$ or $\psi(z) = 0$ for all $z \in \Omega$.

Proof. We only need to show that $\psi(z) \leq 0$ for all $z \in \Omega$. Indeed, if we show this, then, if $\psi(z_0) = 0$ for some $z_0 \in \Omega$, then we can apply Theorem 5.10 to obtain that ψ is identically zero in Ω, and our claim follows.

Let us then prove that $\psi \leq 0$ in Ω. Assume, towards a contradiction, that $\psi(z) > \varepsilon$ for some $\varepsilon > 0$ and $z \in \Omega$. The set $A := \{z \in \Omega : \psi(z) \geq \varepsilon\}$ is therefore nonempty. Observe that A is bounded, because otherwise $\infty \in \partial\Omega$ and $\infty \in \overline{A}^{\mathbb{C}_\infty}$, implying that

$$0 \geq \limsup_{z \to w,\, z \in \Omega} \psi(z) \geq \limsup_{z \to w,\, z \in A} \psi(z) \geq \varepsilon,$$

a contradiction. Let us now check that A is closed in $\mathbb{C}$. Indeed, let $\{z_n\}_n \subset A$ be convergent to $w \in \mathbb{C}$. In the case $w \in \Omega$, by the continuity of ψ we have that $\psi(w) = \lim_{n\to\infty} \psi(z_n) \geq \varepsilon$, and so $w \in A$. And if $w \notin \Omega$, then $w \in \partial\Omega$, and by (5.2.1) we have

$$\varepsilon \leq \lim_{n\to\infty} \psi(z_n) \leq 0,$$

a contradiction. We have shown that A is closed in $\mathbb{C}$. Therefore A is compact, and so we can find $z_0 \in A$ with

$$\psi(z_0) = \max\{\psi(z) \,:\, z \in A\} = \max\{\psi(z) \,:\, z \in \Omega\}.$$

By Theorem 5.10, ψ is constant in Ω. Since $\psi(z_0) \geq \varepsilon$, this contradicts (5.2.1). □

Applying Lemma 5.13 for a difference of two functions, we easily derive the Second Maximum Principle.

Theorem 5.14 (Maximum Principle II)**.** *Let $\Omega \subset \mathbb{C}$ be open and connected, and $u, v \in C(\Omega, \mathbb{R})$ with the Mean Value Property in Ω. Assume further that*

$$\limsup_{z\to w,\, z\in\Omega} u(z) \leq \liminf_{z\to w,\, z\in\Omega} v(z), \quad \textit{for all} \quad w \in \partial_{\mathbb{C}_\infty}(\Omega). \tag{5.2.2}$$

Then either $u(z) < v(z)$ for all $z \in \Omega$ or $u(z) = v(z)$ for all $z \in \Omega$.

Proof. Recall that

$$\limsup_{z\to w,\, z\in\Omega} u(z) = \lim_{\varepsilon\to 0^+} \left(\sup\{u(z) \,:\, z \in \Omega \text{ and } \widehat{d}(z, w) \leq \varepsilon\}\right),$$

for each $w \in \partial_{\mathbb{C}_\infty}\Omega$. By (5.2.2), we have

$$\limsup_{z\to w,\, z\in\Omega} (u - v)(z) \leq \limsup_{z\to w,\, z\in\Omega} u(z) + \limsup_{z\to w,\, z\in\Omega} (-v(z)) \leq \liminf_{z\to w,\, z\in\Omega} v(z) + \limsup_{z\to w,\, z\in\Omega} (-v(z)) = 0.$$

Thus we can apply Lemma 5.13 to the function $\psi = u - v$. □

Consequently, two continuous functions with the Mean Value Property that agree on the boundary, must agree on the interior as well.

Corollary 5.15. *Let $\Omega \subset \mathbb{C}$ be open, connected and bounded, and $u \in C(\overline{\Omega}, \mathbb{R})$ with the Mean Value Property in Ω. Then, if $u(z) = 0$ for all $z \in \partial\Omega$, then $u(z) = 0$ for all $z \in \overline{\Omega}$.*

Proof. By Theorem 5.14, we have that $u < 0$ in Ω or $u \equiv 0$ in Ω. Applying Theorem 5.14 with $-u$ instead of u, we get that either $u > 0$ in Ω or $u \equiv 0$ in Ω. We conclude that $u \equiv 0$ in $\overline{\Omega}$. □

5.3 The Dirichlet Problem in the Disk

Recall the various formulae for the Poisson Kernel:

$$P_r(\theta) := \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta} = \mathrm{Re}\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}, \quad \text{for all} \quad r \in [0, 1),\ \theta \in \mathbb{R}.$$

This kernel essentially solves a fundamental Partial Differential Equation with boundary data.

Theorem 5.16 (Solution to Dirichlet's Problem)**.** *Let $g : \mathbb{T} \to \mathbb{R}$ be a continuous function, then there exists a unique $u \in C(\overline{\mathbb{D}}, \mathbb{R}) \cap \mathrm{Har}(\mathbb{D})$ with $u = g$ in $\mathbb{T}$. Moreover, u can be expressed as*

$$u(z) = \mathrm{Re}(F)(z), \quad F(z) := \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \cdot u(e^{it})\, \mathrm{d}t, \quad \textit{for all} \quad z \in \mathbb{D}. \tag{5.3.1}$$

Or, in polar coordinates, as

$$u(re^{i\theta}) := \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} u(e^{it})\, \mathrm{d}t, \quad \textit{for all} \quad r \in [0, 1),\ \theta \in \mathbb{R}. \tag{5.3.2}$$

Proof. Defining $u = g$ in $\mathbb{T}$ and u as in (5.3.1) or (5.3.2), one has that $u \in C(\overline{\mathbb{D}}, \mathbb{R}) \cap \operatorname{Har}(\mathbb{D})$; see [5, Theorem 6.23]. To show that a function with the conditions of u is unique, assume that $v \in C(\overline{\mathbb{D}}, \mathbb{R}) \cap \operatorname{Har}(\mathbb{D})$ and $u = v$ in $\mathbb{T}$. Since harmonic functions satisfy the Mean Value Property (see Proposition 5.9), we can apply Corollary 5.15 to deduce that $u = v$ in $\overline{\mathbb{D}}$. □

As expected, Theorem 5.16 can be generalized to arbitrary disks, by translating centers and dilating radii.

Corollary 5.17. *Let $z_0 \in \mathbb{C}$, $R > 0$ and $u : \partial D(z_0, R) \to \mathbb{R}$ be continuous. Then the formula*

$$u(z_0 + re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - t) + r^2} u(z_0 + Re^{it})\,\mathrm{d}t, \quad \text{for all} \quad 0 \le r < R,\ \theta \in \mathbb{R}; \tag{5.3.3}$$

defines the unique extension of u to a function $C(\overline{D}(z_0, R)) \cap \operatorname{Har}(D(z_0, R))$.

Consequently, if we are given $u \in C(\overline{D}(z_0, R)) \cap \operatorname{Har}(D(z_0, R))$, then u is as in formula (5.3.3) *in $D(z_0, R)$.*

Proof. If $u : \partial D(z_0, R) \to \mathbb{R}$ is continuous, we can define $v : \mathbb{D} \to \mathbb{R}$ by the formula $v(z) = u(z_0 + Rz)$, for all $z \in \mathbb{T}$, where clearly $v \in C(\mathbb{T})$. By Theorem 5.16, v admits a unique extension to $\overline{\mathbb{D}}$ with $v \in C(\overline{\mathbb{D}}, \mathbb{R}) \cap \operatorname{Har}(\mathbb{D})$, and the extension is given by formula (5.3.2). Given $0 \le r < R$, $\theta \in \mathbb{R}$, denote $s = r/R \in [0, 1)$, and define $u(z_0 + re^{i\theta}) := v(se^{i\theta})$. We deduce that

$$\begin{aligned} u(z_0 + re^{i\theta}) = v(se^{i\theta}) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - s^2}{1 - 2s\cos(\theta - t) + s^2} v(e^{it})\,\mathrm{d}t \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - t) + r^2} u(z_0 + Re^{it})\,\mathrm{d}t, \end{aligned}$$

giving us formula (5.3.3). And if $\widetilde{u} : \overline{D}(z_0, R) \to \mathbb{R}$ is another function $C(\overline{D}(z_0, R)) \cap \operatorname{Har}(D(z_0, R))$ with $\widetilde{u} = u$ in $\partial D(z_0, R)$, then the function $\widetilde{v}(z) = \widetilde{u}(z_0 + Rz)$, $z \in \overline{\mathbb{D}}$ belongs to $C(\overline{\mathbb{D}}) \cap \operatorname{Har}(\mathbb{D})$ and satisfies $\widetilde{v} = v$ in $\mathbb{T}$. By the uniqueness of such functions (guaranteed by Theorem 5.16), we infer that $\widetilde{v} = v$ in $\overline{\mathbb{D}}$, whence $\widetilde{u} = u$ in $\overline{D}(z_0, R)$. □

We are finally ready to state the characterization of Harmonic Functions via the Mean Value Property.

Theorem 5.18 (Mean Value Characterization for Harmonic Functions)**.** *Let $\Omega \subset \mathbb{C}$ be open and $u : \Omega \to \mathbb{R}$ be continuous in Ω. The following statements are equivalent.*

(i) $u \in \operatorname{Har}(\Omega)$.

(ii) u has the Mean Value Property in Ω.

Proof. The implication $(i) \implies (ii)$ was proven in Proposition 5.9.

Conversely, assume that u enjoys the Mean Value Property in Ω. Given $z_0 \in \Omega$, let $r > 0$ so that $\overline{D}(z_0, r) \subset \Omega$. By Corollary 5.17, there is a unique $v \in C(\overline{D}(z_0, r)) \cap \operatorname{Har}(D(z_0, r))$ with $v = u$ in $\partial D(z_0, r)$. Thus the function $u - v$ is continuous in $\overline{D}(z_0, r)$ and satisfies the Mean Value Property in $D(z_0, r)$. By Corollary 5.15, we have that $u = v$ in $D(z_0, r)$, thus deducing that u is harmonic in $D(z_0, r)$. Since $z_0 \in \Omega$, this shows that $u \in \operatorname{Har}(\Omega)$. □

This allows to show that locally uniform limits of harmonic functions are harmonic.

Corollary 5.19. *Let $\Omega \subset \mathbb{C}$ be open, and let τ be the Compact-Open topology in $C(\Omega, \mathbb{C})$. Then* $\operatorname{Har}(\Omega)$ *is a complete and closed subspace of $(C(\Omega, \mathbb{C}), \tau)$. Consequently, if $\{u_n\}_n \subset \operatorname{Har}(\Omega)$ converges locally uniformly in Ω to some $u : \Omega \to \mathbb{R}$, then $u \in \operatorname{Har}(\Omega)$ as well.*

Proof. Since $(C(\Omega,\mathbb{C}),\tau)$ is a complete metric (metrizable) space, it suffices to show that $\mathrm{Har}(\Omega)$ is closed in $(C(\Omega,\mathbb{C}),\tau)$. To see this, let $\{u_n\}_n\subset \mathrm{Har}(\Omega)$ converging to some $u\in C(\Omega,\mathbb{R})$ in the metric ρ. Given any closed disk $\overline{D}(z_0,r)$ contained in Ω, we have that $\{u_n\}_n$ converges uniformly to u on $\overline{D}(z_0,r)$, and by Proposition 5.9 for each u_n we have

$$u(z_0)=\lim_{n\to\infty}u_n(z_0)=\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}u_n(z_0+re^{it})\,\mathrm{d}t=\frac{1}{2\pi}\int_0^{2\pi}u(z_0+re^{it})\,\mathrm{d}t.$$

We have shown that u has the Mean Value Property in Ω. Since u is continuous in Ω, Theorem 5.18 implies that $u\in\mathrm{Har}(\Omega)$. We conclude that $\mathrm{Har}(\Omega)$ is closed in $(C(\Omega,\mathbb{C}),\tau)$. □

5.4 Harnack's Inequalities and Theorem

The Harnack's Inequalities permits to estimate the value of a nonnegative harmonic function in the interior in terms of those at the boundary of the circle.

Theorem 5.20 (Harnack's Inequalities). *Let $z_0\in\mathbb{C}$, $R>0$ and $u\in C(\overline{D}(z_0,R))\cap\mathrm{Har}(D(z_0,R))$ with $u\geq 0$ in $\overline{D}(z_0,R)$. Then*

$$\frac{R-r}{R+r}u(z_0)\leq u(z_0+re^{i\theta})\leq\frac{R+r}{R-r}u(z_0),\quad \textit{for all}\quad 0\leq r<R,\ \theta\in\mathbb{R}. \tag{5.4.1}$$

Proof. First observe that

$$\frac{R^2-r^2}{R^2-2rR\cos(\theta-t)+r^2}=\frac{R^2-r^2}{|R-re^{i(\theta-t)}|^2}\quad\text{for all}\quad 0\leq r<R,\ t,\theta\in\mathbb{R},$$

which, together with $(R-r)^2\leq|R-re^{i(\theta-t)}|^2\leq(R+r)^2$, leads us to

$$\frac{R-r}{R+r}u(z_0+Re^{it})\leq\frac{R^2-r^2}{R^2-2rR\cos(\theta-t)+r^2}u(z_0+Re^{it})\leq\frac{R+r}{R-r}u(z_0+Re^{it}).$$

Integrating over $t\in[0,2\pi]$, using the Mean Value Property for harmonic functions (see Proposition 5.9), we get

$$\begin{aligned}\frac{R-r}{R+r}u(z_0)&=\frac{1}{2\pi}\int_0^{2\pi}\frac{R-r}{R+r}u(z_0+Re^{it})\,\mathrm{d}t\\&\leq\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2rR\cos(\theta-t)+r^2}u(z_0+Re^{it})\,\mathrm{d}t\\&\leq\frac{1}{2\pi}\int_0^{2\pi}\frac{R+r}{R-r}u(z_0+Re^{it})\,\mathrm{d}t=\frac{R+r}{R-r}u(z_0).\end{aligned}$$

Then (5.4.1) follows by observing that the integral in the second equation line above is precisely $u(z_0+re^{i\theta})$ by virtue of formula (5.3.3).

□

Let us consider a particular (but fundamental) case of Theorem 5.20.

Remark 5.21. Let $R>0$, $z_0\in\mathbb{C}$, and $v\in C(\overline{D}(z_0,R))\cap\mathrm{Har}\,(D(z_0,R))$ with $v\geq 0$. Then,

$$\frac{1}{3}v(z_0)\leq v(z)\leq 3v(z_0),\quad\text{for all}\quad z\in\overline{D}(z_0,R/2). \tag{5.4.2}$$

Indeed, by Theorem 5.20, we have that if $|z-z_0|\leq R/2$, then

$$\frac{1}{3}v(z_0)=\frac{R-R/2}{R+R/2}v(z_0)\leq\frac{R-|z-z_0|}{R+|z-z_0|}v(z_0)\leq v(z)\leq\frac{R+|z-z_0|}{R-|z-z_0|}v(z_0)\leq\frac{R-R/2}{R+R/2}v(z_0)\leq 3v(z_0).$$

The following theorem classifies completely the convergence of a non-decreasing sequence of harmonic maps in a domain.

Theorem 5.22 (Harnack's Theorem). *Let $\Omega \subset \mathbb{C}$ be open and connected and $\{u_n\}_n \subset \mathrm{Har}(\Omega)$ a sequence so that $u_n \leq u_{n+1}$ in Ω for all $n \in \mathbb{N}$. Then either*

(A) $\lim_{n\to\infty} u_n(z) = \infty$ uniformly on compact subsets of Ω, or else

(B) $\{u_n\}_n$ converges to some $u \in \mathrm{Har}(\Omega)$ uniformly on compact subsets of Ω.

Proof. We can assume (e.g., considering the sequence $\{u_n - u_1\}_n$ instead of $\{u_n\}_n$) that $u_n \geq 0$ in Ω. We define the function $u : \Omega \to \mathbb{R} \cup \{+\infty\}$ by

$$u(z) := \lim_{n\to\infty} u_n(z) = \sup_{n\in\mathbb{N}} u_n(z), \quad z \in \Omega.$$

Clearly u takes values in $\mathbb{R} \cup \{+\infty\}$ and $\{u_n\}_n$ increases to u in Ω. The sets

$$V := \{z \in \Omega \,:\, u(z) = +\infty\}, \quad W := \{z \in \Omega \,:\, u(z) \in \mathbb{R}\}$$

are disjoint, with $\Omega = V \cup W$. We next show that both V and W are open sets.

Indeed, given $z_0 \in \Omega$ and $R > 0$ with $\overline{D}(z_0, R) \subset \Omega$, we can apply Remark 5.21 to obtain that

$$\frac{1}{3} u_n(z_0) \leq u_n(z) \leq 3u_n(z_0), \quad \text{for all} \quad n \in \mathbb{N},\, z \in \overline{D}(z_0, R/2). \tag{5.4.3}$$

This tells us that if $z_0 \in V$ (resp. if $z_0 \in W$), then $\overline{D}(z_0, R/2) \subset V$ (resp. $\overline{D}(z_0, R/2) \subset W$).

This shows that V and W are open sets. Since their disjoint union is Ω, which is connected, then either $\Omega = V$ or $\Omega = W$.

In the first case, $\Omega = V$, we have that $\lim_{n\to\infty} u_n(z) = \infty$ for all $z \in \Omega$. Let us show that this convergence is locally uniform in Ω, from which we obtain the altervative (A). Indeed, given $z_0 \in \Omega$, let $R > 0$ be so that $\overline{D}(z_0, R) \subset \Omega$. Given $L > 0$ we can find $N \in \mathbb{N}$ so that $u_n(z_0) \geq 3L$ for all $n \geq N$. Now, if $z \in \overline{D}(z_0, R/2)$, we can apply (5.4.3) to obtain

$$u_n(z) \geq \frac{1}{3} u_n(z_0) \geq L, \quad \text{for all} \quad n \geq N.$$

This shows that $\{u_n\}_n$ converges uniformly to ∞ in $\overline{D}(z_0, R/2)$.

In the second case, $\Omega = W$, we have that $u(z) = \lim_{n\to\infty} u_n(z) \in \mathbb{R}$ for all $z \in \Omega$. We will now show that $\{u_n\}_n$ is Cauchy-locally uniformly in Ω, that is, given $z_0 \in \Omega$, there exists $r > 0$ so that $\overline{D}(z_0, r) \subset \Omega$ and for every $\varepsilon > 0$ we can find $N \in \mathbb{N}$ with

$$\sup\{|u_m(z) - u_n(z)| \,:\, m \geq n \geq N,\, z \in \overline{D}(z_0, r)\} \leq \varepsilon. \tag{5.4.4}$$

If $z_0 \in \Omega$, let $R > 0$ be so that $\overline{D}(z_0, R) \subset \Omega$. Given $\varepsilon > 0$, since $\lim_{n\to\infty} u_n(z_0)$ exists (is equal to $u(z_0) \in \mathbb{R}$), there is $N \in \mathbb{N}$ with

$$|u_m(z_0) - u_n(z_0)| = u_m(z_0) - u_n(z_0) \leq \frac{\varepsilon}{3}, \quad \text{for all} \quad m \geq n \geq N. \tag{5.4.5}$$

Now, if $z \in \overline{D}(z_0, R/2)$, and $m \geq n \geq N$, we can use the esimates (5.4.2) for $u_m - n_n \geq 0$, obtaining

$$u_m(z) - u_n(z) \leq 3\left(u_m(z_0) - u_n(z_0)\right), \quad m \geq n \geq N.$$

Using (5.4.5), the above gives precisely (5.4.4). Since $\{u_n\}_n$ is Cauchy-locally uniformly in Ω, we have that $\{u_n\}_n$ is a Cauchy sequence of the space $\mathrm{Har}(\Omega)$ with the Compact-Open Topology. By Corollary 5.19, this space is complete, and so there exists $v \in \mathrm{Har}(\Omega)$ to which $\{u_n\}_n$ converges uniformly on compact subsets of Ω. Since the function $u : \Omega \to \mathbb{R}$ was the pointwise limit of $\{u_n\}_n$, we must have $u = v$, thus concluding $u \in \mathrm{Har}(\Omega)$ and that $\{u_n\}_n$ converges uniformly on compact subsets to u. This is precisely the alternative (B). $\square$

5.5 Exercises

Exercise 5.1. *Let $\Omega \subset \mathbb{C}$ be open, and $f \in \mathcal{H}(\Omega)$ with $f(z) \neq 0$ for all $z \in \Omega$. Show that* $\log|f|$ *is a harmonic function in Ω.*

Exercise 5.2. *Let $\Omega \subset \mathbb{C}$ be open and connected. Show that if both $u \in \mathrm{Har}(\Omega)$ and $u^2 \in \mathrm{Har}(\Omega)$, then u is constant in Ω.*

Exercise 5.3. *Let $\Omega \subset \mathbb{C}$ be open and connected. Show that if $f \in \mathcal{H}(\Omega)$ and $|f|^2 \in \mathrm{Har}(\Omega)$, then f is constant in Ω.*

Exercise 5.4. *Let $\Omega \subset \mathbb{C}$ be open, and $u \in \mathrm{Har}(\Omega)$. Prove that, if $\overline{D}(z_0, R) \subset \Omega$, then*

$$u(z_0) = \frac{1}{\pi R^2} \int_{D(z_0,R)} u(x, y)\, \mathrm{d}x\, \mathrm{d}y.$$

Exercise 5.5. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ a polynomial given by $f(x, y) = \sum_{k,j=0}^{n} a_{k,j} x^k y^j$, for all $(x, y) \in \mathbb{R}^2$, with $a_{k,j} \in \mathbb{R}$ for all $0 \leq k, j \leq n$. Show that $f \in \mathrm{Har}(\mathbb{C})$ if and only if*

(i) $k(k-1)a_{k,j-2} + j(j-1)a_{k-2,j} = 0$ for all $2 \leq k, j \leq n$, and

(ii) $a_{k,j} = 0$ for all $k, j \in \{n-1, n\}$.

Exercise 5.6. *Let $\Omega \subset \mathbb{C}$ be open and connected, and $u \in \mathrm{Har}(\Omega)$ non-constant. Prove that $u : \Omega \to \mathbb{R}$ is an open map.*

Exercise 5.7. *Consider the function $u(z) = \mathrm{Im}\left(\left(\frac{1+z}{1-z}\right)^2\right)$ for all $z \in \mathbb{D}$. Show that $u \in \mathrm{Har}(\mathbb{D})$ and that $\lim_{r \to 1^-} u(re^{i\theta}) = 0$ for all $\theta \in \mathbb{R}$.*

Exercise 5.8. *Let $\Omega \subset \mathbb{C}$ be open and connected, $R > 0$ so that $\overline{D}(0, R) \subset \Omega$ and $f \in \mathcal{H}(\Omega)$ non-constant in Ω. Prove that the function*

$$[0, R] \ni r \mapsto M(r) := \max\{\mathrm{Re}(f)(z) \,:\, |z| = r\}$$

is strictly increasing in $[0, R]$.

Exercise 5.9. *Let $f : \overline{\mathbb{D}} \to \mathbb{C}$ be continuous with $f = u + iv$, and $u, v \in \mathrm{Har}(\mathbb{D})$. First show that*

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} f(e^{it})\, \mathrm{d}t, \quad r \in [0, 1), \theta \in \mathbb{R}.$$

Use this to prove that $f \in \mathcal{H}(\mathbb{D})$ if and only if

$$\int_0^{2\pi} f(e^{it}) e^{int}\, \mathrm{d}t = 0, \quad \text{for all} \quad n \in \mathbb{N}.$$

Exercise 5.10. *Let $\Omega \subset \mathbb{C}$ be open and connected, and let $K \subset \Omega$ be compact. Prove that there exists a constant $L = L(K, \Omega) > 0$ so that*

$$\max_{z \in K} u(z) \leq L \min_{z \in K} u(z)$$

for all $u \in \mathrm{Har}(\Omega)$ with $u \geq 0$ in Ω.

Exercise 5.11. *Let $\Omega \subset \mathbb{C}$ be open and connected, $\{f_n\}_n \subset \mathcal{H}(\Omega)$, and denote $u_n := \mathrm{Re}(f_n)$ for all $n \in \mathbb{N}$. Assume further that there exists $z_0 \in \Omega$ so that $\{f_n(z_0)\}_n$ converges in $\mathbb{C}$, and that $\{u_n\}_n$ converges locally uniformly in Ω to some $u : \Omega \to \mathbb{R}$ continuous. Prove that $\{f_n\}_n$ converges locally uniformly to some $f \in \mathcal{H}(\Omega)$.*

Chapter 6

Analytic Continuation. Approximation

Given an open set $\Omega \subset \mathbb{C}$ and a holomorphic function f in Ω, *can we extend f to a holomorphic function in a larger open set*? *Can we at least extend f continuously to the boundary $\partial\Omega$ of Ω*? The first part of this chapter takes care of some results on this direction. The Schwarz's Reflection Principle (Theorem 6.3) tells us that on open sets Ω that are *symmetric about* $\mathbb{R}$, one can extend holomorphic functions from the positive part to the negative part of Ω, provided the function is continuous up to the real part of Ω and takes real values there. This idea of *reflecting* is generalized to domains with analytic boundary, meaning that their boundaries can be locally parameterized by resctrictions of biholomorphic maps from a disk; see Theorem 6.5.

The second main result of is Carathéodory's Theorem 6.6, whose proof is rather difficult, involving deep topological theorems (such as Jordan's curve theorem) and a bit of measure theory in the arguments. It states that a biholomorphic map between the disk $\mathbb{D}$ and a *Jordan Domain* Ω always admits an extension $\overline{\mathbb{D}} \to \overline{\Omega}$ that is an homeomorphism. We then use this big result in combination with the Riemann Mapping Theorem 4.43 to solve the Dirichlet Problem (on the existence of harmonic extensions of boundary data) in Jordan Domains; see Theorem 6.9.

Examples of holomorphic functions in the disk that do not admit extensions *around any point* of the unit circle are provided by some *lacunary series*; see Example 6.13. Such functions are said to have natural boundary equal to the unit circle.

Then we study *analytic continuations along disks and chains*, being the Monodromy Theorem 6.20 the main result. This theorem says essentially that in a simply-connected domain Ω, one can extend a holomorphic function f from a disk $D \subset \Omega$ to a global holomorphic function in Ω, if one is able to find analytic continuations along all paths in Ω that starts at the center of the disk.

The second part of this chapter is devoted to locally uniform approximation of holomorphic functions by rational functions (with localized poles) or polynomials. This is the content of the Runge's Theorems 6.24 and 6.29. The first one says that if a compact set K has connected complement, then any holomorphic function on a neighbourhood of K can be uniformly approximated in K by polynomials. The second theorem is much more general, and says that one can approximate holomorphic functions in an open set Ω uniformly on compact subsets of Ω by rational functions whose poles are in the connected components of $\mathbb{C}_\infty \setminus \Omega$; see also Corollary 6.31. In particular, in a simply connected domain, all holomorphic maps admit approximations by polynomials uniformly on compact sets.

6.1 The Schwarz Reflection Principle

Symmetric sets about $\mathbb{R}$ are defined (naturally) as follows.

Definition 6.1 (Symmetric Sets)**.** *A set $A \subset \mathbb{C}$ is* ***symmetric about*** $\mathbb{R}$ *if A satisfies the property*

$$z \in A \iff \overline{z} \in A.$$

Also, for A symmetric about $\mathbb{R}$, we denote the sets

$$A^+ := A \cap \{z \in \mathbb{C} \,:\, \mathrm{Im}(z) > 0\}, \quad A^- := A \cap \{z \in \mathbb{C} \,:\, \mathrm{Im}(z) < 0\}, \quad A^0 := A \cap \mathbb{R}.$$

The *union of two* holomorphic functions respectively on Ω^+ and Ω^- that agree (with continuity) at Ω^0 gives a holomorphic function in Ω.

Lemma 6.2. *Let $\Omega \subset \mathbb{C}$ be open and symmetric about $\mathbb{R}$, and consider functions*

$$f : \Omega^+ \cup \Omega^0 \to \mathbb{C}, \quad g : \Omega^- \cup \Omega^0 \to \mathbb{C},$$

with $f \in C\left(\Omega^+ \cup \Omega^0\right) \cap \mathcal{H}(\Omega^+)$ and $g \in C\left(\Omega^- \cup \Omega^0\right) \cap \mathcal{H}(\Omega^-)$. Then, if $f = g$ in Ω^0, the function $F : \Omega \to \mathbb{C}$ given by

$$F(z) = \begin{cases} f(z) & \text{if } z \in \Omega^+ \cup \Omega^0, \\ g(z) & \text{if } z \in \Omega^-, \end{cases}$$

is holomorphic in Ω.

Proof. Since both f and g are continuous in their respective domains, and $f = g$ in Ω^0, then F is continous in Ω. Also, F is holomorphic in $\Omega^+ \cup \Omega^-$, as $f \in \mathcal{H}(\Omega^+)$ and $g \in \mathcal{H}(\Omega^-)$. To show that F is holomorphic in Ω, we verify that

$$\int_T F(z)\,\mathrm{d}z = 0, \quad \text{for every triangle } T \text{ with } \mathrm{co}(T) \subset \Omega. \tag{6.1.1}$$

Here, by a triangle T we mean the boundary of the solid triangle $\mathrm{co}(T)$, which is the convex envelope of three non-align points of Ω. Since $F : \Omega \to \mathbb{C}$ is continuous, (6.1.1) would imply that $F \in \mathcal{H}(\Omega)$ by Morera's Theorem. Let us distinguish four cases of triangles T.

Case 1. $\mathrm{co}(T) \cap \Omega^0 = \emptyset$. Since $\mathrm{co}(T)$ is connected, this implies that either $\mathrm{co}(T) \subset \Omega^+$ or $\mathrm{co}(T) \subset \Omega^-$. Assuming that $\mathrm{co}(T) \subset \Omega^+$ (the other case is analogous), we have that

$$\int_T F(z)\,\mathrm{d}z = \int_T f(z)\,\mathrm{d}z = 0.$$

The last inequality is a consequence of Cauchy's Theorem (see e.g. Corollary 1.14), as $f \in \mathcal{H}(\Omega)$ and $\mathrm{co}(T) \subset \Omega$.

Case 2. $\mathrm{co}(T) \subset \Omega^+ \cup \Omega^0$ or $\mathrm{co}(T) \subset \Omega^- \cup \Omega^0$. Assume the first case $\mathrm{co}(T) \subset \Omega^+ \cup \Omega^0$, where the latter case is studied identically. Denote $\Delta := \mathrm{co}(T)$. Since Ω^+ is open, and Δ is compact, we can find $\varepsilon_0 > 0$ so that

$$\Delta_\varepsilon := \{z + \varepsilon i \,:\, z \in \mathrm{co}(T)\} \subset \Omega^+, \quad 0 < \varepsilon \leq \varepsilon_0.$$

Clearly Δ_ε is a new solid triangle contained in Ω^+, and we denote its boundary by T_ε. By **Case 1** applied to Δ_ε, we have that

$$\int_{T_\varepsilon} F(z)\,\mathrm{d}z = 0 \quad \text{for all} \quad 0 < \varepsilon \leq \varepsilon_0.$$

We have, for every $0 < \varepsilon \leq \varepsilon_0$,

$$\begin{aligned}
\left|\int_T F(z)\,\mathrm{d}z\right| &= \left|\int_T F(z)\,\mathrm{d}z - \int_{T_\varepsilon} F(z)\,\mathrm{d}z\right| = \left|\int_T F(z)\,\mathrm{d}z - \int_T F(z+\varepsilon i)\,\mathrm{d}z\right| \\
&= \left|\int_T \left(F(z) - F(z+\varepsilon i)\right)\,\mathrm{d}z\right| \leq \int_T |F(z) - F(z+\varepsilon i)|\,|\mathrm{d}z| \\
&\leq \ell(T) \sup\{|F(z) - F(w)| \,:\, z, w \in K,\, |z-w| \leq \varepsilon\}, \quad \text{where} \quad K := \Delta \cup \bigcup_{0<t\leq\varepsilon_0} \Delta_t.
\end{aligned}$$

Since K is a compact subset of Ω, we have that F is uniformly continuous in K, and so the last term tends to 0 as $\varepsilon \to 0^+$. Consequently, $\int_T F = 0$.

Case 3. $\mathrm{co}(T) \cap \Omega^+ \neq \emptyset$, $\mathrm{co}(T) \cap \Omega^- \neq \emptyset$ and a vertex of T belongs to Ω^0.

We may of course assume that T has the positive orientation. The line segment that joins the mentioned vertex with the other point in the intersection $\Omega^0 \cap T$ splits $\mathrm{co}(T)$ into two triangles $\mathrm{co}(T_1) \subset \Omega^+ \cup \Omega^0$, $\mathrm{co}(T_2) \subset \Omega^- \cup \Omega^0$. By **Case 2**, we get that

$$\int_{T_1} F(z)\,\mathrm{d}z = \int_{T_2} F(z)\,\mathrm{d}z = 0.$$

Giving the positive orientation to both T_1 and T_2, we get that

$$0 = \int_{T_1} F(z)\,\mathrm{d}z + \int_{T_2} F(z)\,\mathrm{d}z = \int_T F(z)\,\mathrm{d}z.$$

In the second equality we used that F is integrated over ℓ and ℓ^-, where ℓ is the common edge of T_1 and T_2.

Case 4. $\mathrm{co}(T) \cap \Omega^+ \neq \emptyset$, $\mathrm{co}(T) \cap \Omega^- \neq \emptyset$ and no vertex of T belongs to Ω^0.

Assume, without loss of generality that two vertices of T are in Ω^+ and one in Ω^-. The segment $\mathrm{co}(T) \cap R$ splits $\mathrm{co}(T)$ into a solid quadrilateral $Q \subset \Omega^+ \cup \Omega^0$ and a solid triangle $\mathrm{co}(T_1) \subset \Omega^- \cup \Omega^0$. We can further divide Q into two solid triangles $\mathrm{co}(T_2)$ and $\mathrm{co}(T_3)$ contained in $\Omega^+ \cup \Omega^0$. By **Case 2**, we have

$$\int_{T_1} F(z)\,\mathrm{d}z = \int_{T_2} F(z)\,\mathrm{d}z = \int_{T_3} F(z)\,\mathrm{d}z = 0.$$

Giving the positive orientation to all the triangles T, T_1, T_2, T_3, it is then clear that

$$\int_T F(z)\,\mathrm{d}z = \int_{T_1} F(z)\,\mathrm{d}z + \int_{T_2} F(z)\,\mathrm{d}z + \int_{T_3} F(z)\,\mathrm{d}z = 0.$$

□

This is the most basic (but fundamental) result on analytic continuation.

Theorem 6.3 (Schwarz's Reflection Principle)**.** *Let $\Omega \subset \mathbb{C}$ be open and symmetric about $\mathbb{R}$, and $f : \Omega^+ \cup \Omega^0 \to \mathbb{C}$ continuous in $\Omega^+ \cup \Omega^0$, holomorphic in Ω^+, and with the property that $f(z) \in \mathbb{R}$ for all $z \in \Omega^0$. Then the function $F : \Omega \to \mathbb{C}$ given by*

$$F(z) = \begin{cases} f(z) & \text{if } z \in \Omega^+ \cup \Omega^0 \\ \overline{f(\overline{z})} & \text{if } z \in \Omega^-. \end{cases} \tag{6.1.2}$$

is holomorphic in Ω and $F = f$ in $\Omega^+ \cup \Omega^0$.

Proof. Defining $g : \Omega^- \cup \Omega^0 \to \mathbb{C}$ by $g(z) = \overline{f(\overline{z})}$ for all $z \in \Omega^- \cup \Omega^0$, the continuity of f implies that of g in $\Omega^- \cup \Omega^0$. Also, for all $z \in \Omega^0$,

$$g(z) = \overline{f(\overline{z})} = f(\overline{z}) = f(z).$$

Thus, $f = g$ in Ω^0. Also, since $f \in \mathcal{H}(\Omega^+)$, Exercise 6.1 tells us that $g \in \mathcal{H}(\Omega^-)$. All the assumptions of Lemma 6.2 are satisfied for f and g, and so $F \in \mathcal{H}(\Omega)$. □

6.2 Continuation through Analytic Boundaries

In this section we generalize the idea of reflecting from the Schwarz's Theorem 6.3 to a more general setting.

Definition 6.4 (Analytic Boundary). *If $\Omega \subset \mathbb{C}$ is open, we say that Ω has* ***analytic boundary*** *if for every $z_0 \in \partial\Omega$ there exists an open set $U \subset \mathbb{C}$ and a biholomorphic map $\varphi : \mathbb{D} \to U$ with*

$$\varphi(\mathbb{D}^+) = U \cap \Omega, \quad \varphi(\mathbb{D}^0) = U \cap \partial\Omega, \quad \varphi(0) = z_0.$$

Note that then $\varphi(\mathbb{D}^-) = U \setminus \overline{\Omega}$.

To check the last comment in the previous definition, note that if Ω, z_0, U, φ are as above, and $w \in \mathbb{D}^-$ is so that $\varphi(w) \in \overline{\Omega}$, using that φ is bijective, we get that $w \in \mathbb{D}^+ \cup \mathbb{D}^0$, a contradiction.

Theorem 6.5 (Holomorphic Extensions from Analytic Boundaries). *Let $\Omega \subset \mathbb{C}$ be open, and $f : \overline{\Omega} \to \mathbb{C}$ with the following properties:*

- *f is continuous in $\overline{\Omega}$ and holomorphic in Ω.*
- *$f(\Omega)$ is open, and $f(\partial\Omega) \subset \partial f(\Omega)$.*
- *Both Ω and $f(\Omega)$ have analytic boundary.*

Then there exist an open set $W \subset \mathbb{C}$ with $\overline{\Omega} \subset W$, and a holomorphic function $F : W \to \mathbb{C}$ with $F = f$ in $\overline{\Omega}$.

Proof. We will construct W as a union of open neighbourhoods around each point of the boundary, where f has a holomorphic extension, and F as the union of these local extensions.

Let $z \in \partial\Omega$, $U, V \subset \mathbb{C}$ open sets, $\varphi : \mathbb{D} \to U$, $\psi : \mathbb{D} \to V$ biholomorphic so that

$$\varphi(\mathbb{D}^+) = U \cap \Omega, \quad \varphi(\mathbb{D}^0) = U \cap \partial\Omega, \quad \varphi(\mathbb{D}^-) = U \setminus \overline{\Omega}, \quad \varphi(0) = z \tag{6.2.1}$$

$$\psi(\mathbb{D}^+) = V \cap f(\Omega), \quad \psi(\mathbb{D}^0) = V \cap \partial f(\Omega), \quad \psi(\mathbb{D}^-) = V \setminus \overline{f(\Omega)} \quad \psi(0) = f(z). \tag{6.2.2}$$

Since V is open and contains $f(z)$, we can find $\delta > 0$ so that $\overline{D}(f(z), \delta)) \subset V$. The continuity of f in $\overline{\Omega}$ implies that we can find $\mathcal{O} \subset \mathbb{C}$ open containing z so that $f^{-1}(D(f(z), \delta)) = \mathcal{O} \cap \overline{\Omega}$. Defining $A := U \cap \mathcal{O}$, we have that

$$A \subset \mathbb{C} \text{ is open}, \quad z \in A, \quad f(A \cap \Omega) \subset V \cap f(\Omega), \quad f(A \cap \partial\Omega) \subset V \cap \partial f(\Omega). \tag{6.2.3}$$

Let us verify (6.2.3). The first two properties are immediate. The first inclusion follows by writing

$$f(A \cap \Omega) = f(U \cap \mathcal{O} \cap \Omega) \subset D(f(z), \delta) \cap f(\Omega) \subset V \cap f(\Omega).$$

For the second inclusion, the assumption $f(\partial\Omega) \subset \partial f(\Omega)$ gives $f(A \cap \partial\Omega) \subset \partial f(\Omega)$. Also, if $w \in A \cap \partial\Omega$, there is a sequence $\{w_n\}_n \subset A \cap \Omega$ convergent to w. By the continuity of f in $\overline{\Omega}$, we have that $\{f(w_n)\}_n$ is a sequence contained in $f(\Omega) \cap D(f(z), r)$ that converges to $f(w) \in \overline{D}(f(z), r) \subset V$. Thus, we have $f(w) \in V \cap \partial f(\Omega)$, and (6.2.3) is proven.

Now, since $\varphi : \mathbb{D} \to U$ is biholomorphic and $0 \in \varphi^{-1}(A)$, there exists $\varepsilon > 0$ so that the set $\varphi^{-1}(A)$ contains the disk $\mathbb{D}_\varepsilon := D(0, \varepsilon)$. Since obviously $\mathbb{D}_\varepsilon^\pm \subset \mathbb{D}^\pm$ and $\mathbb{D}_\varepsilon^0 \subset \mathbb{D}^0$, by (6.2.1), we get that

$$\varphi(\mathbb{D}_\varepsilon^+) \subset A \cap \Omega, \quad \varphi(\mathbb{D}_\varepsilon^0) \subset A \cap \partial\Omega, \quad \varphi(\mathbb{D}_\varepsilon^-) \subset A \setminus \overline{\Omega}. \tag{6.2.4}$$

Now define the function

$$g : \mathbb{D}_\varepsilon^+ \cup \mathbb{D}_\varepsilon^0 \to \mathbb{D}^+ \cup \mathbb{D}^0, \quad g = \psi^{-1} \circ f \circ \varphi. \tag{6.2.5}$$

Taking into account that $f \in C(\overline{\Omega}) \cap \mathcal{H}(\Omega)$, that φ, ψ are biholomorphic, and (6.2.1)-(6.2.5), it is clear that g is continuous in $\mathbb{D}_\varepsilon^+ \cup \mathbb{D}_\varepsilon^0$, holomorphic in $\mathbb{D}_\varepsilon^+$, with $g(\mathbb{D}_\varepsilon^+) \subset \mathbb{D}^+$ and $g(\mathbb{D}_\varepsilon^0) \subset \mathbb{D}_0 \subset \mathbb{R}$. By Theorem 6.3, there exists $G : \mathbb{D}_\varepsilon \to \mathbb{D}$ with

$$G \in \mathcal{H}(\mathbb{D}_\varepsilon), \quad G = g \text{ on } \mathbb{D}_\varepsilon^+, \quad G(\mathbb{D}_\varepsilon^+) \subset \mathbb{D}^+, \quad G(\mathbb{D}_\varepsilon^0) \subset \mathbb{D}^0, \quad G(\mathbb{D}_\varepsilon^-) \subset \mathbb{D}^-. \tag{6.2.6}$$

We finally define

$$W := \varphi(\mathbb{D}_\varepsilon), \quad F : W \to \mathbb{C}, \quad F := \psi \circ G \circ \varphi^{-1}. \tag{6.2.7}$$

Clearly W is open, and by (6.2.6) and (6.2.5), we get that $F \in \mathcal{H}(W)$, that $F = f$ on $\varphi(\mathbb{D}_\varepsilon^+ \cup \mathbb{D}_\varepsilon^0)$, where, by (6.2.4), $W \cap \overline{\Omega} = \varphi(\mathbb{D}_\varepsilon) \cap \overline{\Omega} \subset \varphi(\mathbb{D}_\varepsilon^+ \cup \mathbb{D}_\varepsilon^0)$.

To summarize, for each $z \in \partial\Omega$, we have found:

$$W_z \subset \mathbb{C} \text{ open with } z \in W_z, \quad F_z : W_z \to \mathbb{C} \text{ holomorphic in } W_z, \quad F_z = f \text{ on } W_z \cap \overline{\Omega}. \tag{6.2.8}$$

To construct an extension of f to a open neighbourhood of $\overline{\Omega}$, we proceed as follows. For each $z \in \partial\Omega$, let W_z and F_z as in (6.2.8), and let $r_z > 0$ so that $D(z, 2r_z) \subset W_z$. We define

$$W := \Omega \cup \bigcup_{z \in \partial\Omega} D(z, r_z),$$

and $F : W \to \mathbb{C}$ as

$$F(w) = \begin{cases} f(w) & \text{if } w \in \Omega, \\ F_z(w) & \text{if } w \in D(z, r_z), \text{ for some } z \in \partial\Omega. \end{cases}$$

Let us check that F is well-defined. Let $w \in D(z, r_z) \cap \Omega$ for some $z \in \partial\Omega$. By (6.2.8), $F_z(w) = f(w)$. Also, if $w \in D(z, r_z) \cap D(\xi, r_\xi)$ for $z, \xi \in \partial\Omega$, where we may assume that $r_z \geq r_\xi$, then the triangle inequality gives that $\xi \in D(z, 2r_z) \cap D(\xi, 2r_\xi)$. Since this intersection defines an open set, there exists $\delta > 0$ with $D(\xi, r) \subset D(z, 2r_z) \cap D(\xi, 2r_\xi)$. But $\xi \in \partial\Omega$, and so we have that $D(\xi, r) \cap \Omega$ is a nonempty open set, thus containing a disk D. Because $D \subset \Omega \cap D(z, 2r_z) \cap D(\xi, 2r_\xi)$, by (6.2.8), we get that $F_z = f = F_\xi$ in D. The Identity Principle for holomorphic maps establishes that $F_z = F_\xi$ in $D(z, 2r_z) \cap D(\xi, 2r_\xi)$, and this set contains w, so $F_z(w) = F_\xi(w)$.

We conclude that F is well-defined in W. It is obvious that $F = f$ in W, and that $F \in \mathcal{H}(W)$, as it coincides locally with a holomorphic map. $\square$

6.3 Carathéodory's Theorem

We split the proof of Carathéodory's Theorem into two main claims. We will use multiple times the *Jordan Curve's Theorem*: if $\gamma : [0, 1] \to \mathbb{C}$ is a closed, continuous and not self-intersecting curve, then $\mathbb{C} \setminus \gamma^*$ consists of two open connected components, of which one is one (called the *inside* of γ) and the other unbounded (calle the *outside* of γ). The boundary of these components is precisely γ^*. Such a curve γ is called a **Jordan curve**, and satisfies that γ^* is homeomorphic to $\mathbb{T} = \partial\mathbb{D}$, the unit circle.

Theorem 6.6 (Carathéodory's Theorem)**.** *Let $\Omega \subset \mathbb{C}$ be open, bounded, and connected, and let $f : \mathbb{D} \to \Omega$ be biholomorphic. The following statements are equivalent.*

(i) $\partial\Omega$ is a Jordan curve, that is, there exists an homeomorphism $h : \mathbb{T} \to \partial\Omega$.

(ii) There exists a unique homeomorphism $F : \overline{\mathbb{D}} \to \overline{\Omega}$ with $F = f$ in $\mathbb{D}$.

Proof. The implication $(ii) \implies (i)$ is easy, because the homeomorphism $F : \overline{\mathbb{D}} \to \overline{\Omega}$ maps boundaries to boundaries, and thus we can use the restriction $F\restriction_{\mathbb{T}} : \mathbb{T} \to \partial\Omega$ to $\mathbb{T}$ to define an homeomorphism between $\mathbb{T}$ and $\partial\Omega$.

Conversely, assume that (i) holds, and let $h : \mathbb{T} \to \partial\Omega$ be an homeomorphism.

Claim 1: $f : \mathbb{D} \to \partial\Omega$ *is uniformly continuous in* $\mathbb{D}$. Suppose, for the sake of contradiction, that there exists $\varepsilon > 0$ and sequences $\{z_n\}_n, \{w_n\}_n \subset \mathbb{D}$ with $\lim_{n\to\infty} |z_n - w_n| = 0$ and $|f(z_n) - f(w_n)| \geq \varepsilon$ for all $n \in \mathbb{N}$. Passing to subsequences, we may assume that there exists $\xi \in \overline{\mathbb{D}}$ so that

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} w_n = \xi, \quad \text{and} \quad |f(z_n) - f(w_n)| \geq \varepsilon \quad \text{for all} \quad n \in \mathbb{N}. \tag{6.3.1}$$

Notice that $\xi \in \partial\mathbb{D}$, as otherwise, $\xi \in \mathbb{D}$ and we would have

$$\lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} f(w_n) = f(\xi),$$

contradicting (6.3.1). Now, for every $r \in (0,1)$ consider the circle paths $[0, 2\pi] \ni t \mapsto \Gamma_r(t) = \xi + re^{it}$ and let $a_r, b_r \in [0, 2\pi]$ be so that the restriction $\gamma_r : [a_r, b_r] \to \mathbb{C}$ of Γ_r to $[a_r, b_r]$ satisfies

$$\gamma_r([a_r, b_r]) = \overline{\mathbb{D}} \cap \partial D(\xi, r), \quad \gamma_r((a_r, b_r)) = \mathbb{D} \cap \partial D(\xi, r), \quad \text{for all} \quad r \in (0,1). \tag{6.3.2}$$

Now, by (6.3.2), we have that $f \circ \gamma_r : (a_r, b_r) \to \Omega$, and by Cauchy-Schwarz Inequality,

$$\int_{a_r}^{b_r} |(f \circ \gamma_r)'(t)| \, \mathrm{d}t = r \int_{a_r}^{b_r} |f'(\gamma_r(t))| \, \mathrm{d}t$$

$$\leq r \left(\int_{a_r}^{b_r} |f'(\gamma_r(t))|^2 \, \mathrm{d}t \right)^{1/2} \left(\int_{a_r}^{b_r} 1^2 \, \mathrm{d}t \right)^{1/2} \leq \sqrt{2\pi r} \left(\int_{a_r}^{b_r} |f'(\gamma_r(t))|^2 \, r \, \mathrm{d}t \right)^{1/2}.$$

Squaring both sides and reorganizing the terms we derive

$$\frac{1}{2\pi r} \left(\int_{a_r}^{b_r} |(f \circ \gamma_r)'(t)| \, \mathrm{d}t \right)^2 \leq \int_{a_r}^{b_r} |f'(\gamma_r(t))|^2 \, r \, \mathrm{d}t.$$

The paths $\{\gamma_r\}_{r\in(0,1)}$ parametrize the set $\mathbb{D} \cap D(\xi, 1) = \{\gamma_r(t) = \xi + re^{it} \,:\, t \in (a_r, b_r),\, r \in (0,1)\}$, and if we integrate the previous estimate over $r \in (0,1)$, we get

$$\begin{aligned}
\frac{1}{2\pi} \int_0^1 \frac{1}{r} \left(\int_{a_r}^{b_r} |(f \circ \gamma_r)'(t)| \, \mathrm{d}t \right)^2 \mathrm{d}r &\leq \int_0^1 \int_{a_r}^{b_r} |f'(\gamma_r(t))|^2 \, r \, \mathrm{d}t \, \mathrm{d}r = \int_{\mathbb{D}\cap D(\xi,1)} |f'(x+iy)|^2 \, \mathrm{d}x \, \mathrm{d}y \\
&\leq \int_{\mathbb{D}} |f'(x+iy)|^2 \, \mathrm{d}x \, \mathrm{d}y = \int_{\mathbb{D}} |\det(Df(x+iy))| \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_{f(\mathbb{D})} \mathrm{d}x \, \mathrm{d}y = \mathcal{L}^2(f(\mathbb{D})) = \mathcal{L}^2(\Omega) < \infty;
\end{aligned}$$

where $\mathcal{L}^2(\Omega)$ denotes the area of Ω. This shows that

$$\frac{1}{2\pi} \int_0^1 \frac{1}{r} \left(\int_{a_r}^{b_r} |(f \circ \gamma_r)'(t)| \, \mathrm{d}t \right)^2 \mathrm{d}r < \infty. \tag{6.3.3}$$

If we had that $\int_{a_r}^{b_r} |(f \circ \gamma_r)'(t)| \, \mathrm{d}t \geq \delta$ for all $0 < r \leq r_0$ and some $r_0, \delta \in (0,1)$, then (6.3.3) would give a contradiction, since

$$\int_0^1 \frac{1}{r} \left(\int_{a_r}^{b_r} |(f \circ \gamma_r)'(t)| \, \mathrm{d}t \right)^2 \geq \int_0^{r_0} \frac{1}{r} \left(\int_{a_r}^{b_r} |(f \circ \gamma_r)'(t)| \, \mathrm{d}t \right)^2 \mathrm{d}r \geq \int_0^{r_0} \frac{\delta^2}{r} \, \mathrm{d}r = \infty.$$

Therefore, there exists a sequence $\{r_n\}_n \subset (0,1)$ decreasing to 0 so that, denoting $a_n = a_{r_n}$, $b_n = a_{r_n}$, $\gamma_n = \gamma_{r_n}$, we have

$$\lim_{n\to\infty} \int_{a_n}^{b_n} |(f \circ \gamma_n)'(t)| \, \mathrm{d}t = 0. \tag{6.3.4}$$

We now use (6.3.4) to extend continuously each $f \circ \gamma_n$ to $[a_n, b_n]$. In order to do this, note that if $a_n < s < t < b_n$, we have

$$|(f \circ \gamma_n)(t) - (f \circ \gamma_n)(s)| = \left| \int_s^t (f \circ \gamma_n)'(\theta)\, \mathrm{d}\theta \right| \leq \int_s^t |(f \circ \gamma_n)'(\theta)|\, \mathrm{d}\theta = \int_{a_n}^{b_n} |(f \circ \gamma_n)'(\theta)| \mathcal{X}_{(s,t)}(\theta)\, \mathrm{d}\theta.$$

Note that

$$\lim_{s,t \to a_n^+} |(f \circ \gamma_n)'(\theta)| \mathcal{X}_{(s,t)}(\theta) = 0 \quad \text{for all} \quad \theta \in [a_n, b_n]$$

and that

$$|(f \circ \gamma_n)'(\theta)| \mathcal{X}_{(s,t)}(\theta) \leq |(f \circ \gamma_n)'(\theta)| \quad \text{for all} \quad a_n < s < t < b_n,$$

and the function $(a_n, b_n) \ni \theta \mapsto |(f \circ \gamma_n)'(\theta)|$ is integrable by (6.3.4). By the Lebesgue Dominated Convergence Theorem, we get

$$\lim_{s,t \to a_n^+} |(f \circ \gamma_n)(t) - (f \circ \gamma_n)(s)| = 0.$$

This shows that the limit $\tau_n := \lim_{t \to a_n^+} (f \circ \gamma_n)(t)$ exists, and similarly we check that the limit $\eta_n := \lim_{t \to b_n^-} (f \circ \gamma_n)(t)$ exists. This allows to extend $f \circ \gamma_n : [a_n, b_n] \to \mathbb{C}$ to $[a_n, b_n]$ with continuity by defining $(f \circ \gamma_n)(a_n) := \tau_n$ and $(f \circ \gamma_n)(b_n) := \eta_n$. Moreover, (6.3.4) implies that

$$\begin{aligned} \lim_{n \to \infty} |\tau_n - \eta_n| &\leq \lim_{n \to \infty} \operatorname{diam}((f \circ \gamma_n)^*) = \lim_{n \to \infty} \sup_{a_n \leq s \leq t \leq b_n} |(f \circ \gamma)(t) - (f \circ \gamma)(s)| \\ &= \lim_{n \to \infty} \sup_{a_n \leq s \leq t \leq b_n} \left| \int_s^t (f \circ \gamma_n)'(\theta)\, \mathrm{d}\theta \right| \leq \lim_{n \to \infty} \int_{a_n}^{b_n} |(f \circ \gamma_n)'(\theta)|\, \mathrm{d}\theta = 0. \end{aligned} \tag{6.3.5}$$

We observe that $\tau_n \in \partial\Omega$, as otherwise $\tau_n \in \Omega$, and since $f^{-1} : \Omega \to \mathbb{D}$ is continuous,

$$\mathbb{D} \ni f^{-1}(\tau_n) = \lim_{t \to a_n^+} f^{-1}((f \circ \gamma_n)(t)) = \lim_{t \to a_n^+} \gamma_n(t) = \gamma_n(a_n),$$

contradicting (6.3.2). Similarly $\eta_n \in \partial\Omega$ for all $n \in \mathbb{N}$. Since $h : \mathbb{T} \to \partial\Omega$ is an homeomorphism, for every $n \in \mathbb{N}$ we can find $p_n, q_n \in \mathbb{T}$ with $h(p_n) = \tau_n$ and $h(q_n) = \eta_n$. Since $h^{-1} : \partial\Omega \to \mathbb{T}$ is uniformly continuous, by (6.3.5), we have that $\lim_{n \to \infty} |p_n - q_n| = 0$. Thus, if we denote by δ_n the arc in $\mathbb{T}$ joining p_n and q_n, then $\lim_{n \to \infty} \operatorname{diam}(\delta_n^*) = 0$. In particular, for n large enough, δ_n is the shortest arc in $\mathbb{T}$ that joins p_n to q_n. We define the continuous curve $\sigma_n := h \circ \delta_n$, whose trace is contained in $\partial\Omega$ and joins τ_n and η_n. And observe that

$$\lim_{n \to \infty} \operatorname{diam}(\sigma_n^*) = 0, \tag{6.3.6}$$

as otherwise, we would have $c > 0$ and points $\tau_n', \eta_n' \in \sigma_n^*$ with $|\tau_n' - \eta_n'| \geq c$ for infinitely many n. But then the continuity of h would imply $|h(\tau_n') - h(\eta_n')| \geq d > 0$ for infinitely many n, where the points $h(\tau_n'), h(\tau_n') \in \delta_n^*$, a contradiction.

We finally define the composite curves $\Sigma_n := \sigma_n \star (f \circ \gamma_n)$ for all $n \in \mathbb{N}$. By the continuity of $f \circ \gamma_n$ at the extreme points, we see that σ_n is a closed, simple, and continuous curve contained in $\overline{\Omega}$. By the Jordan's Curve Theorem, $\mathbb{C} \setminus \Sigma_n^*$ has precisely one bounded connected component (the *inside* of Σ_n), and one unbounded connected component (the *outside* of Σ_n) connected component, both of which have boundary equal to Σ_n^*. We denote by Ω_n the bounded one, where $\partial\Omega_n = \Sigma_n^*$. By (6.3.5) and (6.3.6), we get that

$$\lim_{n \to \infty} \operatorname{diam}(\Omega_n) = \lim_{n \to \infty} \operatorname{diam}(\Sigma_n^*) = 0. \tag{6.3.7}$$

We next observe that $\Omega_n \subset \Omega$. Indeed, since $\partial\Omega$ is a Jordan curve and Ω is bounded, then $\mathbb{C} \setminus \overline{\Omega}$ is the unbounded connected component of $\mathbb{C} \setminus \partial\Omega$. Since $\Sigma_n^* \cap (\mathbb{C} \setminus \overline{\Omega}) = \emptyset$ (by the construction of Σ_n), this implies that $\mathbb{C} \setminus \overline{\Omega}$ is contained in one of the connected components of $\mathbb{C} \setminus \Sigma_n^*$, that is, either $\mathbb{C} \setminus \overline{\Omega} \subset \Omega_n$ or $\mathbb{C} \setminus \overline{\Omega} \subset \mathbb{C} \setminus \overline{\Omega_n}$. The first situation is impossible, as Ω_n is bounded. Therefore, the second possibility holds, and therefore $\overline{\Omega_n} \subset \overline{\Omega}$. But if we had that $\xi \in \Omega_n$ and still $\xi \in \partial\Omega$, then we can find $\{\xi_j\}_j \subset \mathbb{C} \setminus \overline{\Omega} \subset \mathbb{C} \setminus \overline{\Omega_n}$ converging to ξ. Since Ω_n is open, there are ξ_j's belonging to Ω_n and $\mathbb{C} \setminus \overline{\Omega_n}$, a contradiction. We have shown the inclusion $\Omega_n \subset \Omega$.

Now, notice that the trace of $\gamma_n : (a_n, b_n) \to \mathbb{D} \cap D(\xi, r_n)$ splits $\mathbb{D}$ into two disjoint connected open subsets $U_n := \mathbb{D} \cap D(\xi, r_n)$, $V_n = \mathbb{D} \setminus \overline{D}(\xi, r_n)$ of $\mathbb{D} \setminus \gamma_n^*$, where $\lim\limits_{n\to\infty} \operatorname{diam}(U_n) = 0$. Since $f : \mathbb{D} \to \Omega$ is a homeomorphism, we get that $(f \circ \gamma_n)^*$ splits $\Omega \setminus (f \circ \gamma_n)^*$ into two connected components $f(U_n)$ and $f(V_n)$. The set Ω_n is a connected component of $\Omega \setminus (f \circ \gamma_n)^*$ for the following reason. As we observed before, Ω_n is an open and connected subset of $\Omega \setminus (f \circ \gamma_n)^*$. And if $\{\xi_j\}_j \subset \Omega_n$ converges to some $\xi \in \Omega \setminus (f \circ \gamma_n)^*$ with $\xi \notin \Omega_n$, then $\xi \in \partial\Omega_n = \Sigma_n^*$, and hence $\xi \in \sigma_n^* \subset \partial\Omega$, contradicting that $\xi \in \Omega$. Consequently, Ω_n is connected, and both open and (relatively) closed in $\Omega \setminus (f \circ \gamma_n)^*$, thus it is a connected component of $\Omega \setminus (f \circ \gamma_n)^*$, and so either $\Omega_n = f(U_n)$ or $\Omega_n = f(V_n)$. But for n large enough, we have that $D(0, 1/2) \subset V_n$, and so $f(D(0, 1/2)) \subset f(V_n)$, which shows that $f(V_n)$ has diameter bounded below by a positive number independent of n. Due to (6.3.7), we deduce that $\Omega_n = f(U_n)$ for all n large enough. The sequences $\{z_n\}_n, \{w_n\}_n$ from (6.3.1) can be assumed to be contained in U_n. But precisely (6.3.1) contradicts that $\lim\limits_{n\to\infty} f(U_n) = 0$.

Claim 2: *The unique extension $F : \overline{\mathbb{D}} \to \overline{\Omega}$ of f is a homeomorphism.* By **Claim 1**, $f : \mathbb{D} \to \Omega$ is uniformly continuous, and so has a unique uniformly continuous extension $F : \overline{\mathbb{D}} \to \mathbb{C}$ to $\overline{\mathbb{D}}$.

We next show that $F(\partial\mathbb{D}) = \partial\Omega$. Indeed, if $z \in \partial\mathbb{D}$, there is a sequence $\{z_n\}_n \subset \mathbb{D}$ convergent to z, and so $\{F(z_n)\}_n \subset \Omega$ is convergent to $F(z)$ by the continuity, thus $F(z) \in \overline{\Omega}$. But if we had $F(z) \in \Omega$, since $f : \mathbb{D} \to \Omega$ is bijective, there would be $w \in \mathbb{D}$ with $f(w) = F(z)$, and the continuity of f^{-1} yields that $\{z_n\}_n$ converges to $w \in \mathbb{D}$, a contradiction. This shows the inclusion $F(\partial\mathbb{D}) \subset \partial\Omega$. For the reverse inclusion, let $w \in \partial\Omega$ and $\{w_n\}_n \subset \Omega$ convergent to w. Again by the surjectivity of $f : \mathbb{D} \to \Omega$, there is a sequence (passing to a subsequence if necessary) $\{z_n = f^{-1}(w_n)\}_n \subset \mathbb{D}$ convergent to some $\xi \in \overline{\mathbb{D}}$. If we had $\xi \in \mathbb{D}$, then $\{w_n = f(z_n)\}_n$ would converge to $f(\xi) = w$, necessarily implying that $w \in \Omega$, and this is a contradiction. Thus $\xi \in \partial\mathbb{D}$, and $\partial\Omega \subset T(\partial\mathbb{D})$.

Consequently, $F(\overline{\mathbb{D}}) = F(\mathbb{D}) \cup F(\partial\mathbb{D}) = f(\mathbb{D}) \cup \partial\Omega = \Omega \cup \partial\Omega = \overline{\Omega}$, showing that $F : \overline{\mathbb{D}} \to \overline{\Omega}$ is surjective. Since $\overline{\mathbb{D}}$ is compact, $\overline{\Omega}$ is a Haussdorff space, F is continuous and surjective, we will prove that F is a homeomorphism as soon as we check that F is injective. The injectivity of F in $\mathbb{D}$ follows from that of f, and since $F(\partial\mathbb{D}) = \partial\Omega$, it suffices to verify the injectivity of F at points of $\partial\mathbb{D}$. Let $z_1, z_2 \in \partial\mathbb{D}$ be so that $z_1 \neq z_2$ and $F(z_1) = F(z_2)$. Let γ be the arc in $\partial\mathbb{D}$ joining z_1 and z_2, let $\ell_1 := [0, z_1]$, $\ell_2 = [0, z_2]$, and the composite path $\Gamma = \ell_1 \star \gamma \star \ell_2^-$. Obviously Γ is a Jordan curve which divides $\mathbb{D}$ into two connected components (of $\mathbb{D} \setminus \Gamma^*$), say U (the *inside* of Γ) and V (the *outside* of Γ in $\mathbb{D}$). Then $\Sigma := F(\ell_1 \cup \ell_2^-)$ defines a Jordan curve contained in $\overline{\Omega}$, with $\Sigma \cap \partial\Omega = \{F(z_1)\}$, as F maps $\mathbb{D}$ to Ω and $\partial\mathbb{D}$ to $\partial\Omega$. Denoting by W the *inside* of Σ, the same argument we used in **Claim 1** permits to show that $W \subset \Omega$. Moreover, we must have either $F(U) = W$ or $F(V) = W$. In the first case, we get that

$$F(\gamma^*) = F(\partial U \cap \partial\mathbb{D}) \subset \partial W \cap \partial\Omega = \{F(z_1)\}.$$

In the latter case, with the same argument,

$$F(\partial\mathbb{D} \setminus \gamma^*) = F(\partial V \cap \partial\mathbb{D}) \subset \partial W \cap \partial\Omega \subset \{F(z_1)\}.$$

Thus, we have that F is constant in one of the arcs γ^*, $\partial\mathbb{D} \setminus \gamma^*$. Denote by σ this arc, and notice that $\sigma \neq \partial\mathbb{D}$. If $z_0 \in \partial\mathbb{D} \setminus \sigma$, let $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ a Möbius Transformation with

$$T(z_0) = \infty, \quad T(\partial\mathbb{D}) = \mathbb{R}_\infty, \quad T(\mathbb{D}) = \mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}.$$

Since $z_0 \notin \sigma$, then $T(\overline{U}) \subset \mathbb{C}$, the restriction $T\restriction_{\overline{U}} : \overline{U} \to T(\overline{U})$ is a homeomorphism (between compact subsets of $\mathbb{C}$). Moreover $T(U)$ is an open subset of $\mathbb{H}$, so that $T\restriction_U : U \to T(U)$ being biholomorphic, and $T(\sigma) \subset \partial(T(U)) \cap \mathbb{R}$. Since T is continuous and σ is connected, $T(\sigma)$ is connected, and therefore an interval $[a,b]$ of $\mathbb{R}$. Thus there exist $c \in (a,b)$ and a disk $D_\varepsilon = D(c,\varepsilon)$ with $D_\varepsilon^+ \subset T(U) \subset \mathbb{H}$ and $D_\varepsilon^0 \subset T(\sigma) \subset \mathbb{R}$. Finally, if $\xi \in \mathbb{C}$ is the constant value of F over σ, we define the continuous (and holomorphic in $T(U)$) function $g(z) := F \circ T^{-1}(z) - \xi$ for all $z \in T(\overline{U})$, which, in particular, satisfies

$$g \in C(D_\varepsilon^+ \cup D_\varepsilon^0), \quad g \in \mathcal{H}(D_\varepsilon^+), \quad g(D_\varepsilon^0) \subset \{F(w) - \xi \, : \, w \in \sigma\} = \{0\}.$$

By Theorem 6.3, g has a holomorphic extension $G : D_\varepsilon \to \mathbb{C}$. But then G vanishes in D_ε^0, so by the Identity Principles for holomorphic maps, g is null in D_ε^+, and so in $T(\overline{U})$, as this set is connected. Since T^{-1} is a bijection, we deduce that F is constantly equal to ξ in $\overline{U}$, a contradiction, because $U \subset \mathbb{D}$ is an open set, where F coincides with the bijective map f.

We conclude that F is injective in $\partial\mathbb{D}$, and so, by the previous comments, $F : \overline{\mathbb{D}} \to \overline{\Omega}$ is a homeomorphism.

□

If we now combine the Riemann Mapping Theorem 4.43 with Theorem 6.6, and then also with Theorem 6.5.

Corollary 6.7. *Let Ω be open and bounded so that $\partial\Omega$ is a Jordan curve. Then there exists a homeomorphism $\psi : \overline{\mathbb{D}} \to \overline{\Omega}$, with $\psi\restriction_{\mathbb{D}} : \mathbb{D} \to \Omega$ biholomorphic.*

If, in addition, $\partial\Omega$ is analytic, then there exists $W \subset \mathbb{C}$ open with $\overline{\mathbb{D}} \subset W$ and a holomorphic map $\Psi : W \to \mathbb{C}$ with $\Psi = \psi$ in $\overline{\mathbb{D}}$.

Proof. Since $\partial\Omega$ is a Jordan curve, the set $\mathbb{C} \setminus \overline{\Omega}$ is connected. Any cycle Γ in Ω is constant on each connected subset of $\mathbb{C} \setminus \Gamma^*$, thus constant in $\mathbb{C} \setminus \overline{\Omega}$. We know that $W(\Gamma, z) = 0$ for all $|z| \geq r$, and for some large enough $r > 0$; see Remark 1.7. Since there points $|z| \geq r$ belonging to $\mathbb{C} \setminus \overline{\Omega}$ (as Ω is bounded), we have that $W(\Gamma, w) = 0$ for all $w \in \mathbb{C} \setminus \overline{\Omega}$. By the continuity of $W(\Gamma, \cdot) : \mathbb{C} \setminus \Gamma^* \to \mathbb{Z}$, the windinng numbers are zero at points of $\overline{\mathbb{C} \setminus \overline{\Omega}} \supset \mathbb{C} \setminus \Omega$. In other words, $\Gamma \simeq 0$ in Ω. By Riemann's Theorem 4.43, there exists $\varphi : \mathbb{D} \to \Omega$ biholomorphic. By Carathéodory's Theorem 6.6, there exists a unique extension $\psi : \overline{\mathbb{D}} \to \overline{\Omega}$ of φ, which is a homeomorphism.

If, in addition, $\partial\Omega$ is analytic, by Theorem 6.5, there is an open subset $W \supset \overline{\mathbb{D}}$ and a holomorphic map $\Psi : W \to \mathbb{C}$ extending ψ from $\overline{\mathbb{D}}$. □

6.4 The Dirichlet Problem in Jordan Domains

Lemma 6.8. *Let $\Omega, U \subset \mathbb{C}$ be open sets, $f : \Omega \to U$ holomorphic in Ω, and $u : U \to \mathbb{R}$ harmonic in U. Then the function $u \circ f : \Omega \to \mathbb{R}$ is harmonic in Ω.*

Proof. We may assume that Ω is connected, as otherwise we can verify the harmonicity of $u \circ f$ on each connected component. Also, if f is constant the result is obvious. If f is non-constant in Ω, and $z_0 \in \Omega$ and $r > 0$ is so that $D(z_0, r) \subset \Omega$, then $f(D(z_0, r))$ is an open subset of U by Theorem 4.4. Thus there exists $\delta > 0$ with $D(f(z_0), \delta) \subset U$. Since $D(f(z_0), \delta)$ is simply connected, by Corollary 5.6, there exists $g \in \mathcal{H}(D(f(z_0), \delta))$ with $u = \operatorname{Re} g$ in $D(f(z_0), \delta)$. Therefore $u \circ f = \operatorname{Re}(g) \circ f = \operatorname{Re}(g \circ f)$ in the open set $f^{-1}(D(f(z_0), \delta))$, thus $u \circ f$ is harmonic in that set (as $g \circ f$ is holomorphic). Since $f^{-1}(D(f(z_0), \delta))$ contains a disk $D(z_0, \varepsilon)$, we have shown that $u \circ f$ is locally harmonic in Ω, that is harmonic in Ω. □

What we learnt from Corollary 6.7 can be used to solve the Dirichlet Problem for Harmonic Functions in every Jordan domain.

Theorem 6.9 (Dirichlet's Problem in Jordan Domains)**.** *Let Ω be open and bounded, so that $\partial\Omega$ is a Jordan curve, and let $g : \partial\Omega \to \mathbb{R}$ be continuous. Then there exists a unique $u \in C(\overline{\Omega}) \cap \operatorname{Har}(\Omega)$ with $u = g$ in $\partial\mathbb{D}$.*

Proof. By Corollary 6.7, there exists a homeomorphism $\psi : \overline{\mathbb{D}} \to \overline{\Omega}$ with $\psi : \mathbb{D} \to \Omega$ biholomorphic. The composition $h := g \circ \psi\restriction_{\partial\mathbb{D}}: \partial\mathbb{D} \to \mathbb{R}$ is continuous, and by Theorem 5.16 there exists $v \in C(\overline{\mathbb{D}}) \cap \operatorname{Har}(\mathbb{D})$ with $v = h$ in $\mathbb{T}$. If we define $u := v \circ \psi^{-1} : \overline{\Omega} \to \mathbb{R}$ we obtain that $v \in C(\overline{\Omega})$ and $v \in \operatorname{Har}(\Omega)$ by virtue of Lemma 6.8. It is also clear that $u = g$ in $\partial\Omega$.

For the uniqueness, if two functions $u_1, u_2 \in C(\overline{\Omega}) \cap \operatorname{Har}(\Omega)$ satisfy $u_1 = u_2 = g$ in $\partial\mathbb{D}$, then by Corollary 5.15 we get that $u_1 = u_2$ in $\overline{\Omega}$. □

6.5 Singular Points and Natural Boundary

In this section we study analytic continuations of holomorphic functions in a disk to some neighbourhood of the corresponding circles. Regular points are precisely those points in this circle on which the function admits a holomorphic extension around the point.

Definition 6.10 (Regular and Singular Points)**.** *Let $D \subset \mathbb{C}$ be an open disk, and $f \in \mathcal{H}(D)$. A point $z \in \partial D$ is **a regular point of** f **in** ∂D if there exists an open disk D_1 centered at z, and a holomorphic function $g : D_1 \to \mathbb{C}$ with $f = g$ on $D_1 \cap D$.*

*Also, if $z \in \partial\mathbb{D}$ is said to be a **singular point of** f if z is not a regular point of f.*

Notice that if $z \in \partial D$ is a regular point of f, then the limit $\lim\limits_{D \ni w \to z} f(w)$ must exist. The next theorem shows that a power series cannot be extended to a neighborhood of the closure of its disk of convergence.

Theorem 6.11. *Let $\mathbb{D}$ denote the open unit disk, let $f \in \mathcal{H}(\mathbb{D})$, whose Taylor series at 0*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{D}; \tag{6.5.1}$$

has radius of convergence equal to 1. Then f has a singular point $\xi \in \partial\mathbb{D}$.

Proof. For the sake of contradiction, assume that all points $z \in \partial\mathbb{D}$ are regular for f. Then, for every $z \in \partial\mathbb{D}$ there exists $D_z = D(z, r_z)$ and a function $g_z \in \mathcal{H}(D(z, r_z))$ so that $g_z = f$ on $D(z, r_z)$. We can define

$$W := \mathbb{D} \cup \bigcup_{z \in \partial\mathbb{D}} D_z,$$

and $F : W \to \mathbb{C}$ as

$$F(w) = \begin{cases} f(w) & \text{if } w \in \mathbb{D}, \\ g_z(w) & \text{if } w \in D(z, r_z/2), \text{ for some } z \in \partial\mathbb{D}. \end{cases}$$

An argument identical to the one at the end of the proof of Theorem 6.5 shows that F is well-defined and holomorphic in the open set W, and (obviously) $F = f$ in W. Since $\overline{\mathbb{D}} \subset W$ and W is open, there exists $\varepsilon > 0$ so that $\overline{D}(0, 1 + \varepsilon) \subset W$. Thus the Taylor Series of F at 0,

$$\sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

has radius of convergence at least $1 + \varepsilon/2$. Since the coefficients $\{a_n\}_n$ in the series (6.5.1) are precisely $\{f^{(n)}(0)/n!\}_n$, we get a contradiction. □

And when a function on an open disk does not admit an extension around *any* point of the corresponding circle, we say that this circle is the natural boundary of f.

Definition 6.12. *Let D be an open disk, and $f \in \mathcal{H}(D)$. We say that ∂D is **the natural boundary of** f if every point $\xi \in \partial D$ is a singular point of f.*

The third of the following examples is particularly interesting.

Example 6.13. Consider the functions f, g given by the series

$$f(z) = \sum_{n=0}^{\infty} z^n, \quad g(z) = \sum_{n=0}^{\infty} z^{2n}.$$

The radius of convergence of both series is equal to 1, and so $f, g \in \mathcal{H}(\mathbb{D})$. But we can write

$$f(z) = \frac{1}{1-z}, \quad g(z) = \frac{1}{1-z^2}, \quad z \in \mathbb{D},$$

where we see that f is holomorphic in $\mathbb{C} \setminus \{1\}$ and g is holomorphic in $\mathbb{C} \setminus \{-1, 1\}$. The limit at these singularities of the functions do not exist, and so we may conclude that 1 is a singular point of f, that all $\xi \in \partial\mathbb{D} \setminus \{1\}$ is a regular point of f, that $-1, 1$ are singular points of g, and that all $\xi \in \partial\mathbb{D} \setminus \{-1, 1\}$ is a regular point of g.

We next consider the following *lacunary series*

$$F(z) = \sum_{n=0}^{\infty} z^{2^n}, \quad z \in \mathbb{D}.$$

The series have radius of convergence 1, and so $F \in \mathcal{H}(\mathbb{D})$. We claim that all points $\xi \in \partial\mathbb{D}$ are singular points of F, and so $\partial\mathbb{D}$ is the natural boundary of F. To see this, consider the set of dyadic points

$$\mathcal{S} := \{e^{2\pi i \frac{k}{2^m}} \,:\, 0 \le k \le 2^m,\, k, m \in \mathbb{N}\} \subset \partial\mathbb{D}.$$

The set $\mathcal{S}$ is dense in $\partial\mathbb{D}$. Since the set of singular points of a function is closed (see Exercise 6.6), if we show that each $\xi \in \mathcal{S}$ is singular for F, then each $\xi \in \partial\mathbb{D}$ will be singular as well.

Let us then fix $0 \le k \le 2^m$ with $k, m \in \mathbb{N}$, and $\xi := e^{2\pi i \frac{k}{2^m}}$. To see that ξ is a singular point for F, it suffices to check that

$$\text{the limit } \lim_{r \to 1^-} F(r\xi) \text{ does not exist.} \tag{6.5.2}$$

Indeed, we can write

$$\begin{aligned}
F(r\xi) &= \sum_{n=0}^{\infty} (r\xi)^{2^n} = \sum_{n=0}^{m-1} (r\xi)^{2^n} + \sum_{n=m}^{\infty} \left(re^{2\pi i \frac{k}{2^m}}\right)^{2^n} \\
&= \sum_{n=0}^{m-1} (r\xi)^{2^n} + \sum_{n=m}^{\infty} r^{2^n} e^{2\pi k i \, 2^{n-m}} \\
&= \sum_{n=0}^{m-1} (r\xi)^{2^n} + \sum_{n=m}^{\infty} r^{2^n}.
\end{aligned}$$

Since the first sum is finite, we will show (6.5.2) as soon as we prove that

$$\text{the limit } \lim_{r \to 1^-} \sum_{n=m}^{\infty} r^{2^n} \text{ does not exist.} \tag{6.5.3}$$

But note that, for all $N \in \mathbb{N}$, we can write

$$\sum_{n=m}^{\infty} r^{2^n} = \sum_{j=0}^{\infty} r^{2^{m+j}} \ge \sum_{j=0}^{N} r^{2^{m+j}} = \ge (N+1) r^{2^{m+N}}.$$

Thus, for every $N \in \mathbb{N}$,

$$\liminf_{r \to 1^-} \sum_{n=m}^{\infty} r^{2^n} \ge \liminf_{r \to 1^-} (N+1) r^{2^{m+N}} = N + 1,$$

showing (6.5.3), and thus (6.5.2).

6.6 Analytic Continuation along Curves

Our next goal is to define a suitable notion of continuation of a map along a paths, and conclude the existence of global extensions provided there are extensions along paths.

6.6.1 Direct Continuations and Chains

We need to define first the concept of element of function and direct continuation along a chain.

Definition 6.14 (Direct Continuation. Chains). *A* ***function element*** *is a pair (D, f), where $D \subset \mathbb{C}$ is an open disk and $f \in \mathcal{H}(D)$.*

Two function elements (D_0, f_0), (D_1, f_1) are said to be a ***direct continuation*** *of each other, which we represent by $(D_0, f_0) \sim (D_1, f_1)$, when*

$$D_0 \cap D_1 \neq \emptyset \quad \text{and} \quad f_0 = f_1 \text{ on } D_0 \cap D_1.$$

Also, a ***chain*** *$\mathcal{C}$ is a finite (ordered) collection $\mathcal{C} = \{D_0, \ldots, D_n\}$ of open disks with $D_j \cap D_{j+1} \neq \emptyset$ for all $j = 0, \ldots, n-1$. Given a function element (D, f) and a chain $\mathcal{C}$ as above, we say that (D, f) admits an* ***analytic continuation along*** *$\mathcal{C}$ of there are function elements $\{(D_0, f_0), \ldots, (D_n, f_n)\}$ so that*

$$(D_0, f_0) = (D, f) \quad \text{and} \quad (D_j, f_j) \sim (D_{j+1}, f_{j+1}) \quad \text{for all} \quad j = 0, \ldots, n-1.$$

In such case, say that (D_n, f_n) is ***the analytic continuation of*** *(D, f)* ***along the chain*** *$\mathcal{C}$*

We will see in the following remark under which conditions the direct analytic continuation is an equivalent relationship between function elements.

Remark 6.15. Concerning Definition 6.14, we observe the following.

(1) The relation $\sim$ between functions elements is obviously symmetric and reflexive.

(2) The relation $\sim$ is not transitive, that is, given function elements (D_0, f_0), (D_1, f_1), (D_2, f_2) with $(D_0, f_0) \sim (D_1, f_1)$ and $(D_1, f_1) \sim (D_2, f_2)$, it is in general not true that $(D_0, f_0) \sim (D_2, f_2)$, even when $D_0 \cap D_2 \neq \emptyset$.

An example is as follows. Let D_0, D_1, and D_2 be respectively the open disks with radius 1 and centered at points $1, w, w^2$, respectively, where $w = e^{2\pi i/3}$. Notice that the three disks are pairwise non-disjoint, but $D_0 \cap D_1 \cap D_2 = \emptyset$. Let f_0 be the principal branch of the logarithm, namely,

$$f_0(z) := \operatorname{Log} z = z + i \operatorname{Arg}(z), \quad z \in \mathbb{C} \setminus (-\infty, 0].$$

Recall that $\operatorname{Arg}(z) \in (-\pi, \pi]$ for all $z \in \mathbb{C} \setminus \{0\}$. In D_1 and D_2 we consider a different branch of the logarithm:

$$f_1(z) = f_2(z) := \log z + i\alpha(z), \quad z \in \mathbb{C} \setminus [0, +\infty),$$

where $\alpha : \mathbb{C}\setminus\{0\} \to [0, 2\pi)$ is a continuous branch of the argument. It is clear that $f_j \in \mathcal{H}(D_j)$ for $j = 0, 1, 2$, and that $f_0 = f_1$ on $D_0 \cap D_1$ and $f_1 = f_2$ in $D_1 \cap D_2$. However, $f_0 \neq f_2$ in $D_0 \cap D_2$, because this intersection is contained in $\{z \in \mathbb{C} : \operatorname{Re}(z) > 0 > \operatorname{Im}(z)\}$, where the branches Arg and α differ.

(3) If (D_0, f_0), (D_1, f_1), (D_2, f_2) are function elements with

$$D_0 \cap D_1 \cap D_2 \neq \emptyset, \quad D_0, f_0) \sim (D_1, f_1), \quad \text{and} \quad (D_1, f_1) \sim (D_2, f_2),$$

then $(D_0, f_0) \sim (D_2, f_2)$.

Indeed, $f_0 = f_1$ in $D_0 \cap D_1$ and $f_1 = f_2$ in $D_1 \cap D_2$ implies that $f_0 = f_2$ in the nonempty open set $D_0 \cap D_1 \cap D_2$. Since $D_0 \cap D_2$ is connected, by the Identity Principles for holomorphic maps, $f_0 = f_2$ in $D_0 \cap D_2$.

6.6.2 Continuation along Curves and One-Parameter Families

Analytic continuation along a curves is defined as the final element of a continuation along a chain that covers the curve.

Definition 6.16 (Continuation along Curves)**.** *Given a continuous curve $\gamma : [0,1] \to \mathbb{C}$ and a chain of disks $\mathcal{C} = \{D_0, \ldots, D_n\}$, we say that $\mathcal{C}$ **covers** γ if $\gamma(0)$ is the center of D_0, $\gamma(1)$ is the center of D_n, and there exist $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ so that*

$$\gamma([t_j, t_{j+1}]) \subset D_j \quad \textit{for all} \quad j = 0, \ldots, n-1.$$

*Also, a function element (D, f) is said to admit an **analytic continuation along** γ if there exists a chain $\mathcal{C} = \{D_0, \ldots, D_n\}$ that covers γ and (D, f) admits an analytic continuation along $\mathcal{C}$. If (D_n, f_n) is such analytic continuation, we say that (D_n, f_n) is **the analytic continuation of** (D, f) **along the chain** γ.*

A curve can be covered by two different chains as above and so, in order to show that the previous definition gives a consistent analytic continuation, we need to check that analytic continuations along two different coverings yield two function elements that are direct analytic continuations of each other.

Theorem 6.17 (Uniqueness of Continuations along Curves)**.** *Let $\gamma : [0,1] \to \mathbb{C}$ be a continuous curve, and (D, f) a function element with D centered at $\gamma(0)$. Let $\mathcal{C} = \{D_0, \ldots, D_n\}$ and $\mathcal{C}' = \{B_0, \ldots, B_m\}$ be two chains covering γ, and (D_n, f_n), (B_m, g_m) be analytic continuations of f along $\mathcal{C}$ and $\mathcal{C}'$ respectively. Then $f_n = g_m$ on $D_n \cap B_m$.*

Proof. There are function elements $\{(D_0, f_0), \ldots, (D_n, f_n)\}$ and $\{(B_0, g_0), \ldots, (B_m, g_m)\}$ with

$$(D_0, f_0) = (B_0, g_0) = (D, f), \tag{6.6.1}$$

$$(D_j, f_j) \sim (D_{j+1}, f_{j+1}), \quad (B_k, g_k) \sim (B_{k+1}, g_{k+1}) \quad \text{for all} \quad j = 0, \ldots, n-1,\ k = 0, \ldots, m-1, \tag{6.6.2}$$

and partitions

$$0 = t_0 < t_1 \ldots < t_{n-1} < t_n = 1, \quad 0 = s_0 < s_1 \ldots < s_{m-1} < s_m = 1$$

of $[0,1]$ such that

$$\gamma([t_j, t_{j+1}]) \subset D_j, \quad \gamma([s_k, s_{k+1}]) \subset B_k \quad \text{for all} \quad j = 0, \ldots, n-1,\ k = 0, \ldots, m-1. \tag{6.6.3}$$

We claim that for all $(j,k) \in \{0, \ldots, n-1\} \times \{0, \ldots, m-1\}$ with $[t_j, t_{j+1}] \cap [s_k, s_{k+1}] \neq \emptyset$, we must have $(D_j, f_j) \sim (B_k, g_k)$.

Indeed, assume that the claim does not hold, and let (j,k) be a couple as above with $j+k$ minimal. Note that (6.6.1) shows that it holds for $j = k = 0$. Thus, $j + k > 0$. Since $[t_j, t_{j+1}] \cap [s_k, s_{k+1}] \neq \emptyset$ we have (without loss of generality) that $s_k \geq t_j$. Using (6.6.3) we get

$$\gamma(s_k) \in B_k \cap B_{k-1} \cap D_j. \tag{6.6.4}$$

But clearly $[t_j, t_{j+1}] \cap [s_{k-1}, s_k] \neq \emptyset$, and so, by the minimality of $j+k$, we have that

$$(B_{k-1}, g_{k-1}) \sim (D_j, f_j). \tag{6.6.5}$$

Since, by (6.6.2), $(B_{k-1}, g_{k-1}) \sim (B_k, g_k)$, combining (6.6.5), (6.6.4) and Remark 6.15(3), we get that $(B_k, g_k) \sim (D_j, f_j)$, contradicting that the claim fails for (j,k).

Therefore, our claim holds, in particular being valid in the trivial case $j = n$, $k = m$, which proves the theorem.

□

We define one-parameter families of curves, via certain *homotopy* function; compare with Definition 1.25.

Definition 6.18 (One-parameter curves with common endpoints)**.** *Let $\Omega \subset \mathbb{C}$ be open, and $z, w \in \Omega$. A family of continuous curves $\{\gamma_t : [0,1] \to \Omega\}_{t\in[0,1]}$ in Ω so that*

$$\gamma_t(0) = z, \quad \gamma_t(1) = w \quad \textit{for all} \quad t \in [0,1]$$

is a ***one-parameter family of curves from*** *z* ***to*** *w* ***in*** *Ω when there exists a continuous mapping $H : [0,1] \times [0,1] \to \Omega$ with*

$$H(s,t) = \gamma_t(s), \quad \textit{for all} \quad (s,t) \in [0,1] \times [0,1].$$

If a function elements admits analytic continuation along each curve of a one-parameter family, then the initial and final continuations of the family are direct continuations of each other,

Theorem 6.19 (Analytic Continuation via One-Paremeter Curves)**.** *Let $\{\gamma_t\}_{t\in[0,1]}$ be a one-parameter family of curves from z to w in $\mathbb{C}$, and let (D,f) a function element with D centered at z, and so that (D,f) admits an analytic continuation (D_t, f_t) along γ_t for each $t \in [0,1]$. Then $f_1 = f_0$ on $D_0 \cap D_1$.*

Proof. Given $t \in [0,1]$, there are function elements $(D,f) = (B_0^t, g_0^t), \dots, (B_{n_t}^t, g_{n_t}^t) = (D_t, f_t)$ so that the chain $\{B_0^t, \dots, B_{n_t}^t\}$ covers γ_t. Thus, there are points $0 = s_0^t < s_1^t < \cdots < s_{n_t}^t = 1$ with

$$K_j^t := \gamma_t\left([s_j^t, s_{j+1}^t]\right) \subset B_j^t, \quad j = 0, \dots, n_t - 1. \tag{6.6.6}$$

By the compactness of each K_j^t, there exists $\varepsilon > 0$ with

$$\varepsilon < \min\{\operatorname{dist}(K_j^t, \partial B_j^t) \,:\, j = 0, \dots, n_t - 1\}. \tag{6.6.7}$$

Now, if $H : [0,1] \times [0,1] \to \mathbb{C}$ is a continuous map as in Definition 6.18, then H is of course uniformly continuous in $[0,1]^2$, and so we can find $\delta_t > 0$ (depending on ε, thus on t) so that

$$|\gamma_t(s) - \gamma_r(s)| < \varepsilon, \quad \text{for all} \quad s, r \in [0,1],\ |r - t| \le \delta_t. \tag{6.6.8}$$

By (6.6.6), (6.6.6), and (6.6.8), whenever $|r - t| \le \delta_t$, we can use the chain $\{B_0^t, \dots, B_{n_t}^t\}$ to cover γ_r. By Theorem 6.17, we have that $f_t = f_r$ on $D_t \cap D_r$. We have shown that

$$\text{For every } \ t \in [0,1], \ \text{ there is } \ \delta_t > 0 \ \text{ so that } \ f_t = f_r \ \text{ on } \ D_t \cap D_r \ \text{ whenever } \ |r - t| \le \delta_t. \tag{6.6.9}$$

By the compactness of $[0,1]$, finitely many intervals $I_k := (t_k - \delta_k, t_k + \delta_k)$, $k = 0, \dots, N$, as above cover $[0,1]$, where $0 \in I_0$ and $1 \in I_N$. But then (6.6.9) implies that

$$(D_0, f_0) \sim (D_{t_0}, f_{t_0}) \sim (D_{t_1}, f_{t_1}) \sim \cdots \sim (D_{t_{N-1}}, f_{t_{N-1}}) \sim (D_{t_N}, f_{t_N}) \sim (D_1, f_1).$$

Since all these disks have the same center w, the intersection

$$D_0 \cap \left(\bigcap_{k=1}^{N} D_{t_k} \right) \cap D_1$$

is nonempty, and by Remark 6.15(3), we may conclude that $(D_0, f_0) \sim (D_1, f_1)$, thus $f_0 = f_1$ on $D_0 \cap D_1$. $\square$

6.6.3 The Monodromy Theorem

We are now ready to show the main theorem of this section, where the domain is assumed to be simply-connected.

Theorem 6.20 (Monodromy Theorem). *Let $\Omega \subset \mathbb{C}$ be open and simply connected, (D, f) a function element with $D \subset \Omega$. Assume that (D, f) admits an analytic continuation along every continuous curve starting from the center of D and contained in Ω. Then there exists $F \in \mathcal{H}(\Omega)$ with $F = f$ on D.*

Proof. Denote by z_0 the center of D, and for each $z \in \Omega$, let $\gamma_z : [0,1] \to \Omega$ be a polygonal curve with $\gamma_z(0) = z_0$ and $\gamma_z(1) = z$. The element (D, f) admits an analytic continuation (D_z, F_{γ_z}) along γ_z (through a chain $\mathcal{C}_z$) with $D_z = D(z, r_z)$, for some $r_z > 0$. In the case $z = z_0$, we simply understand that $(D_{z_0}, F_{z_0}) = (D, f)$. We define $F : \Omega \to \mathbb{C}$ by the formula

$$F(w) = F_{\gamma_z}(w) \quad \text{whenever} \quad w \in D(z, r_z/2),\ z \in \Omega. \tag{6.6.10}$$

Let us verify that F is well-defined. Given $w \in D(z, r_z/2) \cap D(\xi, r_\xi/2)$ for $w, \xi \in \Omega$, we wish to show that $F_{\gamma_z}(w) = F_{\gamma_\xi}(w)$. Obviously the disks $D_{z,w} := D(w, r_z/2)$ and $D_{\xi,w} := D(w, r_\xi/2)$ are contained respectively in $D(z, r_z)$ and $D(\xi, r_\xi)$. We extend γ_z and γ_ξ to w as

$$\Gamma_z := \gamma_z \star [z, w], \quad \Gamma_\xi := \gamma_\xi \star [\xi, w].$$

And we add the function elements $(D_{z,w}, F_{\gamma_z})$ and $(D_{\xi,w}, F_{\gamma_\xi})$ to the chains $\mathcal{C}_z$ and $\mathcal{C}_\xi$. We thus obtain new chains covering Γ_z and Γ_ξ respectively, with analytic continuations $(D_{z,w}, F_{\gamma_z})$, $(D_{\xi,w}, F_{\gamma_\xi})$ of (D, f) along Γ_z and Γ_ξ. Both Γ_z and Γ_ξ have initial point z_0 and end-point w. Because Ω is simply connected, by Exercise 6.8, there exists a one-parameter family of continuous curves $\{\Gamma_t\}_{t\in[0,1]}$ in Ω with $\Gamma_0 = \Gamma_z$ and $\Gamma_1 = \Gamma_\xi$. By the assumption (D, f) admits an analytic continuation along each Γ_t. Applying Theorem 6.19, we get that

$$F_{\gamma_z} = F_{\gamma_\xi} \quad \text{on} \quad D_{z,w} \cap D_{\xi,w} = D(w, r_z/2) \cap D(w, r_\xi/2),$$

as desired.

Now, observe that the formula (6.6.10) for F defines a holomorphic function, as it agrees locally with holomorphic functions. Also, note that $F = F_{z_0} = f$ in $D(z_0, r_{z_0}/2)$, where $D = D(z_0, r_{z_0})$, and $F, f \in \mathcal{H}(D)$. Thus $F = f$ on D, by the Identity Principles for holomorphic maps. □

6.7 Approximation by Polynomials and Rational Functions

By the Weierstrass's Approximation Theorem, any continuous function $g : K \to \mathbb{R}$ on a compact subset of $K \subset \mathbb{R}^2$ can be uniformly approximated by polynomials in K. These real polynomials are functions of the form

$$Q(x, y) = \sum_{k,j=0}^{n} a_{k,j} x^k y^j, \quad a_{k,j} \in \mathbb{R}.$$

However, in order to approximate an $f : K \to \mathbb{C}$ by *complex* polynomials of the form

$$P(z) = \sum_{k=0}^{n} c_k z^k, \quad z_k \in \mathbb{C},$$

it is not enough to separately approximate the real u and imaginary v parts of f by real polynomials Q_u, Q_v, as the result $Q_u + iQ_v$ might not be a complex polynomial, and in fact not even a holomorphic function.

On the other hand, recall that by Wierstrass Approximation Theorem 3.1 says that uniform convergence in compact sets of holomorphic functions gives a holomorphic function. This indicates that the approximation problem (analogous to Weierstrass Approximation Theorem) should be formulated as follows:

given $\Omega \subset \mathbb{C}$ open, $K \subset \Omega$ compact, and $f \in \mathcal{H}(\Omega)$, under what conditions does there exist, for every $\varepsilon > 0$, a complex polynomial P so that

$$|f(z) - P(z)| \leq \varepsilon \quad \text{for all} \quad z \in K\,?$$

We will see that the answer to this questions depends heavily on the topology of Ω, and that in many cases, the best we can achieve is an approximation by rational functions with *well-localized* poles.

6.7.1 Rational Approximation with Poles inside Cycles

The first result provides approximation of *Cauchy Integral Formulas* along a path by rational functions with poles contained in the path.

Proposition 6.21. *Let $\gamma : [0,1] \to \mathbb{C}$ be a piecewise C^1 path, and let $K \subset \mathbb{C}$ be a compact set with $\gamma^* \cap K = \emptyset$. Then, for every continuous function $f : \gamma^* \to \mathbb{C}$ and $\varepsilon > 0$, there exists a rational function $R \in \mathcal{M}(\mathbb{C}_\infty)$, whose (possible) poles are contained in γ^*, so that*

$$\left| \int_\gamma \frac{f(w)}{w-z} - R(z) \right| \leq \varepsilon, \quad \textit{for all} \quad z \in K.$$

Proof. By compactness, the sets $\gamma^* = \gamma([0,1])$, $f(\gamma^*)$, and K, are contained in $\overline{D}(0,r)$. Also, the functions γ and $f \circ \gamma$ are uniformly continuous on $[0,1]$. Therefore, denoting $\delta = \operatorname{dist}(\gamma^*, K) > 0$, there are numbers

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$$

so that

$$\max\{|\gamma(t_j) - \gamma(t)|, |(f\circ\gamma)(t_j) - (f\circ\gamma)(t)|\} \leq \frac{\varepsilon\delta^2}{4r\ell(\gamma)}, \quad \text{for all} \quad t \in [t_{j-1}, t_j],\ j = 1, \dots, n.$$

This implies that, for all $z \in K$ and $t \in [t_{j-1}, t_j]$, one has

$$\begin{aligned}
\left| \frac{(f\circ\gamma)(t)}{\gamma(t)-z} - \frac{(f\circ\gamma)(t_{j-1})}{\gamma(t_{j-1})-z} \right| &= \frac{|(f\circ\gamma)(t)(\gamma(t_{j-1})-z) - (f\circ\gamma)(t)(\gamma(t_{j-1})-z)|}{|\gamma(t)-z||\gamma(t_{j-1})-z|} \\
&= \frac{|\left((f\circ\gamma)(t) - (f\circ\gamma)(t_{j-1})\right)(\gamma(t_{j-1})-z) + (f\circ\gamma)(t_{j-1})\left(\gamma(t_{j-1}) - \gamma(t)\right)}{|\gamma(t)-z||\gamma(t_{j-1})-z|} \\
&\leq \frac{2r|(f\circ\gamma)(t) - (f\circ\gamma)(t_{j-1})| + r|\gamma(t_{j-1}) - \gamma(t)|}{\delta^2} \leq \frac{3\varepsilon r\delta^2}{4r\ell(\gamma)\delta^2} = \frac{3\varepsilon}{4\ell(\gamma)}.
\end{aligned} \tag{6.7.1}$$

We define the rational function

$$R(z) = \sum_{j=1}^{n} (f\circ\gamma)(t_{j-1})\left(\gamma(t_j) - \gamma(t_{j-1})\right) \frac{1}{\gamma(t_{j-1}) - z}.$$

This formula and (6.7.1) gives the estimates

$$\begin{aligned}
\left| \int_\gamma \frac{f(w)}{w-z} - R(z) \right| &= \left| \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \frac{(f\circ\gamma)(t)}{\gamma(t)-z}\gamma'(t)\,\mathrm{d}t - \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \frac{(f\circ\gamma)(t_{j-1})}{\gamma(t_{j-1})-z}\gamma'(t)\,\mathrm{d}t \right| \\
&\leq \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \left| \frac{(f\circ\gamma)(t)}{\gamma(t)-z} - \frac{(f\circ\gamma)(t_{j-1})}{\gamma(t_{j-1})-z} \right| \mathrm{d}t \\
&\leq \frac{3\varepsilon}{4\ell(\gamma)} \sum_{j=1}^{m-1} \int_{t_{j-1}}^{t_j} |\gamma'(t)|\,\mathrm{d}t = \frac{3\varepsilon}{4\ell(\gamma)}\ell(\gamma) < \varepsilon.
\end{aligned}$$

□

Using the previous proposition and the Integral Representation Theorem 1.22, we obtain the following.

Corollary 6.22. *Let $\Omega \subset \mathbb{C}$ be open, $K \subset \Omega$ compact and $f \in \mathcal{H}(\Omega)$. Then, for every $\varepsilon > 0$, there exists a rational function $R \in \mathcal{M}(\mathbb{C}_\infty)$, whose poles are contained in $\mathbb{C} \setminus K$ (none of them at ∞) so that*

$$|f(z) - R(z)| \leq \varepsilon, \quad \textit{for all} \quad z \in K.$$

Proof. By Theorem 1.22, we can find line segments $L_1, \ldots, L_m$ with traces contained in $\Omega \setminus K$, and so that

$$f(z) = \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{L_j} \frac{f(w)}{w-z} \, \mathrm{d}w, \quad \text{for all} \quad z \in K. \tag{6.7.2}$$

Now, given $\varepsilon > 0$, by Proposition 6.21 there exists, for each $0 \leq j \leq m$, a rational function R_j, whose poles are contained in L_j, and so that

$$\left| \frac{1}{2\pi i} \int_{L_j} \frac{f(w)}{w-z} \, \mathrm{d}w - R_j(z) \right| \leq \frac{\varepsilon}{m}, \quad \text{for all} \quad z \in K. \tag{6.7.3}$$

Combining (6.7.2) and (6.7.2), we deduce that

$$\left| f(z) - \sum_{j=1}^{m} R_j(z) \right| \leq \varepsilon, \quad z \in K.$$

The function $R(z) := \sum_{j=1}^m R_j(z)$ is clearly rational with poles contained in $\bigcup_{j=1}^m \gamma_j^* \subset \mathbb{C} \setminus K$. □

6.7.2 Runge's Theorem: first version

Given a compact set K, denote

$$\mathcal{P}(K) = \{P\restriction_K : \, P \text{ polynomial in } \mathbb{C}\}, \quad \mathcal{A}(K) = \overline{\mathcal{P}(K)}^{(C(K), \|\cdot\|_\infty)}. \tag{6.7.4}$$

That is, $\mathcal{P}(K)$ are restrictions of all polynomials to K, and $\mathcal{A}(K)$ is the closure of $\mathcal{P}(\mathcal{K})$ in the Banach space $(C(K), \|\cdot\|_\infty)$. In other words, $\mathcal{P}(\mathcal{K})$ is the collection of all (continuous) functions in K, that can be uniformly approximated in K by polynomials.

It is easy to see that if $f, g \in \mathcal{A}(K)$, and $\lambda \in \mathbb{C}$, then

$$\lambda f + g \in \mathcal{A}(K), \quad f \cdot g \in \mathcal{A}(K).$$

In the next lemma we prove that inversions with pole outside a compact set K are in the algebra $\mathcal{A}(K)$, if $\mathbb{C} \setminus K$ is connected.

Lemma 6.23. *Let $K \subset \mathbb{C}$ be compact, with $\mathbb{C} \setminus K$ connected. Denote, for each $a \in \mathbb{C} \setminus K$, the function*

$$f_a(z) = \frac{1}{z-a}, \quad z \in \mathbb{C} \setminus \{a\}.$$

Then $f_a \in \mathcal{A}(K)$ for all $a \in \mathbb{C} \setminus K$.

Proof. Define

$$U := \{a \in \mathbb{C} \setminus K \, : \, f_a \in \mathcal{A}(K)\}.$$

Let us first verify that U is nonempty. Let $R > 0$ so that $|z| < R$ for all $z \in K$. If $|z_0| > R$, then we have that, for all $z \in K$,

$$f_{z_0}(z) = \frac{1}{z - z_0} = -\frac{1}{z_0} \cdot \frac{1}{1 - \frac{z}{z_0}} = -\frac{1}{z_0} \sum_{n=0}^{\infty} \left(\frac{z}{z_0} \right)^n.$$

For each $n \in \mathbb{N} \cup \{0\}$ and $z \in K$, we have the estimate

$$\left|\frac{z}{z_0}\right|^n \leq \left(\frac{R}{|z_0|}\right)^n;$$

where $|z_0| > R$ implies

$$\sum_{n=0}^{\infty} \left(\frac{R}{|z_0|}\right)^n < \infty.$$

By the Weierstrass M-test, the series (which is a limit of polynomials) converges uniformly in K. This shows that $z_0 \in U$.

We next prove that U is closed relative to $\mathbb{C} \setminus K$. Let $\{a_n\}_n \subset U$ convergent to some $a \in \mathbb{C} \setminus K$. For n large enough, we have that $\operatorname{dist}(a_n, K) \geq \frac{1}{2} \operatorname{dist}(a, K) > 0$, thus

$$\sup_{z \in K} |f_a(z) - f_{a_n}(z)| = \sup_{z \in K} \left|\frac{1}{z-a} - \frac{1}{z-a_n}\right| \leq \sup_{z \in K} \frac{|a-a_n|}{|z-a||z-a_n|} \leq \frac{2|a-a_n|}{\operatorname{dist}(a,K)^2}.$$

The last term goes to 0 as n goes to infinity, which shows that f_a is approximated uniformly on K by the functions f_{a_n}. Since each f_{a_n} can be uniformly approximated on K by polynomials, we conclude that $a \in U$.

Since $\mathbb{C} \setminus K$ is open and connected, we will have shown our lemma as soon as we prove that U is an open set. To see this, let $a \in U$, and notice that then $a \notin K$, whence $\operatorname{dist}(a, K) > 0$. We will prove that $D(a, \operatorname{dist}(a, K))$ is contained in U. For every $b \in D(a, \operatorname{dist}(a, K))$ with $a \neq b$, one has that

$$f_b(z) = \frac{1}{z-b} = \frac{1}{z-a} \cdot \frac{1}{1 - \frac{b-a}{z-a}} = \frac{1}{z-a} \sum_{n=0}^{\infty} \left(\frac{b-a}{z-a}\right)^n, \quad z \in K. \tag{6.7.5}$$

The series converges uniformly on $z \in K$ by virtue of the Weierstrass M-test, as $|b-a| < \operatorname{dist}(a, K)$, and for all $z \in K$ we have the bound

$$\left|\frac{b-a}{z-a}\right|^n \leq \left(\frac{|b-a|}{\operatorname{dist}(a,K)}\right)^n, \quad \text{with} \quad \sum_{n=0}^{\infty} \left(\frac{|b-a|}{\operatorname{dist}(a,K)}\right)^n < \infty.$$

Therefore, from (6.7.5) we deduce that the f_b is the uniform limit of functions of the form

$$g_N(z) = \sum_{n=0}^{N} (b-a)^n (f_a(z))^{n+1}, \quad z \in K,\ N \in \mathbb{N}.$$

Since $f_a \in \mathcal{A}(K)$, by the remark right after the definition (6.7.4), each $g_N \in \mathcal{A}(K)$, and therefore $f_b \in \mathcal{A}(K)$ and $b \in U$. We conclude that U is open, which completes the proof of the lemma. □

It is now easy to prove the first version of Runge's Approximation Theorem.

Theorem 6.24 (Runge's Theorem, v.1)**.** *Let $\Omega \subset \mathbb{C}$ be open, $K \subset \Omega$ compact and $f \in \mathcal{H}(\Omega)$. Given $\varepsilon > 0$ there exists a rational function R, with poles in $\mathbb{C} \setminus K$, so that*

$$|f(z) - R(z)| \leq \varepsilon, \quad \textit{for all} \quad z \in K.$$

If, in addition, $\mathbb{C} \setminus K$ is connected, each R can be taken to be a polynomial.

Proof. The first part was already shown in Corollary 6.22. Assume that $\mathbb{C} \setminus K$ is connected. Given $f \in \mathcal{H}(\Omega)$ and $\varepsilon > 0$, by the first part, there exists a rational function R with poles outside K so that

$$|f(z) - R(z)| \leq \frac{\varepsilon}{2}, \quad z \in K.$$

But the function R can be written as

$$R(z)=\sum_{k=1}^{n}\sum_{j=0}^{m_k}\frac{c_{k,j}}{(z-a_k)^j},\quad a_1,\dots,a_n\in\mathbb{C}\setminus K,\ c_{k,j}\in\mathbb{C},\ 0\le j\le m_k,\ 1\le k\le n,$$

by Corollary 2.20. Thus, by Lemma 6.23 (and the comment right after definition (6.7.4)), $R\in\mathcal{A}(K)$, and so there exists a polynomial $P:\mathbb{C}\to\mathbb{C}$ with

$$|R(z)-P(z)|\le\frac{\varepsilon}{2},\quad z\in K.$$

We conclude that $|f(z)-R(z)|\le\varepsilon$ for all $z\in K$. □

6.7.3 Runge's Theorem: second version

Before stating and proving a more general version of Theorem 6.24, we first need to two purely topologycal lemmas.

Lemma 6.25. *Let $K\subset\mathbb{C}$ be compact. Then $W\subset\mathbb{C}_\infty\setminus K$ is a connected component of $\mathbb{C}_\infty\setminus K$ if and only if $W\setminus\{\infty\}$ is a connected component of $\mathbb{C}\setminus K$.*

Proof. The set $\mathbb{C}_\infty\setminus K$ can be written as a disjoint union of its connected components $\{W_j\}_{j\in J}$:

$$\mathbb{C}_\infty\setminus K=W_{j_0}\uplus\biguplus_{j\neq j_0}W_j,$$

where W_{j_0} is the connected component containing ∞. Therefore W_j is a connected open subset of $\mathbb{C}$ for all $j\in J$ with $j\neq j_0$. Setting $V_{j_0}:=W_{j_0}\setminus\{\infty\}$ and $V_j:=W_j$ for all $j\neq j_0$, we can write

$$\mathbb{C}\setminus K=\biguplus_{j\in J}V_j;$$

where each V_j is open and connected in $\mathbb{C}\setminus K$. Observe that if a set $A\subset\mathbb{C}$ is connected in the topology of $\mathbb{C}_\infty$, then it is connected in the usual topology of $\mathbb{C}$, as there are fewer open sets (thus fewer possible separations) in $\mathbb{C}$ than in $\mathbb{C}_\infty$. Let us show that the above implies that the $\{V_j\}_j$ are precisely the connected components of $\mathbb{C}\setminus K$. Indeed, let V a connected component of $\mathbb{C}\setminus K$. Obviously V must contain some V_k, and if we suppose that $V_k\subsetneq V$, we have that V must intersect another V_j. Thus we have the inclusion

$$V\subset V_k\uplus\biguplus_{j\neq k}(V_j\cap V);$$

where V_k and $\biguplus_{j\neq k}V_j\cap V$ are disjoint open sets, having non-empty intersection with V. This would imply that V is not connected, a contradiction. Thus $\{V_j\}_j$ are precisely the connected components of $\mathbb{C}\setminus K$, from which the lemma now follows immediately. □

Lemma 6.26. *Let $\Omega,U\subset\mathbb{C}$ two open sets with $U\subset\Omega$ and $\partial U\cap\Omega=\emptyset$. If W is a connected component of Ω intersecting U, then $W\subset U$.*

Proof. Let $z\in U\cap W$, and let $A\subset U$ be a connected component of U with $z\in A$. Since W and A have a common point, both A and W are connected subsets of Ω, and W is a connected component of Ω, we must have $A\subset W$. Since $\mathbb{C}$ is locally connected (actually connected), and A is a connected component of U, with U open, we have that $\partial A\subset\partial W$. But by the hypothesis $\partial U\cap\Omega=\emptyset$ and the fact that $W\subset\Omega$, this implies that $\partial A\cap W=\emptyset$. This allows us to write

$$W\setminus A=W\cap(\mathbb{C}\setminus A)=W\cap\left(\partial A\cup\mathbb{C}\setminus\overline{A}\right)=W\cap\left(\mathbb{C}\setminus\overline{A}\right).$$

Therefore $W\setminus A$ is both open and relatively closed in W, thus $W\setminus A=\emptyset$, whence $W=A\subset U$ as desired. □

Consider now the following family of functions.

Definition 6.27. *Let $K \subset \mathbb{C}$ be compact, and $E \subset \mathbb{C}_\infty \setminus K$. We define*

$$\mathcal{R}_E(K) := \{R\restriction_K : R \text{ rational with poles in } E\}, \quad \mathcal{A}_E(K) = \overline{\mathcal{R}_E(K)}^{(C(K), \|\cdot\|_\infty)}.$$

In other words, $\mathcal{R}_E(K)$ is the restriction to K of all those rational functions whose poles are exclusively in E, and $\mathcal{A}_E(K)$ is the family of continuous functions in K that can be uniformly approximated on K by functions of the family $\mathcal{R}_E(K)$.

As in the previous section, it is easy to see that if K and E are as above, and $f, g \in \mathcal{A}_E(K)$, and $\lambda \in \mathbb{C}$, then

$$\lambda f + g \in \mathcal{A}_E(K), \quad f \cdot g \in \mathcal{A}_E(K).$$

The following lemma generalizes Lemma 6.23, since rational functions with poles only at ∞ are precisely the polynomials.

Lemma 6.28. *Let $K \subset \mathbb{C}$ be compact, and $E \subset \mathbb{C}_\infty \setminus K$ a set intersecting all connected components of $\mathbb{C}_\infty \setminus K$. Denote, for each $a \in \mathbb{C} \setminus K$, the function*

$$f_a(z) = \frac{1}{z-a}, \quad z \in \mathbb{C} \setminus \{0\}.$$

Then $f_a \in \mathcal{A}_E(K)$ for all $a \in \mathbb{C} \setminus K$.

Proof. We consider first the case where $\infty \notin E$, so that $E \subset \mathbb{C} \setminus K$. Denote $\Omega = \mathbb{C} \setminus K$, and

$$U := \{a \in \Omega \,:\, f_a \in \mathcal{A}_E(K)\}.$$

It is obvious that $E \subset U \subset \Omega$. Let us show that

$$a \in U \implies D(a, \operatorname{dist}(z, K)) \subset U, \tag{6.7.6}$$

which in particular implies that U is an open set. Given $a \in U$, note that $a \notin K$, whence $\operatorname{dist}(a, K) > 0$. For every $b \in D(a, \operatorname{dist}(a, K))$ with $a \neq b$, one has that

$$f_b(z) = \frac{1}{z-b} = \frac{1}{z-a} \cdot \frac{1}{1 - \frac{b-a}{z-a}} = \frac{1}{z-a} \sum_{n=0}^{\infty} \left(\frac{b-a}{z-a}\right)^n, \quad z \in K. \tag{6.7.7}$$

The series converges uniformly on $z \in K$ by virtue of the Weierstrass M-test, as $|b-a| < \operatorname{dist}(a, K)$, and for all $z \in K$ we have the bound

$$\left|\frac{b-a}{z-a}\right|^n \leq \left(\frac{|b-a|}{\operatorname{dist}(a, K)}\right)^n, \quad \text{with} \quad \sum_{n=0}^{\infty} \left(\frac{|b-a|}{\operatorname{dist}(a, K)}\right)^n < \infty.$$

Therefore, from (6.7.7) we deduce that the f_b is the uniform limit of functions of the form

$$g_N(z) = \sum_{n=0}^{N} (b-a)^n (f_a(z))^{n+1}, \quad z \in K, \; N \in \mathbb{N}.$$

Since $f_a \in \mathcal{A}_E(K)$, by the remark right after the definition (6.7.4), each $g_N \in \mathcal{A}_E(K)$, and therefore $f_b \in \mathcal{A}_E(K)$ and $b \in U$. This confirms the validity of (6.7.6) and that U is open. Also, observe that if $a \in \partial U$, then there is a sequence $\{a_n\}_n \subset U$ convergent to a, and by (6.7.6) we must have

$$|a - a_n| \geq \operatorname{dist}(a_n, K), \quad n \in \mathbb{N}.$$

Taking limits as $n \to \infty$ in both sides we get that $\operatorname{dist}(a, K) = 0$, and therefore $a \in K$. We have shown that $\partial U \cap \Omega = \emptyset$. Since E intersects all the connected components of $\mathbb{C}_\infty \setminus K$, by Lemma

6.25, E (and so U) intersects all the connected components of $\Omega = \mathbb{C} \setminus K$. By Lemma 6.26, we get that all connected components of Ω are in U, thus showing that $\Omega = U$, as desired.

Now, assume that $\infty \in E$, and let $\{W_j\}_{j\in J}$ be the connected components of $\mathbb{C}_\infty \setminus K$, with W_{j_0} being the one containing ∞. Thus $W_{i_0} = \mathbb{C}_\infty \setminus L$ for some compact set $L \subset \mathbb{C}$ containing K. The rest of the connected components W_j, $j \neq j_0$ are disjoint with W_{j_0}, and so $W_j \subset L$ for all $j \neq j_0$. We can find

$$a_0 \in \mathbb{C} \setminus K \quad \text{with} \quad |a_0| \geq 2\max\{1 + |z| \,:\, z \in L\} \geq 2\max\{|z| \,:\, z \in K\}. \tag{6.7.8}$$

Defining $E_0 := (E\setminus\{\infty\})\cup\{a_0\}$ and bearing in mind that E intersects all the connected components of $\mathbb{C}_\infty \setminus K$, (6.7.8) tells us that E_0 intersects all those components as well, and by the first part of the proof of the current lemma, we obtain $f_a \in \mathcal{A}_{E_0}(K)$ for all $a \in \mathbb{C} \setminus K$. Since each rational function with poles in E_0 can be written as a linear combination of one with poles in E and one of the form $\sum_{n=0}^m f_{a_0}^n$ (see e.g. Corollary 2.20), it only remains to show that $f_{a_0} \in \mathcal{A}_E(K)$. To see this, write

$$f_{a_0}(z) = \frac{1}{z - a_0} = -\frac{1}{a_0}\sum_{n=0}^{\infty}\left(\frac{z}{a_0}\right)^n = \sum_{j=0}^{\infty} -\frac{z^n}{a_0^{n+1}}, \quad z \in K.$$

By the choice of a_0 in (6.7.8), the Weierstrass M-test guarantees the uniform convergence of the series in K, thus showing that f_{a_0} is the uniformm limit (in K) of polynomials, that is, rational functions with poles at infinity, that is functions in $\mathcal{R}_E(K)$. We conclude that $f_{a_0} \in \mathcal{A}_E(K)$.

□

We are now ready to show the announced generalization of Theorem 6.24.

Theorem 6.29 (Runge's Theorem, v.2). *Let $\Omega \subset \mathbb{C}$ be open, $K \subset \Omega$ compact and $E \subset \mathbb{C}_\infty \setminus K$ a set intersecting each connected component of $\mathbb{C}_\infty \setminus K$. Then, for every $f \in \mathcal{H}(\Omega)$ and $\varepsilon > 0$, there exists a rational function R with poles in E, so that*

$$|f(z) - R(z)| \leq \varepsilon, \quad \textit{for all} \quad z \in K.$$

Proof. By Corollary 6.22, there exists R rational with poles in $\mathbb{C} \setminus K$ so that

$$|f(z) - R(z)| \leq \frac{\varepsilon}{2}, \quad z \in K.$$

We write the function R as

$$R(z) = \sum_{k=1}^{n}\sum_{j=0}^{m_k} c_{k,j}\left(f_{a_k}(z)\right)^j, \quad a_1, \ldots, a_n \in \mathbb{C} \setminus K,\ c_{k,j} \in \mathbb{C},\ 0 \leq j \leq m_k,\ 1 \leq k \leq n,$$

by Corollary 2.20. Since E intersects all the connected components of $\mathbb{C}_\infty \setminus K$, by Lemma 6.28, each f_{a_k} above belongs to $\mathcal{A}_E(K)$, and so, $R \in \mathcal{A}_E(K)$. Therefore, we can find a rational function S with poles in E so that

$$|R(z) - S(z)| \leq \frac{\varepsilon}{2}, \quad z \in K.$$

We conclude that $|f(z) - S(z)| \leq \varepsilon$ for all $z \in K$. □

6.7.4 Approximation in the Compact-Open Topology

The next results on approximations differ a bit from those in the previous section, since now we look for a sequence of polynomials $\{P_n\}_n$ or rational functions $\{R_n\}_n$ that approximate a given $f \in \mathcal{H}(\Omega)$ uniformly on *each* compact subset of Ω. Note that then the same sequence $\{R_n\}_n$ works simultaneously for all compact subsets. Naturally, this follows by Runge's Theorem 6.29, but we first need to prove an additional topological property on the nested sequences of compact sets from Proposition 3.3.

Lemma 6.30. *Given an open set $\Omega \subset \mathbb{C}$, there exists a nested family $\{K_n\}_{n\in\mathbb{N}}$ of compact sets in Ω with the property that, for every $n \in \mathbb{N}$, every connected component of $\mathbb{C}_\infty \setminus K_n$ contains a connected component of $\mathbb{C}_\infty \setminus \Omega$.*

Proof. In the case $\Omega = \mathbb{C}$, it suffices to take $K_n = \overline{D}(0,n)$ for every $n \in \mathbb{N}$, where the only connected component of $\mathbb{C}_\infty \setminus K_n$ is precisely $\mathbb{C}_\infty \setminus K_n$, which obviously contains $\{\infty\} = \mathbb{C}_\infty \setminus \mathbb{C}$.

Consider now the case $\Omega \subsetneq \mathbb{C}$, and define

$$K_n := \overline{D}(0,n) \cap \{z \in \Omega \,:\, \operatorname{dist}(z, \mathbb{C} \setminus \Omega) \geq 1/n\}, \quad n \in \mathbb{N}.$$

As we saw in the proof of Proposition 3.3, $\{K_n\}_{n\in\mathbb{N}}$ is a nested family of compact sets in Ω. Now, fix $n \in \mathbb{N}$, and

$$\mathbb{C}_\infty \setminus K_n = W_{j_0} \uplus \biguplus_{j \neq j_0} W_j,$$

the decomposition of $\mathbb{C}_\infty \setminus K$ into its connected components, where W_{j_0} is the one containing ∞. If A is the connected component of $\mathbb{C}_\infty \setminus \Omega$ that contains ∞, and $K_n \subset \Omega$, the set A is a connected subset of $\mathbb{C}_\infty \setminus K_n$ intersecting W_{j_0}, thus $A \subset W_{j_0}$. Now we need to verify that each W_j with $j \neq j_0$ contains a connected component of $\mathbb{C}_\infty \setminus \Omega$. Notice that, since $K_n \subset \overline{D}(0,n)$, and $\mathbb{C}_\infty \setminus \overline{D}(0,n)$ is connected in $\mathbb{C}_\infty$ and contains ∞, we have that $\mathbb{C}_\infty \setminus \overline{D}(0,n) \subset W_{j_0}$, and so $W_j \subset D(0,n)$. If we fix a point $z \in W_j$, the inequality $\operatorname{dist}(z_0, \mathbb{C} \setminus \Omega) < 1/n$ gives the existence of $w \in \mathbb{C} \setminus \Omega$ for which $|z - w| < 1/n$. By the definition of K_n, we clearly have the inclusions

$$z \in D(w, 1/n) \subset \mathbb{C}_\infty \setminus K_n.$$

Because $D(w,1/n)$ is connected in $\mathbb{C}_\infty$ (as disks are bounded and connected in $\mathbb{C}$) and contains z, this implies that $D(w,1/n) \subset W_j$, and, in particular $w \in W_j$. Now, if A is the connected component of $\mathbb{C}_\infty \setminus \Omega$ containing w, then A is also a connected subset of $\mathbb{C}_\infty \setminus K_n$ containing w, and therefore $A \subset W_j$, as desired. □

Applying Runge's Theorem 6.29 to each K_n of the nested sequence, we get the following.

Corollary 6.31. *Let $\Omega \subset \mathbb{C}$ be open and $E \subset \mathbb{C}_\infty \setminus \Omega$ a set intersecting each connected component of $\mathbb{C}_\infty \setminus \Omega$. Then, for every $f \in \mathcal{H}(\Omega)$ there exists a sequence of rational functions $\{R_n\}_n$ with poles contained in E, and so that $\{R_n\}_n$ converges to f uniformly on compact subsets of Ω.*

Proof. Let $\{K_n\}_n$ be a nested sequence of compact sets in Ω as in Lemma 6.30. Given $n \in \mathbb{N}$, the set E intersects all the connected components of $\mathbb{C}_\infty \setminus K_n$, and by Runge's Theorem 6.29 there exists R_n rational with poles in E so that

$$|f(z) - R_n(z)| \leq 1/n \quad \text{for all} \quad z \in K_n.$$

We obtain a sequence $\{R_n\}_n$ of rational functions with poles in E satisfying the above estimates. For every $K \subset \Omega$ compact and every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ with $K \subset K_n$ and $1/n < \varepsilon$ for all $n \geq n_0$. The estimates above shows that

$$\sup\{|f(z) - R_n(z)| \,:\, z \in K\} \leq \sup\{|f(z) - R_n(z)| \,:\, z \in K_n\} \leq 1/n < \varepsilon, \quad \text{for all} \quad n \geq n_0.$$

□

6.7.5 Polynomic Approximation in Simply Connected Domains

It is immediate from Corollary 6.31 that if $\mathbb{C}_\infty \setminus \Omega$ is already connected, then the approximating functions can be taken to be polynomials, as one can choose $E = \{\infty\}$, so that rational functions with poles in E are precisely polynomials. The connectedness of the $\mathbb{C}_\infty \setminus \Omega$ (for a connected open set Ω) is equivalent to the simple-connectedness of Ω, but this purely topological equivalence is not very non-trivial to prove. Nevertheless, we can use a refinment of the Integral Represenatation Theorem 1.22 to show this.

We begin with the following combinatorial lemma.

Lemma 6.32. *Let $a_1,\dots,a_m,b_1,\dots,b_m\in\mathbb{C}$ be numbers satisfying the identity*

$$\sum_{j=1}^{m} P(a_j)=\sum_{j=1}^{m} P(b_j) \quad \textit{for all polynomial} \quad P. \tag{6.7.9}$$

Then there exists a permutation (a bijection) $\phi:\{1,\dots,m\}\to\{1,\dots,m\}$ so that $a_{\phi(j)}=b_j$ for all $j=1,\dots,m$.

Proof. We use induction on m. The case $m=1$ is trivial, because in such case we could take $P(z)=z$ and the hypothesis implies that $a=b$. Now, assume our conclusion is true for $m-1$, and let $a_1,\dots,a_m,b_1,\dots,b_m\in\mathbb{C}$ be satisfying (6.7.9). Our first claim is that $a_m\in\{b_1,\dots,b_m\}$. Indeed, assume that $a_m\neq b_j$ for all $j=1,\dots,m$, and define the sets

$$J=\{j\in\{1,\dots,m\}\,:\,a_j\neq a_m\},\quad I=\{j\in\{1,\dots,m\}\,:\,a_j=a_m\},$$

and the polynomial

$$P(z):=\prod_{j\in J}(z-a_j)\cdot\prod_{j=1}^{m}(z-b_j)$$

(understanding that the first product is constantly equal to 1 when J is empty). Then $P(b_j)=0$ for all $j\in\{1,\dots,m\}$, $P(a_j)=0$ for all $j\in J$, and $P(a_j)=P(a_k)\neq 0$ for all $j\in I$. Applying (6.7.9), we get that

$$0=\sum_{j=1}^{m}P(b_j)=\sum_{j\in J}P(a_j)+\sum_{j\in I}P(a_j)=\operatorname{card}(I)P(a_k),$$

implying that $P(a_k)=0$, a contradiction.

We have shown that there exists $k\in\{1,\dots,m\}$ with $a_m=a_k$. Using (6.7.9), we get that, for all polynomial P,

$$\sum_{j=1}^{m-1}P(a_j)=\sum_{j=1}^{m}P(a_j)-P(b_k)=\sum_{j=1}^{m}P(b_j)-P(b_k)=\sum_{j=1,\,j\neq k}^{m}P(b_j).$$

By the induction hypothesis, there exists a bijection $\varphi:\{1,\dots,m\}\setminus\{k\}\to\{1,\dots,k-1\}$ with $a_{\varphi(j)}=b_j$ for all $j\in\{1,\dots,m\}\setminus\{k\}$. We can therefore define a new bijection ϕ of $\{1,\dots,m$ by setting $\phi=\varphi$ on $\{1,\dots,m\}\setminus\{k\}$ and $\phi(k):=m$. Clearly $a_{\phi(j)}=b_j$ for all $j\in\{1,\dots,m\}$. □

The mentioned variation of Theorem 1.22 consists of replacing the line segments by *closed polygonal lines*, thus obtaining a cycle.

Theorem 6.33. *Let $\Omega\subset\mathbb{C}$ be open, and $K\subset\Omega$ a nonempty compact set. There are closed polygonal lines $\gamma_1,\dots,\gamma_N\subset\Omega\setminus K$, such that for every $f:\Omega\to\mathbb{C}$ holomorphic, we have*

$$f(z)=\frac{1}{2\pi i}\sum_{j=1}^{N}\int_{\gamma_j}\frac{f(w)}{w-z}\,\mathrm{d}w,\quad \textit{for all}\quad z\in K. \tag{6.7.10}$$

Proof. Let $L_1,\dots,L_m$ be the line segments from Theorem 1.22, and denote $L_j:=[a_j,b_j]$, with (as we may assume) $a_j\neq b_j$, for all $j\in\{1,\dots,m\}$. For every polynomial P and a fixed $z_0\in K$, we can apply Theorem 6.33 (formula (1.2.8)) to the holomorphic map $f(z)=P'(z)(z-z_0)$ to obtain that

$$0=f(z_0)=\frac{1}{2\pi i}\sum_{j=1}^{m}\int_{L_j}\frac{f(w)}{w-z_0}\,\mathrm{d}w=\frac{1}{2\pi i}\sum_{j=1}^{m}\int_{L_j}P'(w)\,\mathrm{d}w=\frac{1}{2\pi i}\sum_{j=1}^{m}\left(P(b_j)-P(a_j)\right),$$

and thus

$$\sum_{j=1}^{m} P(a_j) = \sum_{j=1}^{m} P(b_j) \quad \text{for all polynomial} \quad P.$$

By Lemma 6.32, there is a permutation ϕ of $\{1,\ldots,m\}$ with $a_{\phi(j)} = b_j$ for all $j \in \{1,\ldots,m\}$. Note that, since $a_j \neq b_j$, we have that $\phi(j) \neq j$ for all $j \in \{1,\ldots,m\}$. We can write ϕ as a composition of disjoint cycles $\varphi_1,\ldots,\varphi_N$, that is,

$$\phi = \varphi_N \circ \cdots \circ \varphi_1, \quad \varphi_k = \begin{pmatrix} j_k & \phi(j_k) & \cdots & \phi^{\ell_k}(j_k) & j_k \end{pmatrix}, \quad k = 1,\ldots,N.$$

Each cycle φ_k has length at least 2, as ϕ does not have fixed points. Therefore, $\ell_k \geq 1$ for all $k = 1,\ldots,N$. For each line segment

$$L_j = [a_j, b_j] = [a_j, a_{\phi(j)}],$$

there are unique (by the disjointness of the cycles) $k \in \{1,\ldots,N\}$, $j_k \in \{1,\ldots,m\}$ and $0 \leq l \leq \ell_k$ so that

$$[a_j, a_{\phi(j)}] = [a_{\phi^l(j_k)}, a_{\phi^{l+1}(j_k)}],$$

understanding that $\phi^{l+1}(j_k) = j_k$ when $l = \ell_k$. Conversely, each line segment $[a_{\phi^l(j_k)}, a_{\phi^{l+1}(j_k)}]$ formed by two consecutive indices of the cycle φ_k is of course of the form $[a_j, b_j] = L_j$ for some $j \in \{1,\ldots,m\}$. Therefore, we can group the line segments $L_1,\ldots,L_m$ using the cycles $\varphi_1,\ldots,\varphi_N$ as

$$\bigcup_{j=1}^{m} L_j = \bigcup_{j=1}^{m} [a_j, b_j] = \bigcup_{k=1}^{N} \gamma_k; \quad \gamma_k := [a_{j_k}, a_{\phi(j_k)}] \cup \cdots \cup [a_{\phi^{\ell_k}(j_k)}, a_{j_k}], \quad k = 1,\ldots,N.$$

Each γ_k is a closed polygonal line, and for any continuous $h : \bigcup_{j=1}^{m} L_j^* \to \mathbb{C}$, we have that

$$\sum_{j=1}^{m} \int_{L_j} h(w)\,\mathrm{d}w = \sum_{k=1}^{N} \int_{\gamma_k} h(w)\,\mathrm{d}w.$$

Together with formula (1.2.8), this implies (6.7.10). $\square$

We are now ready to prove the polynomic approximation of holomorphic functios in simply-connected domains, showing that actually this property characterizes simple-connectedness.

Theorem 6.34. *Let $\Omega \subset \mathbb{C}$ be open and connected. The following statements are equivalent.*

(i) *Ω is simply connected.*

(ii) *$\mathbb{C}_\infty \setminus \Omega$ is connected.*

(iii) *For every $f \in \mathcal{H}(\Omega)$ there is a sequence $\{P_n\}_n$ of polynomials converging to f uniformly on compact subsets of Ω.*

Proof. We begin with the implication $(i) \implies (ii)$. Assume that $\mathbb{C}_\infty \setminus \Omega$ is not connected, and let us find a cycle Γ in Ω with $\Gamma \not\approx 0$ in Ω. This will imply that Ω is not simply-connected by Corollary 4.44. Since $\mathbb{C}_\infty \setminus \Omega$ is not connected and closed, there are disjoint and nonempty closed subsets $F_1, F_2 \subset \mathbb{C}_\infty$ so that $\mathbb{C}_\infty \setminus \Omega = F_1 \cup F_2$. One of them, say F_1, contains ∞, and so F_2, being closed in $\mathbb{C}_\infty$ without ∞, must be a bounded and closed subset of $\mathbb{C}$, and therefore F_2 is compact in $\mathbb{C}$. We define $U := \Omega \cup F_2$, and note that

$$U = \Omega \cup ((\mathbb{C}_\infty \setminus \Omega) \cap (\mathbb{C}_\infty \setminus F_1)) = \Omega \cup (\mathbb{C}_\infty \setminus F_1) = \mathbb{C}_\infty \setminus F_1,$$

which is an open set of $\mathbb{C}_\infty$. By Theorem 6.33 there is a cycle Γ (union of finitely many closed poligonal lines) in U with $\Gamma^* \subset U \setminus F_2 \subset \Omega$ and so that (taking the function $f(w) = 1$ for all $w \in U$)

$$1 = \frac{1}{2\pi i} \int_\Gamma \frac{\mathrm{d}w}{w - z} = W(\Gamma, z), \quad \text{for all} \quad z \in F_2.$$

Thus Γ is a cycle in Ω with $W(\Gamma, z) \neq 0$ for all $z \in F_2 \subset \mathbb{C} \setminus \Omega$. Since F_2 is nonempty, we get that $\Gamma \not\approx 0$ in Ω, giving the desired contradiction.

Let us now show $(ii) \implies (iii)$. Defining $E := \{\infty\}$, obviously E intersects the connected component of $\mathbb{C}_\infty \setminus \Omega$ (itself), and we can apply Corollary 6.31 to $f \in \mathcal{H}(\Omega)$ in order to find rational functions $\{R_n\}_n$ with poles in E that approximate f uniformly on compact sets of Ω. But rational functions with only one pole at ∞ are precisely the polynomials.

Finally, let us prove $(iii) \implies (i)$. By Corollary 4.44 and Theorem 1.19, it suffices to check that if $f \in \mathcal{H}(\Omega)$ and Γ is a cycle in Ω, then $\int_\Gamma f(z)\,\mathrm{d}z = 0$. But (ii) provides us with polynomials $\{P_n\}_n$ converging to f uniformly on compact sets of Ω, whence

$$\int_\Gamma f(z)\,\mathrm{d}z = \lim_{n\to\infty} \int_\Gamma P_n(z)\,\mathrm{d}z = 0,$$

as polynomials always have primitives, and the Fundamental Theorem of Calculus for the Path-Integral can be applied. □

6.8 Mittag-Leffler's Theorem

We finish this chapter with the following famous theorem of Mittag-Leffler, which tells us that one can construct meromorphic functions choosing their poles and the correponding principal parts of the Laurent Series.

Theorem 6.35 (Mittag-Leffler's Theorem)**.** *Let $\Omega \subset \mathbb{C}$ be open, and $S \subset \Omega$ a set with $S' \cap \Omega = \emptyset$. Given a family $\{P_w\}_{w\in S}$ of rational functions of the form*

$$P_w(z) = \frac{A_{1,w}}{z-w} + \frac{A_{2,w}}{(z-w)^2} + \cdots + \frac{A_{m_w,w}}{(z-w)^{m_w}}, \quad A_{1,w}, \ldots, A_{m_w,w} \in \mathbb{C},\ m_w \in \mathbb{N}, \quad \textit{for all} \quad w \in S,$$

there exists a function $f \in \mathcal{H}(\Omega \setminus S)$ whose Laurent Series centered at w has principal part equal to P_w for all $w \in S$. In particular, $f \in \mathcal{M}(\Omega)$ with a pole of order $m_w \in \mathbb{N}$ at w, for all $w \in S$.

Proof. First notice that S must me countable, e.g., by Proposition 2.12. Let $\{K_n\}_n$ a nested sequence of compact subsets in Ω as in Lemma 6.30. We define a partition of S into the finite sets $\{S_n\}_{n\in}$ given by

$$S_1 := S \cap K_1, \quad S_n := S \cap (K_n \setminus K_{n-1}) \quad \text{for all} \quad n \geq 2.$$

Each S_n is a bounded subset of $K_n \subset \Omega$, and since $S' \cap \Omega = \emptyset$, we have that indeed S_n is finite. Also, since $K_{n-1} \subset \operatorname{int}(K_n)$ and S_k is finite, it is clear that there exists, for every $n \geq 2$, an open set U_n with

$$K_{n-1} \subset U_n \subset \operatorname{int}(K_n) \quad \text{and} \quad S_n \cap U_n = \emptyset, \tag{6.8.1}$$

Defining

$$f_n(z) := \sum_{w\in S_n} P_w(z), \quad n \in \mathbb{N}, \tag{6.8.2}$$

we get from (6.8.1) that f_n is a rational function with poles in S_n, and that $f_n \in \mathcal{H}(U_n)$ for all $n \geq 2$. Let $E \subset \mathbb{C}_\infty \setminus \Omega$ be a set intersecting all the connected components of $\mathbb{C}_\infty \setminus \Omega$. By Corollary 6.31 applied to each $f_n \in \mathcal{H}(U_n)$ and the compact subset K_{n-1}, there are rational functions $\{R_n\}_{n\geq 2}$ with poles contained in E and so that

$$|f_n(z) - R_n(z)| \leq \frac{1}{2^n}, \quad \text{for all} \quad z \in K_{n-1},\ n \geq 2. \tag{6.8.3}$$

Since the poles of R_n are outside of Ω, we have that $R_n \in \mathcal{H}(\Omega)$ for all $n \geq 2$. We define

$$f(z) := f_1(z) + \sum_{n=2}^{\infty} (f_n(z) - R_n(z)), \quad z \in \Omega \setminus S. \tag{6.8.4}$$

This function is the pointwise limit of holomorphic functions in $\Omega \setminus S$. Let us check that the convergence of the series is uniform on compact subsets of $\Omega \setminus S$. Let $K \subset \Omega \setminus S$, and let $N \in \mathbb{N}$ with $K \subset K_N$. Then $K \subset K_n$ for all $n \geq N$, and so (6.8.3) leads us to

$$\sup\{|f_n(z) - R_n(z)| \,:\, z \in K\} \leq \frac{1}{2^n}, \quad n \geq N+1, \quad \text{where} \quad \sum_{n=N+1}^{\infty} \frac{1}{2^n} < \infty.$$

By Weierstrass M-test, the series $\sum_{n=2}^{\infty}(f_n - R_n)$ converges uniformly in K. Therefore, we see from (6.8.4) that f is the locally uniform limit of holomorphic functions in $\Omega \setminus S$, thus $f \in \mathcal{H}(\Omega \setminus S)$ by Weierstrass Theorem 3.1. Since S has no accumulation points in Ω, f has isolated singularities at each $w \in S$.

Finally, given $w_0 \in S$, let us show that the principal part of the Laurent Series of f at w_0 coincides with P_{w_0} for each $w_0 \in S$. Indeed, since $S' \cap \Omega = \emptyset$ and $w_0 \in S \subset \Omega$, there exists $\varepsilon > 0$ so that $w \notin D(w_0, \varepsilon) \setminus \{w_0\}$ for all $w \in S$. There exists a unique $N \in N$ with $w_0 \in S_N$, and note that then the functions $\{f_n\}_{n \neq N}$, $\{R_n\}_n$ are holomorphic in $D(w_0, \varepsilon)$. Thus, from (6.8.4), there exists $g \in \mathcal{H}(D(w_0, \varepsilon))$ so that

$$f(z) = g(z) + f_N(z), \quad z \in D(w_0, \varepsilon) \setminus \{z_0\}.$$

But, since $w \notin D(w_0, \varepsilon)$ for all $w \in S \setminus \{w_0\}$, and w is the only singularity of P_w, the formula (6.8.2) shows that there is $h \in \mathcal{H}(D(w_0, \varepsilon))$ with

$$f(z) = g(z) + f_N(z) = g(z) + h(z) + P_{w_0}(z), \quad z \in D(w_0, \varepsilon) \setminus \{z_0\}.$$

Because $g + h$ is holomorphic in $\mathcal{H}(D(w_0, \varepsilon))$, and the Laurent Series expansion of f is unique around w_0, the principal part of this series must be equal to P_{w_0}. $\square$

6.9 Exercises

Exercise 6.1. *Let $\Omega \subset \mathbb{C}$ be open and symmetric about the origin, and $f : \Omega^+ \to \mathbb{C}$ be holomorphic. Prove that the function $g : \Omega^- \to \mathbb{C}$ given by*

$$g(z) = \overline{f(\overline{z})}, \quad \textit{for all} \quad z \in \Omega^-,$$

is holomorphic in Ω^-.

Exercise 6.2. *Let $\Omega \subset \mathbb{C}$ be open, and $f : \Omega \to \mathbb{C}$ continuous in Ω and holomorphic in $\Omega \setminus \mathbb{R}$. Prove that f is holomorphic in Ω.*

Exercise 6.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be real analytic. Prove that there exists $\Omega \subset \mathbb{C}$ open with $\mathbb{R} \subset \Omega$ and $F \in \mathcal{H}(\Omega)$ with $F = f$ on $\mathbb{R}$.*

Exercise 6.4. *Give an example of an open, bounded and simply connected set Ω whose boundary $\partial\Omega$ is not a Jordan curve.*

Exercise 6.5. *Let $\Omega \subset \mathbb{C}$ be open and bounded, and $f : \overline{\Omega} \to \mathbb{C}$ continuous in $\overline{\Omega}$ and holomorphic in Ω. Prove the inclusion $\partial f(\Omega) \subset f(\partial\Omega)$.*

Exercise 6.6. *Let $D \subset \mathbb{C}$ be an open disk, and $f \in \mathcal{H}(D)$. Prove that the set $S \subset \partial D$ of singular points of f is a closed subset of $\mathbb{C}$.*

Exercise 6.7. *Consider the function* $f:\mathbb{D}\to\mathbb{C}$ *given by*

$$f(z)=\sum_{n=0}^{\infty} z^{n!},\quad z\in\mathbb{D}.$$

Prove that the natural boundary of f *is* $\partial\mathbb{D}$*, that is, each* $\xi\in\partial\mathbb{D}$ *is a singular point for* f*.*

Exercise 6.8. *Let* $\Omega\subset\mathbb{C}$ *be open and simply-connected, and let* $\Gamma_0,\Gamma_1:[0,1]\to\Omega$ *two continuous curves in* Ω *so that*

$$z_0:=\Gamma_0(0)=\Gamma_1(0),\quad z_1:=\Gamma_0(1)=\Gamma_1(1).$$

Prove that there exists a one-parameter family $\{\gamma_t\}_{t\in[0,1]}$ *of continuous curves in* Ω *with initial point* z_0*, end point* z_1*, and* $\gamma_0=\Gamma_0$*,* $\gamma_1=\Gamma_1$*.*

Exercise 6.9. *Show that there is no sequence* $\{P_n\}_n$ *of polynomials with* $P_n(0)=1$ *for all* $n\in\mathbb{N}$ *and converging uniformly to* $1/2$ *on* $\partial\mathbb{D}$*.*

Exercise 6.10. *Let* $K\subset\mathbb{C}$ *be compact with* $\mathbb{C}\setminus K$ *connected,* $\Omega\subset\mathbb{C}$ *open containing* K*, and* $z_0\in\mathbb{C}\setminus\Omega$*. Show that for every* $N\in\mathbb{N}$ *and* $\varepsilon>0$*, there exists a* P *polynomial with a zero of order at least* N *at* z_0*, and so that*

$$|f(z)-P(z)|\le\varepsilon,\quad \textit{for all}\quad z\in K.$$

Exercise 6.11. *Denote*

$$\mathbb{H}^+:=\{z\in\mathbb{C}\,:\,\mathrm{Im}(z)>0\},\quad \mathbb{H}^-:=\{z\in\mathbb{C}\,:\,\mathrm{Im}(z)<0\}.$$

Prove that there exists a sequence $\{P_n\}_n$ *of polynomials so that:*

- $\{P_n\}_n$ *converges to* 1 *uniformly on each compact subset of* $\mathbb{H}^+$*, and*
- $\{P_n\}_n$ *converges to* -1 *uniformly on each compact subset of* $\mathbb{H}^-$*.*

Exercise 6.12. *Let* $\Omega=\{z\in\mathbb{C}\,:\,|z|<2$ *and* $|z+1|>1\}$*, and consider the function* $f(z)=\frac{1}{z+2}$*. Prove that:*

(i) There exists a sequence $\{P_n\}_n$ *of polynomials converging to* f *uniformly on each compact subset of* Ω*.*

(ii) There is no sequence of polynomials converging to f *uniformly on* Ω*.*

Then, for the function $g(z)=\frac{1}{z+3}$*, find an explicit sequence of polynomials converging to* g *uniformly on* Ω*.*

Exercise 6.13. *Prove that there exists a sequence* $\{P_n\}_n$ *of polynomials with* $\lim_{n\to\infty}P_n(z)=1$ *if* $z\in\mathbb{R}$ *and* $\lim_{n\to\infty}P_n(z)=0$ *if* $z\in\mathbb{C}\setminus\mathbb{R}$*.*

Exercise 6.14. *Construct a meromorphic function* f *in* $\mathbb{C}$ *with poles* $\mathcal{P}(f)=\mathbb{N}$*, all of them of order* 1*, and so that* $\mathrm{Res}(f,n)=n$ *for all* $n\in\mathbb{N}$*.*

Exercise 6.15. *Construct a meromorphic function* f *in* $\mathbb{C}$ *with poles* $\mathcal{P}(f)=\mathbb{N}$*, and so that the principal part of the Laurent Series of* f *around* n *is equal to*

$$\frac{1}{(z-n)^2},$$

for all $n\in\mathbb{N}$*.*

Chapter 7

Products, Factorization and Interpolation

Given a sequence of numbers $\{z_n\}_n \subset \mathbb{C}$ with no accumulation points, and natural numbers $\{m_n\}_n \subset \mathbb{N}$, *does there exist a holomorphic function f in C whose zeros is precisely the set $\{z_n\}_n$ and so that $m(f, z_j) = z_j$ for all $j \in \mathbb{N}$?*

If the collection is finite $z_1, \ldots, z_n$ the polynomial

$$f(z) = (z - z_1)^{m_1} \cdots (z - z_n)^{m_n}, \quad z \in \mathbb{C},$$

is an example of such function.

It we consider the corresponding *infinite product* as the limit of these (finite) partial products, we need to check whether the limit converges to a holomorphic function. In this chapter, we begin by studying infinite products of numbers and functions and give criteria for their convergence or uniform convergence. Theorem 7.8 is the main result of the chapter in that direction.

The answer to the initial question is given by an infinite product of suitable *Weierstrass Factors*; see Theorem 7.11. A converse of this result is the *Weierstrass Factorization* Theorem 7.13, which permits to write *every* entire function as the product of an exponential and an infinite product of Weierstrass factors, based on the zeros of the function. In the Exercise section 7.5, we will use this theorem to obtain factorizations of trigonometric functions in terms of their zeros.

Then we prove *Jensen's Formula* (Theorem 7.15) on the distribution of zeros of analytic functions in terms of certain averaged-logaritmic integral in a circle.

Finally, we define the *Blaschke Product* and show that they provide bounded analytic functions in the unit disk prescribing zeros and multiplicities, provided the desired zeros satisfy the *Blaschke Condition*; see Theorem 7.16. Using Jensen's Formula, we show in Theorem 7.17 that the *Blaschke Condition* is actually necessary.

7.1 Infinite Products

Let $\{z_n\}_{n\geq 1}$ be a sequence of complex numbers. The **infinite product of** $\{z_n\}_n$ is the expression

$$\prod_{n=1}^{\infty} z_n.$$

The sequence of **partial products** is $\{\prod_{n=1}^{N} z_n\}_{N\in\mathbb{N}}$.

7.1.1 Convergence: Definition and Criteria

The definition of convergence of a product depends on the presence of zeros in the sequence.

Definition 7.1 (Convergence of Infinite Products)**.** *Let $\{z_n\}_{n\geq 1}$ be a sequence of complex numbers. To define the convergence or divergence of the infinite product*

$$\prod_{n=1}^{\infty} z_n,$$

we distinguish three cases

(1) *Assume that $z_n \neq 0$ for all $n \in \mathbb{N}$. We say that $\prod_{n=1}^{\infty} z_n$ **converges** if*

$$p := \lim_{N\to\infty} \prod_{n=1}^{N} z_n \ \text{ exists and } \ p \in \mathbb{C}\setminus\{0\}.$$

*In this case, we say that p is the **the product of** $\prod_{n=1}^{\infty} z_n$.*

*Now, if the limit p above is 0 (resp. ∞), then we say that $\prod_{n=1}^{\infty} z_n$ **diverges** to 0 (resp. to ∞).*

*Finally, if the limit does not exists, we of course say that $\prod_{n=1}^{\infty} z_n$ **diverges**.*

(2) *Assume that there is $N \in \mathbb{N}$ so that $z_n \neq 0$ for all $n \geq N+1$ and that $z_n = 0$ for at least some $n \leq N$.*

*We say that $\prod_{n=1}^{\infty} z_n$ **converges to** 0 if the infinite product $\prod_{n=N+1}^{\infty} z_n$ converges in the sense of case (1).*

*Now, if $\prod_{n=N+1}^{\infty} z_n$ diverges, then we say that $\prod_{n=1}^{\infty} z_n$ **diverges**, even if the divergence of $\prod_{n=N+1}^{\infty} z_n$ is to 0 or ∞.*

(3) *Assume that $z_n = 0$ for infinitely many $n \in \mathbb{N}$. Then we say that $\prod_{n=1}^{\infty} z_n$ **diverges**.*

Let us give two simple criteria for convergence which are immediate from Definition 7.1.

Remark 7.2. If $\{z_n\}_n \subset \mathbb{C}$, from Definition 7.1 we note the following.

(i) The infinite product $\prod_{n=1}^{\infty} z_n$ converges if and only if there exists $N \in \mathbb{N}$ so that $z_n \neq 0$ for all $n \geq N+1$ and

$$\lim_{m\to\infty} \prod_{n=N+1}^{m} z_n \in \mathbb{C}\setminus\{0\}.$$

(ii) If $\prod_{n=1}^{\infty} z_n$ converges, then $\lim_{n\to\infty} z_n = 1$.

Indeed, by the above there exists $N \in \mathbb{N}$ so that $z_n \neq 0$ for all $n \geq N+1$, and

$$p := \lim_{m\to\infty} \prod_{n=N+1}^{m} z_n \in \mathbb{C}\setminus\{0\}.$$

This allows to write, for $m \geq N+2$,

$$z_m = \frac{\prod_{n=N+1}^{m} z_n}{\prod_{n=N+1}^{m-1} z_n},$$

whence
$$\lim_{m\to\infty} z_m = \frac{p}{p} = 1.$$

The following inequalities, though elementary, are very useful when studying convergence of products.

Lemma 7.3. *Let $\{z_n\}_{n\geq 1} \subset \mathbb{C}$ be a sequence, and denote*
$$p_n := (1+z_1)\cdots(1+z_n), \quad p_n^* := (1+|z_1|)\cdots(1+|z_n|), \quad n \in \mathbb{N}.$$
Then,
$$1 + \sum_{k=1}^{n} |z_k| \leq p_n^* \leq \exp\left(\sum_{k=1}^{n} |z_k|\right), \quad n \in \mathbb{N}; \tag{7.1.1}$$
$$|p_n - 1| \leq p_n^* - 1, \quad n \in \mathbb{N}; \tag{7.1.2}$$
$$|p_m - p_n| \leq p_m^* - p_n^*, \quad m \geq n,\ m, n \in \mathbb{N}. \tag{7.1.3}$$

Proof. The left inequality of (7.1.1) is obvious. For the right one, it suffices to recall that $e^x \geq 1+x$ for all $x \in \mathbb{R}$, and so $e^{|z_k|} \geq 1 + |z_k|$ for all $k = 1, \ldots, n$.

Onto (7.1.2), we use induction on n. The case $n = 1$ is immediate, and then assume that (7.1.2) holds for certain $n \in \mathbb{N}$. Then,
$$\begin{aligned}|p_{n+1} - 1| &= |p_n(1+z_{n+1}) - 1| = \big|(p_n - 1)(1+z_{n+1}) + z_{n+1}\big| \leq |p_n - 1||1+z_{n+1}| + |z_{n+1}| \\ &\leq (p_n^* - 1)(1 + |z_{n+1}|) + |z_{n+1}| = p_n^*(1+|z_{n+1}|) - 1 = p_{n+1}^* - 1.\end{aligned}$$

Finally, to prove (7.1.3), we use the inequality (7.1.2) for the numbers $z_{n+1}, \ldots, z_m$ to obtain
$$\begin{aligned}|p_m - p_n| &= \left|\prod_{k=1}^{n}(1+z_k)\left(\prod_{k=n+1}^{m}(1+z_k) - 1\right)\right| \leq \prod_{k=1}^{n}(1+|z_k|)\left(\prod_{k=n+1}^{m}(1+|z_k|) - 1\right) \\ &= \prod_{k=1}^{m}(1+|z_k|) - \prod_{k=1}^{n}(1+|z_k|) = p_m^* - p_n^*,\end{aligned}$$
as desired. □

One can determine the convergence of certain products by looking at a corresponding series.

Proposition 7.4. *Let $a_n \geq 0$ for all $n \in \mathbb{N}$. Then*
$$\prod_{n=1}^{\infty}(1+a_n) \text{ converges} \iff \sum_{n=1}^{\infty} a_n \text{ converges.}$$

Proof. Let p_N be as in Lemma 7.3 for a_n's in place of the z_n's there. We have that $\{p_N\}_N$ is non-decreasing with $p_N \geq 1$ for all $N \in \mathbb{N}$. Therefore, there exists the limit $\lim\limits_{N\to\infty} p_N \in [1, +\infty]$, and thus the convergence of the product $\prod\limits_{n=1}^{\infty}(1+a_n)$ is equivalent to proving that $\lim\limits_{N\to\infty} p_N < \infty$. From this, it is clear that (7.1.1) gives the desired equivalence. □

The convergence can also be characterized by the convertence of a logarithmic series, if all the terms are in the same half-plane.

Proposition 7.5. *Let* $\{z_n\}_{n\geq 1} \subset \mathbb{C}$ *be so that* $\mathrm{Re}(z_n) > 0$ *for all* $n \in \mathbb{N}$. *Then,*

$$\prod_{n=1}^{\infty} z_n \ \text{ converges } \iff \sum_{n=1}^{\infty} \mathrm{Log}(z_n) \ \text{ converges.}$$

Here Log *denotes the principal branch of the logarithm.*

Proof. Assume first that $\prod_{n=1}^{\infty} z_n$ converges to some $p \in \mathbb{C}\setminus\{0\}$, and denote by p_n the corresponding n^{th}-partial product. By Remark 7.2, one has that $\lim_{n\to\infty} z_n = 1$. Let $\theta \in (-\pi, \pi]$ be so that $p = |p|e^{i\theta}$, let $\alpha : \mathbb{C}\setminus\ell \to (\theta-\pi, \theta+\pi]$ be a continuous branch of the argument (where ℓ is an appropriate half-line starting from the origin), and the associated continuous branch of the logarithm $f : \mathbb{C}\setminus\ell \to \mathbb{C}$:

$$f(w) = \log|w| + i\alpha(w), \quad w \in \mathbb{C}\setminus\ell.$$

Since $\lim_{n\to\infty} p_n = p$, we have that $p_n \in \mathbb{C}\setminus\ell$ for n large enough (as p is contained in the other half line of ℓ, that is, in $-\ell$), and therefore

$$f(p) = \lim_{n\to\infty} f(p_n). \tag{7.1.4}$$

Define the partial sums

$$s_n := \mathrm{Log}(z_1) + \cdots + \mathrm{Log}(z_n), \quad \text{where clearly} \quad e^{s_n} = p_n, \quad n \in \mathbb{N}. \tag{7.1.5}$$

Since $f(p_n)$ is a logarithm of p_n, we get that $e^{s_n} = e^{f(p_n)}$, and thus there exists $k_n \in \mathbb{Z}$ so that $s_n = f(p_n) + 2\pi i k_n$, for all $n \in \mathbb{N}$. By (7.1.5) and (7.1.4), we have that

$$\lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} \mathrm{Log}(z_n) = \mathrm{Log}(1) = 0, \quad \text{and} \quad \lim_{n\to\infty}(f(p_n) - f(p_{n-1})) = 0.$$

Therefore $\lim_{n\to\infty}(k_n - k_{n-1}) = 0$, and thus $k_n = k$ for all $n \geq n_0$, and some $n_0 \in \mathbb{N}$, $k \in \mathbb{Z}$. Using this and again (7.1.4), we conclude that the following limit exists

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}(f(p_n) + 2\pi i k) = f(p) + 2\pi i k.$$

Conversely, if the series $\sum_{n=1}^{\infty} \mathrm{Log}(z_n)$ converges to some $s \in \mathbb{C}$, and $\{s_n\}_n$ denotes its partial sums, taking exponentials we see that

$$e^s = \lim_{n\to\infty} e^{s_n} = \lim_{n\to\infty} \prod_{k=1}^{n} e^{\mathrm{Log}(z_k)} = \lim_{n\to\infty} \prod_{k=1}^{n} z_k.$$

Since $e^s \neq 0$, we may conclude that $\prod_{n=1}^{\infty} z_n$ converges. ☐

Finally, we see that certain version of *absolute convergence of the product* implies the ordinaric convergence.

Proposition 7.6. *Let* $\{z_n\}_n \subset \mathbb{C}$ *be a sequence so that* $\prod_{n=1}^{\infty}(1+|z_n|)$ *converges. Then* $\prod_{n=1}^{\infty}(1+z_n)$ *converges as well.*

Proof. By Remark 7.2, the sequence $\{z_n\}_n$ converges to 0, and thus, $1 + z_n \neq 0$ except for (possibly) finitely many $n \in \mathbb{N}$. Also, by Proposition 7.4, the series $\sum_{n=1}^{\infty}|z_n|$ converges, and so the corresponding tails $\{\sum_{n\geq N}|z_n|\}_N$ converge to 0 as $N \to \infty$. Therefore, starting the sequence from a large enough index, for the convergence of $\prod_{n=1}^{\infty}(1+z_n)$ we may and do assume that

$$1 + z_n \neq 0 \quad \text{for all} \quad n \in \mathbb{N}, \quad \text{and} \quad \exp\left(\sum_{n=1}^{\infty}|z_n|\right) - 1 \leq \frac{1}{2}. \tag{7.1.6}$$

Now, following the notation from Lemma 7.3, we have that

$$|p_m - p_n| \leq p_m^* - p_n^*, \quad m \geq n.$$

Since $\prod_{n=1}^{\infty}(1+|z_n|)$ converges, the sequence $\{p_n^*\}_n$ has the Cauchy property, and the previous inequality shows that $\{p_n\}_n$ is also a Cauchy sequence, thus convergent to some $p \in \mathbb{C}$. But by (7.1.6) and the inequalities (7.1.2) and (7.1.1), we obtain

$$|p_n - 1| \leq p_n^* - 1 \leq \quad \exp\left(\sum_{n=1}^{\infty}|z_n|\right) - 1 \leq \frac{1}{2}, \quad n \in \mathbb{N}.$$

Therefore $|p_n| \geq 1/2$ for all $n \in \mathbb{N}$, and so $p \neq 0$, showing that $\prod_{n=1}^{\infty}(1+z_n)$ converges. $\square$

7.1.2 Infinite Products of Functions

We are now interested in the pointwise or uniform convergence of products of functions, for which we will look again at the corresponding series.

Theorem 7.7. *Let $E \subset \mathbb{C}$ be a set, and $\{f_n : E \to \mathbb{C}\}_n$ a sequence of bounded functions in E. Assume that the series $\sum_{n=1}^{\infty}|f_n|$ converges uniformly on E. Then the product of functions*

$$E \ni z \mapsto \prod_{n=1}^{\infty}(1+f_n(z))$$

converges uniformly on E.

Proof. By Propositions 7.5 and 7.6, we have that

$$p(z) := \prod_{n=1}^{\infty}(1+f_n(z)) := \lim_{n\to\infty} p_n(z) \in \mathbb{C}, \quad z \in E, \quad p_n := \prod_{k=1}^{n}(1+f_k), \quad n \in \mathbb{N}.$$

Denoting the sums

$$s(z) = \sum_{n=1}^{\infty}|f_n(z)|, \quad s_n(z) = \sum_{k=1}^{n}|f_k(z)|, \quad z \in E,\, n \in \mathbb{N}.,$$

the uniform convergence of $\{s_n\}_n$ to s and the fact that the functions f_k's are bounded in E, implies that $s : E \to \mathbb{C}$ is a bounded function, and let $C > 0$ be an upper bound for $|s(z)|$ for all $z \in E$. Now, using first the estimate (7.1.3) and then (7.1.1), we get, for all $m > n$ and $z \in E$ that

$$\begin{aligned}|p_m(z) - p_n(z)| &\leq \prod_{k=1}^{m}(1+|f_k(z)|) - \prod_{k=1}^{n}(1+|f_k(z)|) = \prod_{k=1}^{n}(1+|f_k(z)|)\left(\prod_{k=n+1}^{m}(1+|f_k(z)|) - 1\right) \\ &\leq \exp\left(\sum_{k=1}^{n}|f_k(z)|\right)\left(\exp\left(\sum_{k=n+1}^{m}|f_k(z)|\right) - 1\right) \leq e^C\left(\exp\left(s(z) - s_n(z)\right) - 1\right).\end{aligned}$$

Since the last term is independent of m, letting $m \to \infty$ gives

$$\sup\{|p(z) - p_n(z)| \,:\, z \in E\} \leq e^C\left(\exp\left(\sup\{|s(z) - s_n(z)| \,:\, z \in E\}\right) - 1\right),$$

where the last term goes to 0 as $n \to \infty$, because $\{s_n\}_n$ converges to s uniformly on E. $\square$

As concerns products of holomorphic functions, we have the following theorem, which will be essential in the rest of the chapter.

Theorem 7.8. *Let $\Omega \subset \mathbb{C}$ be open and connected, $\{f_n : \Omega \to \mathbb{C}\}_n \subset \mathcal{H}(\Omega)$, so that none of the f_n are identically 0 on Ω. Assume also that the series $\sum_{n=1}^{\infty} |1 - f_n|$ converges uniformly on each compact subset of Ω. Then the product of functions*

$$F(z) = \prod_{n=1}^{\infty} f_n(z), \quad z \in \Omega,$$

converges uniformly on compact subsets of Ω and $F \in \mathcal{H}(\Omega)$. Also, the following relation between the order of the zeros holds:

$$m(F, z) = \sum_{n=1}^{\infty} m(f_n, z), \quad z \in \Omega, \tag{7.1.7}$$

where the sum is finite for each $z \in \Omega$. Moreover, denoting by $\mathcal{Z}(F)$ the zeros of F in Ω, we have that

$$\frac{F'(z)}{F(z)} = \sum_{n=1}^{\infty} \frac{f_n'(z)}{f_n(z)}, \quad z \in \Omega \setminus \mathcal{Z}(F); \tag{7.1.8}$$

where the series of functions converges uniformly on each compact subset of $\Omega \setminus \mathcal{Z}(F)$.

Proof. By Theorem 7.7 (with $f_n - 1$ in place of f_n) we get that the product $\prod_{n=1}^{\infty} |f_n|$ converges uniformly on compact sets of Ω, thus $F \in \mathcal{H}(\Omega)$ by Weierstrass Theorem 3.1. Clearly $F(z) = 0$ if and only if $f_n(z) = 0$ for some $n \in \mathbb{N}$, that is,

$$\mathcal{Z}(F) = \bigcup_{n \in \mathbb{N}} \mathcal{Z}(f_n).$$

The zeros of each f_n are isolated, as $f_n \not\equiv 0$ in Ω and Ω is connected. The set $\mathcal{Z}(f_n)$ is countable (see e.g. Proposition 2.13), and the above then implies that $\mathcal{Z}(F)$ is countable, and so $f \not\equiv 0$ in Ω by the Identity Principles for holomorphic functions. This is turn shows that $\mathcal{Z}(F)$ has no accumulation points in Ω. We next observe that:

$$\text{for each compact } K \subset \Omega \text{ there is } N = N(K) \text{ with } f_n(z) \neq 0 \text{ for all } n \geq N,\, z \in K. \tag{7.1.9}$$

Indeed, if $K \subset \Omega$, the set $\mathcal{Z}(F) \cap K$ is finite, as $\mathcal{Z}(F)' \cap \Omega = \emptyset$. Denoting those zeros by $z_1, \ldots, z_m$, the pointwise convergence of $\prod_{n=1}^{\infty} f_n$ in Ω implies that, for each z_j, there exist $N_j \in \mathbb{N}$ so that $f_n(z_j) \neq 0$ for all $n \geq N_j$. If N denotes the maximum of those $N_1, \ldots, N_m$, we get that $f_n(z_j) \neq 0$ for all $j \in \{1, \ldots, m\}$ and all $n \geq N$. Since a zero of a function f_n defines automatically a zero of F, we have shown (7.1.9).

To show (7.1.7), let $z \in \Omega$ with $F(z) = 0$ (if $F(z) \neq 0$, then all the orders of zero at z are zero, and the identity trivially holds). Since the zeros of F are isolated, there is $\varepsilon > 0$ so that $\overline{D}(z, \varepsilon) \subset \Omega$ and $\overline{D}(z, \varepsilon) \cap \mathcal{Z}(F) = \{z\}$. Applying (7.1.9) for $K = \overline{D}(z, \varepsilon)$, we find $N = N(z) \in \mathbb{N}$, so that we can factorize F around this disk as

$$F(z) = h(z) \cdot \prod_{n=1}^{N} f_n(z),\ z \in \overline{D}(z, \varepsilon), \text{ where } h := \prod_{n=N+1}^{\infty} f_n \in \mathcal{H}(\Omega),\ h(z) \neq 0 \text{ for all } z \in \overline{D}(z, \varepsilon).$$

Factorizing each $f_n(w) = (w - z)^{m(f_n, z)} h_n(w)$ as $h_n \in \mathcal{H}(D(z, \varepsilon))$ with $h_n(w) \neq 0$ for all $w \in D(z, \varepsilon)$, it immediately follows that

$$m(F, z) = \sum_{n=1}^{N} m(f_n, z) = \sum_{n=1}^{\infty} m(f_n, z),$$

where the sum is actually finite, and this shows (7.1.7).

Let us now prove (7.1.8). For each compact $K \subset \Omega \setminus \mathcal{Z}(F)$, let $N = N(K) \in \mathbb{N}$ as in (7.1.9). The partial products

$$G_m := \prod_{n=N}^{m} f_n, \quad m \in \mathbb{N}, m \geq N,$$

define holomorphic functions converging uniformly to $G := \prod_{n=N}^{\infty} f_n$ uniformly in compact subsets of Ω. In particular, recalling Weierstrass Convergence Theorem 3.1 for the derivatives we get that $\{G_m\}_m$ and $\{G'_m\}_m$ converge uniformly to G and G' on K. But since the function G does not vanish in the compact set K (by (7.1.9)), there exists $c > 0$ so that $|G(z)| \geq c$ for all $z \in K$. Along with the convergence to G and G' of the mentioned sequences, we derive that

$$\frac{G'_m}{G_m} \to \frac{G'}{G} \quad \text{uniformly on } K. \tag{7.1.10}$$

A simple computation shows that

$$\frac{G'_m}{G_m} = \sum_{n=N}^{m} \frac{f'_n}{f_n} \quad \text{on } K,$$

which, together with (7.1.10), yields

$$\frac{G'}{G} = \lim_{m\to\infty} \sum_{n=N}^{m} \frac{f'_n}{f_n} = \sum_{n=N}^{\infty} \frac{f'_n}{f_n} \quad \text{uniformly on } K. \tag{7.1.11}$$

Writing $F = G \cdot \prod_{n=1}^{N-1} f_n$, (7.1.11) tells us that

$$\frac{F'}{F} = \frac{G'}{G} + \sum_{n=1}^{N-1} \frac{f'_n}{f_n} = \sum_{n=1}^{\infty} \frac{f'_n(z)}{f_n(z)} \quad \text{with uniform convergence on } K.$$

Since K is an arbitrary compact subset of $\Omega \setminus \mathcal{Z}(F)$, we have proved (7.1.8). $\square$

7.2 The Weierstrass Factorization Theorem

The key ingredient to solve the problems we discussed at the beginning of the chapter is the family of *Weierstrass Factors.*

Definition 7.9 (Weierstrass Elementary Factors)**.** *The* ***Weierstrass Factors*** *is the sequence* $\{E_n\}_{n\geq 0}$ *of functions given by the formulae*

$$E_0(z) = 1 - z, \quad E_n(z) = (1-z)\exp\left(\sum_{k=1}^{n} \frac{z^k}{k}\right) = (1-z)e^{z+\frac{z^2}{2}+\cdots+\frac{z^n}{n}}, \quad n \in \mathbb{N},\, z \in \mathbb{C}.$$

Note that $E_n \in \mathcal{H}(\mathbb{C})$, $E_n(0) = 1$ *and* $E_n(1) = 0$ *for all* $n \in \mathbb{N} \cup \{0\}$.

There is a key estimate for the Weierstrass factors in the unit disk.

Lemma 7.10. *For all* $n \in \mathbb{N} \cup \{0\}$, *we have*

$$|1 - E_n(z)| \leq |z|^{n+1} \quad \textit{for all} \quad |z| \leq 1.$$

Proof. If $n = 0$, the inequality is trivial, and actually an identity. Assume from now that $n \geq 1$. Observe that, for each $z \in \mathbb{C}$,

$$
\begin{aligned}
-E_n'(z) &= \exp\left(\sum_{k=1}^{n} \frac{z^k}{k}\right) - (1-z)\exp\left(\sum_{k=1}^{n} \frac{z^k}{k}\right) \cdot \sum_{k=0}^{n-1} z^k \\
&= \exp\left(\sum_{k=1}^{n} \frac{z^k}{k}\right) - (1-z^n)\exp\left(\sum_{k=1}^{n} \frac{z^k}{k}\right) = z^n \exp\left(\sum_{k=1}^{n} \frac{z^k}{k}\right) \\
&= z^n \sum_{j=0}^{\infty} \frac{1}{j!}\left(\sum_{k=1}^{n} \frac{z^k}{k}\right)^j = z^n \sum_{j=0}^{\infty} a_j z^j = \sum_{j=0}^{\infty} a_j z^{j+n}, \quad \text{where } a_j > 0, \text{ for all } j \in \mathbb{N} \cup \{0\}.
\end{aligned}
$$

Integrating termwise in the series (as we may do, since the series above has radius of convergence equal to ∞), the result is a holomorphic function in $\mathbb{C}$ whose derivative is $-E_n'$. Thus there exists a constant $w \in \mathbb{C}$ so that

$$
w - E_n(z) = \sum_{j=0}^{\infty} \frac{a_j}{j+n+1} z^{j+n+1} = z^{n+1} \sum_{j=0}^{\infty} b_j z^j, \quad b_j := \frac{a_j}{j+n+1} > 0,\ j \in \mathbb{N} \cup \{0\}.
$$

Since $E_n(0) = 1$, and the last term vanishes at 0, we get that

$$
1 - E_n(z) = z^{n+1} \sum_{j=0}^{\infty} b_j z^j, \quad z \in \mathbb{C}.
$$

The fact that $E_n(1) = 0$ and the previous identity shows that $\sum_{j=0}^{\infty} b_j = 1$. Thus, we may conclude, for $|z| \leq 1$, that

$$
|1 - E_n(z)| \leq |z|^{n+1} \sum_{j=0}^{\infty} b_j |z|^j \leq |z|^{n+1} \sum_{j=0}^{\infty} b_j = |z|^{n+1}.
$$

□

We are now ready to construct entire functions prescribing zeros and multiplicities.

Theorem 7.11. *Let $\{z_n\} \subset \mathbb{C} \setminus \{0\}$ be a sequence with $\lim_{n\to\infty} |z_n| = \infty$, and $\{k_n\}_n \subset \mathbb{N} \cup \{0\}$ another sequence satisfying that*

$$
\sum_{n=1}^{\infty} \left(\frac{r}{|z_n|}\right)^{1+k_n}, \quad \textit{for all} \quad r > 0. \tag{7.2.1}
$$

Then the function

$$
F(z) = \prod_{n=1}^{\infty} E_{k_n}\left(\frac{z}{z_n}\right), \quad z \in \mathbb{C}, \tag{7.2.2}
$$

is well-defined and holomorphic in $\mathbb{C}$, with $\mathcal{Z}(F) = \{z_n\}_n$ and so that the order $m(F, z_j)$ of z_j as zero of F coincides with the number of appearances of z_j in the sequence $\{z_n\}_n$, that is,

$$
m(F, z_j) = \operatorname{card}\{n \in \mathbb{N} : z_n = z_j\}, \quad \textit{for all} \quad j \in \mathbb{N}.
$$

Proof. Define, for each $n \in \mathbb{N}$, the holomorphic functions $f_n(z) := E_{k_n}\left(\frac{z}{z_n}\right)$, $z \in \mathbb{C}$. Given $r > 0$, let $N \in \mathbb{N}$ be so that $|z_n| > r$ for all $n \geq N$. We can apply Lemma 7.10 (for each z/z_n) to obtain that, for all $z \in \overline{D}(0, r)$,

$$
\sum_{n=N}^{\infty} |1 - f_n(z)| = \sum_{n=N}^{\infty} \left|1 - E_{k_n}\left(\frac{z}{z_n}\right)\right| \leq \sum_{n=N}^{\infty} \left|\frac{z}{z_n}\right|^{1+k_n}.
$$

Estimating, for every $z \in \overline{D}(0,r)$ and every $n \geq N$, by

$$\left|\frac{z}{z_n}\right|^{1+k_n} \leq \left(\frac{r}{|z_n|}\right)^{1+k_n},$$

we see that the condition (7.2.1) and Weierstrass M-test guarantee the uniform convergence of the series of functions $\sum_{n=1}^{\infty} |1 - f_n|$ on $\overline{D}(0,r)$. Since $r > 0$ is arbitrary, the series converges uniformly on compact subsets of $\mathbb{C}$, and Theorem 7.8 implies that the infinite product F as in (7.2.2) is holomorphic in $\mathbb{C}$ with

$$\mathcal{Z}(F) = \bigcup_{n=1}^{\infty} \mathcal{Z}(f_n) = \{z_n\}_n,$$

as Weierstrass Factors E_p only vanish at $z = 1$. Moreover, E_p has a zero of order 1 at 1, and then we can use (7.1.7) from Theorem 7.8 to conclude that, for every $j \in \mathbb{N}$,

$$m(F, z_j) = \sum_{n=1}^{\infty} m(f_n, z_j) = \sum_{n=1}^{\infty} \mathcal{X}_{\{n \in \mathbb{N}\,:\, z_n = z_j\}}(n) = \text{card}\{n \in \mathbb{N} \,:\, z_n = z_j\}.$$

□

Remark 7.12. If $\{z_n\}_n \subset \mathbb{C} \setminus \{0\}$ is a sequence with $\lim_{n\to\infty} |z_n| = \infty$, then the condition (7.2.1) of Theorem 7.11 is always satisfied if we take the sequence $k_n = n - 1$, $n \in \mathbb{N}$.

To see this, simply observe that for all $r > 0$ there exists $N \in \mathbb{N}$ so that that $r/|z_n| \leq 1/2$ for all $n \geq N$. Thus

$$\sum_{n=N}^{\infty} \left(\frac{r}{|z_n|}\right)^{1+k_n} \leq \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n < \infty.$$

Here are some concrete examples of couples $\{k_n, z_n\}_n$ satisfying the requirement (7.2.1).

(1) If $\{z_n\}_n$ is so that

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|} < \infty,$$

then one can take $k_n = 0$, for all $n \in \mathbb{N}$, and formula (7.2.2) becomes

$$F(z) = \prod_{n=1}^{\infty} E_0(z/z_n) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right), \quad z \in \mathbb{C}.$$

(2) If $\{z_n\}_n$ is so that

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^2} = \infty,$$

then one can take $k_n = 1$, for all $n \in \mathbb{N}$, and formula (7.2.2) becomes

$$F(z) = \prod_{n=1}^{\infty} E_1(z/z_n) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n}}, \quad z \in \mathbb{C}.$$

(3) If $z_n = \sqrt{n}$ for all n, we can take $k_n = 2$ for all $n \in \mathbb{N}$.

Moreover, every entire function can be written as a product of an exponential function and an infinite product of certain Weierstrass factors, based on the zeros of a function.

Theorem 7.13 (Weierstrass Factorization Theorem)**.** *Let $f \in \mathcal{H}(\mathbb{C})$ and let $\mathcal{Z}(f) \setminus \{0\} = \{z_n\}_{n\in\mathbb{N}}$ the non-zero zeros of f, where each z_n appears precisely $m(f, z_n)$ times in the sequence. Denote by $m = m(f, 0) \in \mathbb{N} \cup \{0\}$ the order of 0 as a zero of f. Then, for any sequence $\{k_n\}_n \subset \mathbb{N} \cup \{0\}$ satisfying condition (7.2.1) for $\{z_n\}_n$, there exists $g \in \mathcal{H}(\mathbb{C})$ so that*

$$f(z) = z^m \cdot e^{g(z)} \cdot \prod_{n=1}^{\infty} E_{k_n}\left(\frac{z}{z_n}\right), \quad z \in \mathbb{C}. \tag{7.2.3}$$

Proof. If $m = m(f, 0) \in \mathbb{N} \cup \{0\}$, we can factorize $f(z) = z^m \cdot h(z)$ for all $z \in \mathbb{N}$, for some $h \in \mathcal{H}(\mathbb{C})$ with $h(0) \neq 0$, and so that $\mathcal{Z}(h) = \mathcal{Z}(f) \setminus \{0\}$. This shows that, to derive a factorization like (7.2.3), we may assume that $f(0) \neq 0$.

Now, assuming that $f(0) \neq 0$, first note that $\lim\limits_{n\to\infty} z_n = \infty$. Indeed, otherwise there would be a bounded subsequence $\{w_n\}_n$ of $\{z_n\}_n$ consisting of mutually distinct points. By Bolzano-Weierstrass theorem, there would be an accumuluation point w of $\{w_n\}_n$, and the continuity of f would lead to $f(w) = 0$, thus showing that w would be a non-isolated zero of f, which would imply that f is identically null in $\mathbb{C}$, a contradiction.

Since $\lim\limits_{n\to\infty} z_n = \infty$, combining Remark 7.12 and Theorem 7.11, we find a sequence $\{k_n\}_n \subset \mathbb{N} \cup \{0\}$ so that the function

$$\mathbb{C} \ni z \mapsto F(z) := \prod_{n=1}^{\infty} E_{k_n}\left(\frac{z}{z_n}\right)$$

is holomorphic in $\mathbb{C}$ and has zeros at each z_j with mulitplicity equal to the number of ocurrences of z_j in the sequence $\{z_n\}_n$. In other words, $m(F, z_j) = m(f, z_j)$. But also note that then, the function $h = f/F$ has a removable singularity at every z_j, and thus there exists a holomorphic extension $H : \mathbb{C} \to \mathbb{C}$ of h. Since $\mathbb{C}$ is simply-connected, by Theorem 1.29 there exists $g \in \mathcal{H}(\mathbb{C})$ so that $H = e^g$. Therefore,

$$f = H \cdot F = e^g \cdot F,$$

which shows formula (7.2.3). □

7.3 Jensen's Formula

To prove Jensen's formula, first we need to evaluate the following improper integral.

Lemma 7.14. *We have that*

$$\mathrm{pv}\int_0^{2\pi} \log|1 - e^{it}|\, dt := \lim_{\varepsilon\to 0^+} \int_{\varepsilon}^{2\pi-\varepsilon} \log|1 - e^{it}|\, dt = 0.$$

Proof. In the simply connected open set $\Omega := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$, the function $z \mapsto 1 - z$ never vanishes, and by Corollary 1.29 there exists $h \in \mathcal{H}(\Omega)$ so that $e^{h(z)} = 1 - z$ for all $z \in \Omega$. This implies that $e^{h(0)} = 1$, and so $e^{h(z)-h(0)} = 1 - z$ for all $z \in \Omega$. Therefore, we may assume that

$$e^{h(z)} = 1 - z, \quad z \in \Omega, \quad h(0) = 0.$$

Then we have that

$$e^{\mathrm{Re}(h(z))} = \left|e^{h(z)}\right| = |1 - z|, \quad e^{\mathrm{Re}(h(z))} \cos(\mathrm{Im}(h(z))) = \mathrm{Re}(1 - z) > 0, \quad z \in \Omega.$$

Therefore

$$\mathrm{Re}(h(z)) = \log|1 - z| \quad \text{and} \quad |\mathrm{Im}(h(z))| < \frac{\pi}{2}, \quad z \in \Omega. \tag{7.3.1}$$

For every $0 < \varepsilon < 1$, let $\gamma_\varepsilon : [\varepsilon, 2\pi - \varepsilon] \to \mathbb{C}$ be the path $\gamma_\varepsilon(t) = e^{it}$, $t \in [\varepsilon, 2\pi - \varepsilon]$. Also, let σ_ε be the circle arc centered at 1 and with radius $|1 - e^{i\varepsilon}|$ joining $e^{i\varepsilon}$ to $e^{-i\varepsilon}$ within $\mathbb{D}$. Clearly

$\Gamma_\varepsilon := \gamma_\varepsilon \star \sigma_\varepsilon^-$ is a closed piecewise C^1 path. Thus, by Cauchy Integral formula (bearing in mind that $h(0) = 0$),

$$\frac{1}{2\pi i}\int_{\gamma_\varepsilon} \frac{h(z)}{z}\,\mathrm{d}z = \frac{1}{2\pi i}\int_{\Gamma_\varepsilon} \frac{h(z)}{z}\,\mathrm{d}z - \frac{1}{2\pi i}\int_{\sigma_\varepsilon} \frac{h(z)}{z}\,\mathrm{d}z = \frac{1}{2\pi i}\int_{\sigma_\varepsilon} \frac{h(z)}{z}\,\mathrm{d}z. \tag{7.3.2}$$

Using (7.3.1) and (7.3.2) we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_\varepsilon^{2\pi-\varepsilon} \log|1-e^{it}|\,\mathrm{d}t &= \frac{1}{2\pi}\int_\varepsilon^{2\pi-\varepsilon} \mathrm{Re}(h(e^{it}))\,\mathrm{d}t = \mathrm{Re}\left(\frac{1}{2\pi}\int_\varepsilon^{2\pi-\varepsilon} h(e^{it})\,\mathrm{d}t\right)\\ &= \mathrm{Re}\left(\frac{1}{2\pi i}\int_{\gamma_\varepsilon}\frac{h(z)}{z}\,\mathrm{d}z\right) = \mathrm{Re}\left(\frac{1}{2\pi i}\int_{\sigma_\varepsilon}\frac{h(z)}{z}\,\mathrm{d}z\right).\end{aligned}$$

And applying again (7.3.1) to the last integral, we get, for ε small enough,

$$\left|\frac{1}{2\pi}\int_\varepsilon^{2\pi-\varepsilon} \log|1-e^{it}|\,\mathrm{d}t\right| \le \frac{1}{2\pi}\int_{\sigma_\varepsilon}\frac{|h(z)|}{|z|}|\,\mathrm{d}z| \le \frac{\sqrt{2}}{2\pi}\int_{\sigma_\varepsilon}\frac{\frac{\pi}{2}+|\log|1-z||}{|z|}|\,\mathrm{d}z| \le \frac{\sqrt{2}}{2\pi}\frac{\left(\frac{\pi}{2}+|\log\varepsilon|\right)}{1/2}\ell(\sigma_\varepsilon).$$

The last term goes to 0 as $\varepsilon \to 0^+$ because

$$\ell(\sigma_\varepsilon) \le 2\pi|1-e^{i\varepsilon}| = 2\pi|e^{-i\varepsilon/2} - e^{i\varepsilon/2}| = 4\pi|\sin(\varepsilon/2)| \le 2\pi\varepsilon.$$

□

The following Jensen's formula relates the zeros of a holomorphic function in a disk with the integral of the logarithm of the function in the corresponding circle. Note that if a function $f \in \mathcal{H}(D(0,R))$ does not vanish, then Exercise 5.1 says that $\log|f|$ is harmonic, and thus satisfies the Mean Value Property, from which

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{it})|\,\mathrm{d}t, \quad 0<r<R.$$

Jensen's formula generalize this identity to the case of functions with zeros.

Theorem 7.15 (Jensen's Formula)**.** *Let $f \in \mathcal{H}(D(0,R))$ with $f(0) \neq 0$, and for $0<r<R$, let $w_1,\dots,w_N$ be the zeros of f in $\overline{D}(0,r)$, listed as many times as their multiplicity. Then,*

$$\sum_{n=1}^N \log\left(\frac{r}{|w_n|}\right) = -\log|f(0)| + \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{it})|\,\mathrm{d}t. \tag{7.3.3}$$

Proof. Given $0<r<R$, we can assume (after rearranging) that $w_1,\dots,w_m$ are the zeros of f in $D(0,r)$, and $w_{m+1},\dots,w_N$ the zeros in $\partial D(0,r)$. There exists $\varepsilon>0$ with $r+\varepsilon<R$ and so that the zeros of f in $D(0,1+\varepsilon)$ are still precisely $w_1,\dots,w_N$. Thus, we can factorize

$$f(z) = \prod_{n=1}^N (z-w_n)\cdot h(z), \quad z\in D(0,r+\varepsilon), \quad h\in\mathcal{H}(D(0,r+\varepsilon)),\ h(z)\neq 0 \text{ for all } z\in D(0,r+\varepsilon).$$

We define

$$g(z) = h(z)\cdot\prod_{n=1}^m r\left(\frac{\overline{w_n}}{r}\cdot\frac{z}{r}-1\right)\cdot\prod_{n=m+1}^N w_n, \quad z\in D(0,r+\varepsilon). \tag{7.3.4}$$

Obviously $h\in\mathcal{H}(D(0,r+\varepsilon))$ and

$$|g(0)| = |h(0)|\cdot\prod_{n=1}^m r\cdot\prod_{n=m+1}^N|w_n| = \frac{|f(0)|}{\prod_{n=1}^N|w_n|}\cdot\prod_{n=1}^m r\cdot\prod_{n=m+1}^N|w_n| = |f(0)|\cdot\prod_{n=1}^m\frac{r}{|w_n|}. \tag{7.3.5}$$

Now observe that if $|z| = r$, then, for $n = 1, \dots, m$,

$$|z - w_n| = r\left|\frac{z}{r} - \frac{w_n}{r}\right| = r\left|1 - \frac{\overline{w_n}}{r}\cdot\frac{z}{r}\right|,$$

as shown for example by Proposition 4.18, if one consider $\varphi_{w_n/r}(z/r)$, with $|w_n/r| < 1$ and $|z/r| = 1$. Using this and (7.3.4), we get that

$$|f(z)| = \prod_{n=1}^{N}|z - w_n|\cdot|h(z)| = \prod_{n=1}^{m} r\left|1 - \frac{\overline{w_n}}{r}\cdot\frac{z}{r}\right|\cdot\prod_{n=m+1}^{N}|z - w_n|\cdot|h(z)| = |g(z)|\prod_{n=m+1}^{N}\frac{|z - w_n|}{|w_n|}.$$

For each $n = m+1, \dots, N$ we write $w_n = re^{it_n}$, and this identity together with (7.3.5) give

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{it})|\,\mathrm{d}t &= \frac{1}{2\pi}\int_0^{2\pi}\log|g(re^{it})|\,\mathrm{d}t + \sum_{n=m+1}^{N}\frac{1}{2\pi}\int_0^{2\pi}\log|1 - e^{i(t-t_n)}|\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^{2\pi}\log|g(re^{it})|\,\mathrm{d}t + \sum_{n=m+1}^{N}\frac{1}{2\pi}\int_0^{2\pi}\log|1 - e^{it}|\,\mathrm{d}t.\end{aligned}$$

By Lemma 7.14, the last sum is null. Moreover, since g does not vanish in $D(0, r+\varepsilon)$, the function $\log|g|$ is harmonic (see Exercise 5.1) in $D(0, r+\varepsilon)$ and so satisfies the Mean Value Property (Proposition 5.9). Thus the above and (7.3.5) show

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{it})|\,\mathrm{d}t &= \frac{1}{2\pi}\int_0^{2\pi}\log|g(re^{it})|\,\mathrm{d}t = \log|g(0)| = \log\left(|f(0)|\cdot\prod_{n=1}^{m}\frac{r}{|w_n|}\right) \\ &= \log|f(0)| + \sum_{n=1}^{N}\log\left(\frac{r}{|w_n|}\right),\end{aligned}$$

from which formula (7.3.3) is proven.

□

7.4 The Blaschke Product

We now consider construction of holomorphic and bounded functions in the unit disk prescribing zeros and multiplicities. The desired zeros $\{w_n\}_n$ must satisfy a summability condition, known as the *Blaschke condition*, and the functions is constructed as an infinite product of automorphisms of the unit disk. This infinite product is called the *Blaschke product*.

Theorem 7.16 (Blaschke Interpolation). *Let $\{w_n\}_{n\in\mathbb{N}} \subset \mathbb{D}$ a sequence with $w_n \neq 0$ for all $n \in \mathbb{N}$, and so that*

$$\sum_{n=1}^{\infty}(1 - |w_n|) < \infty. \tag{7.4.1}$$

Then, the function

$$B(z) = \prod_{n=1}^{\infty}\frac{|w_n|}{w_n}\cdot\frac{w_n - z}{1 - \overline{w_n}z}, \quad z \in \mathbb{D}, \tag{7.4.2}$$

defined a holomorphic function in $\mathbb{D}$, with $B(\mathbb{D}) \subset \mathbb{D}$ with $\mathcal{Z}(B) = \{w_n\}_{n\in\mathbb{N}}$, and $m(B, z_j)$ is the number of appearances of the number z_j in $\{z_n\}_{n\in\mathbb{N}}$ for all $j \in \mathbb{N}$.

Proof. Denote

$$f_n(z) = \frac{|w_n|}{w_n}\cdot\frac{w_n - z}{1 - \overline{w_n}z}, \quad z \in \mathbb{D},$$

and notice that, if $z \in \overline{D}(0,r)$ with $0<r<1$,

$$|1-f_n(z)| = \left|1-\frac{|w_n|}{w_n}\cdot\frac{w_n-z}{1-\overline{w_n}z}\right| = \left|\frac{w_n-|w_n|^2z-w_n|w_n|+z|w_n|}{(1-\overline{w_n}z)w_n}\right|$$

$$= \left|\frac{w_n+|w_n|z}{(1-\overline{w_n}z)w_n}\right|(1-|w_n|) \leq \frac{1+|z|}{1-|z|}(1-|w_n|) \leq \frac{1+r}{1-r}(1-|w_n|).$$

This, together with the condition (7.4.1) and Weierstrass M-test, show that the series

$$\sum_{n=1}^{\infty}|1-f_n(z)|$$

converges uniformly on compact subsets of $\mathbb{D}$. We can therefore apply Theorem 7.8 to deduce that the function B from (7.4.2), is holomorphic in $\mathbb{D}$, and $\mathcal{Z}(B)=\bigcup_{n=1}^{\infty}\mathcal{Z}(f_n)$ with

$$m(\Omega,z)=\sum_{n=1}^{\infty}m(f_n,z), \quad z\in\mathbb{D}.$$

But each f_n vanishes only at z_n and with multiplicity 1, and so the claim about the zeros of B in the present theorem holds true.

Finally, to show that $B(\mathbb{D})\subset\mathbb{D}$, observe that

$$B(z)=\prod_{n=1}^{\infty}f_n(z), \quad f_n(z)=\frac{|w_n|}{w_n}\cdot\left(-\varphi_{w_n}(z)\right), \quad n\in\mathbb{N},\ z\in\mathbb{D},$$

where φ_w, $w\in\mathbb{D}$, denotes the automorphism of $\mathbb{D}$ defined in Definition 4.17. By Proposition 4.18, we have that $|f_n(z)|<1$ for all $z\in\mathbb{D}$ and all $n\in\mathbb{N}$. Therefore, $|B(z)|<1$ for all $z\in\mathbb{D}$, that is, $B(\mathbb{D})\subset\mathbb{D}$. □

Finally, using Jensen's Formula, one can show that the Blaschke condition (7.4.1) is also necessary for analytic and bounded interpolation in the unit disk.

Theorem 7.17. *Let $f\in\mathcal{H}(\mathbb{D})$ with f bounded in $\mathbb{D}$, $f(0)\neq 0$ and f has infinitely many zeros in $\mathbb{D}$. Denote by $\{w_n\}_{n=1}^{\infty}$ the zeros of f in $\mathbb{D}$, listed according to their multiplicities. Then,*

$$\sum_{n=1}^{\infty}(1-|w_n|)<\infty. \tag{7.4.3}$$

Proof. Define $C=\sup\{|f(z)| : z\in\mathbb{D}\}$. The function f has infinitely many zeros in $\mathbb{D}$ and finitely many on each compact subset of $\mathbb{D}$. For every $0<r<1$, denote by $M(r)$ the number of zeros (counted with multiplicity) of f in $\overline{D}(0,r)$, and note that $M(r)$ goes to ∞ as $r\to 1^-$ by the previous observation. Therefore, given $N\in\mathbb{N}$, we can find $0<r_N<1$ so that

$$r_N \geq \max\left\{1-\frac{1}{N^2},\frac{1}{2}\right\} \quad \text{and} \quad M(r_N)\geq N. \tag{7.4.4}$$

Rearranging the sequence of zeros if necessary, we can assume that $w_1,\dots,w_{M(r_N)}$ are precisely the zeros of f in $\overline{D}(0,r_N)$. By Theorem 7.15 (taking exponentials in (7.3.3)) and the fact that $|f|\leq C$ in $\mathbb{D}$ we have that

$$|f(0)|\cdot\prod_{n=1}^{M(r_N)}\frac{r_N}{|w_n|}=\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log|f(r_Ne^{it})|\,\mathrm{d}t\right)\leq\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log C\,\mathrm{d}t\right)=C,$$

and so

$$\prod_{n=1}^{N} \frac{r_N}{|w_n|} \leq \prod_{n=1}^{M(r_N)} \frac{r_N}{|w_n|} = \frac{C}{|f(0)|}.$$

Rewrite this estimate and using (7.4.4)

$$p_N := \prod_{n=1}^{N} |w_n| \geq \frac{r_N^N \cdot |f(0)|}{C} \geq \max\left\{\left(1 - \frac{1}{N^2}\right)^N, \frac{1}{2^N}\right\} \cdot \frac{|f(0)|}{C}, \quad N \in \mathbb{N}.$$

This shows that the partial products above $\{p_N\}_{N\in\mathbb{N}}$ are all nonzero and converge to some $p \in \mathbb{C} \setminus \{0\}$. Exercise 7.5 tells us that $\sum_{n=1}^{\infty}(1 - |w_n|)$ converges. Since the terms of this series are all positive, the rearrangement we chose in the proof does not affect the convergence, and so (7.4.3) holds. □

7.5 Exercises

Exercise 7.1. *Check that* $\prod\limits_{n=1}^{\infty} \left(1 + \frac{(-1)^{n+1}}{\sqrt{n}}\right)$ *is not convergent, despite that* $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ *is convergent. Compare with Proposition 7.4.*

Exercise 7.2. *Check that* $\prod\limits_{n=1}^{\infty} \left(1 + \frac{(-1)^{n+1}}{n}\right)$ *converges and that* $\prod\limits_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)$ *is not convergent. Compare with Proposition 7.6.*

Exercise 7.3. *Let* $\{a_n\}_n \subset \mathbb{R}$. *Prove that*

(a) If $\sum_{n=1}^{\infty} a_n < \infty$, *then*

$$\prod_{n=1}^{\infty}(1 + a_n) \iff \sum_{n=1}^{\infty} a_n^2.$$

(b) If $\sum_{n=1}^{\infty} a_n^2 < \infty$, *then*

$$\prod_{n=1}^{\infty}(1 + a_n) \iff \sum_{n=1}^{\infty} a_n.$$

Exercise 7.4. *Let* $\{z_n\}_n \subset \mathbb{C}$. *Prove that* $\prod\limits_{n=1}^{\infty} z_n$ *converges if and only if*

$$\textit{for every } \varepsilon > 0 \textit{ there is } N \in \mathbb{N} \textit{ so that } |z_{n+1} \cdots z_{n+N} - 1| \leq \varepsilon \textit{ for all } n \geq N, k \in \mathbb{N}.$$

Exercise 7.5. *Let* $\{a_n\}_n \subset [0, 1)$. *Prove that*

$$\prod_{n=1}^{\infty}(1 - a_n) \textit{ converges} \iff \sum_{n=1}^{\infty} a_n \textit{ converges.}$$

Exercise 7.6. *Verify the convergence of the infinite product* $\prod\limits_{n=1}^{\infty}(1 + z^{2^n})$ *for all* $z \in \mathbb{D}$, *and show the identity*

$$(1 - z)\prod_{n=1}^{\infty}(1 + z^{2^n}) = \frac{1}{1 + z}, \quad |z| < 1.$$

Exercise 7.7. *Verify that the infinite product*

$$f(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right), \quad z \in \mathbb{C},$$

defines a holomorphic function in $\mathbb{C}$ *so that* $\mathcal{Z}(f) = \mathbb{C} \setminus \{0\}$. *Then, show that*

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}, \quad z \in \mathbb{C} \setminus \mathcal{Z}(f).$$

Exercise 7.8. *Construct a function* $f \in \mathcal{H}(\mathbb{C})$ *to that* $\mathcal{Z}(f) = \{\log n \,:\, n \in \mathbb{N}, n \geq 2\}$, *and* $m(f, \log n) = 1$ *for all* $n \geq 2$, $n \in \mathbb{N}$.

Exercise 7.9. *Verify the following identities, showing first that the infinite products define holomorphic functions in* $\mathbb{C}$:

$$\sin(\pi z) = \pi z \cdot \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right), \quad \cos(\pi z) = \prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n-1)^2}\right), \quad \sinh(z) = z \cdot \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{\pi^2 n^2}\right).$$

Exercise 7.10. *Use the factorization of* $\sin z$ *from Exercise 7.9 to prove the Wallis's Formula:*

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \dots \cdot \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)}.$$

Exercise 7.11. *Use the factorization of* $\cos(\pi z)$ *from Exercise 7.9 to show that*

$$-\pi \tan(\pi z) = \sum_{n=1}^{\infty}\left(\frac{1}{z - n + \frac{1}{2}} + \frac{1}{z + n - \frac{1}{2}}\right), \quad z \in \mathbb{C} \setminus \{n + \tfrac{1}{2} \,:\, n \in \mathbb{Z}\}$$

Exercise 7.12. *Use the factorization of* $\sinh z$ *from Exercise 7.9 to show that*

$$\coth(z) := \frac{\cosh(z)}{\sinh(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{\pi^2 n^2 + z^2}, \quad z \in \mathbb{C} \setminus \{\pi i k \,:\, k \in \mathbb{Z}\}.$$

Use this formula to derive the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + x^2} = \frac{\pi}{2x} \cdot \frac{e^{\pi x} + e^{-\pi x}}{e^{\pi x} - e^{-\pi x}} - \frac{1}{2x^2}, \quad \textit{for all} \quad x > 0.$$

Exercise 7.13. *Let* $f \in \mathcal{H}(D(0,R))$ *with a zero of order* $m \in \mathbb{N} \cup \{0\}$ *at* 0. *For* $0 < r < R$, *let* $w_1, \dots, w_N$ *be the zeros of* f *in* $\overline{D}(0,r)$, *listed as many times as their multiplicity. Prove the formula*

$$\sum_{n=1}^{N} \log\left(\frac{r}{|w_n|}\right) = -\log\left|\frac{f^{(m)}(0)}{m!}\right| - m\log r + \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{it})|\,\mathrm{d}t.$$

Exercise 7.14. *Let* $f \in \mathcal{H}(D(0,R))$ *with* $f(0) \neq 0$, *let* $0 < s < r < R$ *and denote by* $N(s)$ *the number of zeros (counted with multiplicty) of* f *in* $\overline{D}(0,s)$. *Show the estimate*

$$N(s) \leq \frac{\log\left(M_r/|f(0)|\right)}{\log\left(r/s\right)}; \quad \textit{where} \quad M_r := \max\{|f(z)| \,:\, |z| = r\}.$$

Exercise 7.15. *Find a function* $f \in \mathcal{H}(\mathbb{D})$ *with* $|f(z)| < 1$ *for all* $z \in \mathbb{D}$ *so that*

$$f(1 - \tfrac{1}{n^3}) = 0 \quad \textit{and} \quad m(f, 1 - \tfrac{1}{n^3}) = n, \quad \textit{for all} \quad n \in \mathbb{N}, n \geq 2.$$

Show that there is no $g \in \mathcal{H}(\mathbb{D})$ *bounded in* $\mathbb{D}$ *with*

$$f(1 - \tfrac{1}{n^2}) = 0 \quad \textit{and} \quad m(f, 1 - \tfrac{1}{n^2}) = n, \quad \textit{for all} \quad n \in \mathbb{N}, n \geq 2.$$

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